Theorem 15 - The Alternating Series Test

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Theorem 16 - The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$.

$$s_n = u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n+1}u_n .$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Definition

A series that is convergent but not absolutely convergent is called **conditionally convergent**.

Summary of Tests to Determine Convergence or Divergence

1. **The *n*th-Term Test for Divergence**: Unless $a_n \to 0$, the series diverges.

2. **Geometric Series**: $\sum ar^n$ converges if $|r| < 1$; otherwise diverges.

3. ***p*-series**: $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise diverges.

4. **Series with nonnegative terms**: Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.

5. **Series with some negative terms**: Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.

6 **Alternating series**: $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

4) $\sum_{n=2}^{\infty}(-1)^n\frac{4}{(\ln n)^2}$ $a_n=(-1)^n\frac{4}{(\ln n)^2}$ $u_n=\frac{4}{(\ln n)^2}>0$ for all $n\geq 2$

① for all $n\geq 2$ $u_n=\frac{4}{(\ln n)^2}>0$

② $n\geq 2\Rightarrow n+1\geq n\Rightarrow \ln(n+1)\geq \ln n\Rightarrow(\ln(n+1))^2\geq(\ln n)^2$

$$\Rightarrow\frac{1}{(\ln(n+1))^2}\leq\frac{1}{(\ln n)^2}\Rightarrow\frac{4}{(\ln(n+1))^2}\leq\frac{4}{(\ln n)^2}\Rightarrow u_{n+1}\leq u_n$$

③ $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{4}{(\ln n)^2}=0\Rightarrow\lim_{n\to\infty}u_{n+1}=\lim_{n\to\infty}\frac{4}{(\ln(n+1))^2}\leq\lim_{n\to\infty}u_n=0$

by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{(\ln n)^2}$ converges

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8) $\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{(n+1)!}$ $a_n=(-1)^n\frac{10^n}{(n+1)!}$ $|a_n|=\frac{10^n}{(n+1)!}$

$$|a_{n+1}|=\frac{10^{(n+1)}}{((n+1)+1)!}=\frac{10^{(n+1)}}{(n+2)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{10^{(n+1)}}{(n+2)!}}{\frac{10^n}{(n+1)!}}\right|=\lim_{n\to\infty}\left(\frac{10^{(n+1)}}{(n+2)!}\right)\left(\frac{(n+1)!}{10^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(10^n)(10^1)}{(n+2)(n+1)!}\right)\left(\frac{(n+1)!}{10^n}\right)=\lim_{n\to\infty}\frac{10}{n+2}=0<1$$

by the Ratio Test, $\sum_{n=1}^{\infty}|a_n|=0<1$ converges

and by Absolute Convergence Test $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)\frac{10^n}{(n+1)!}$

converges absolutely

12) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ $\quad a_n = (-1)^n \ln\left(1+\frac{1}{n}\right)$ $\quad u_n = \ln\left(1+\frac{1}{n}\right)$

1. for all $n \geq 1$ $\quad u_n = \ln\left(1+\frac{1}{n}\right) \geq 0$

2. let $f(x) = \ln\left(1+\frac{1}{x}\right) = \ln\left(1+x^{-1}\right)$

$$\frac{df}{dx} = \frac{1}{(1+x^{-1})}\left(-1x^{-2}\right) = \frac{-1}{x^2\left(1+\frac{1}{x}\right)} = \frac{-1}{x^2\left(\frac{x+1}{x}\right)} = \frac{-1}{x(x+1)}$$

for $x > 0$, $\frac{df}{dx} = \frac{-1}{x(x+1)} < 0$ which means that $f(x)$ is decreasing

$\therefore$ $u_n \geq u_{n+1}$

3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln\left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right\} = \ln\{1\} = 0$

by Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ converges

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14) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$ $\quad a_n = (-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$ $\quad |a_n| = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{3\frac{\sqrt{n+1}}{\sqrt{n}}}{\frac{\sqrt{n}+1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3\sqrt{\frac{n+1}{n}}}{\frac{\sqrt{n}}{\sqrt{n}}+\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{3\sqrt{1+\frac{1}{n}}}{1+\frac{1}{\sqrt{n}}} = \frac{3\sqrt{1+0}}{1+0} = 3 \neq 0$$

by nth-Term Test for Divergence, $\sum_{n=1}^{\infty} |a_n|$ diverges

so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$ diverges

20) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$ $\quad a_n=(-1)^{n+1}\frac{n!}{2^n}$ $\quad |a_n|=\frac{n!}{2^n}$

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{n!}{2^n}=\lim_{n\to\infty}\frac{(1)(2)(3)\cdots(n-1)(n)}{\underbrace{(2)(2)(2)\cdots(2)(2)}_{n\text{-Times}}}=+\infty\neq 0$$

for $n\geq 3$, $n!>2^n$

by nth-Term Test for Divergence, $\sum_{n=1}^{\infty}|a_n|$ diverges

so $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$ diverges

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22) $\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$ $\quad a_n=(-1)^n\frac{\sin n}{n^2}$ $\quad |a_n|=\left|\frac{\sin n}{n^2}\right|=\frac{|\sin n|}{n^2}$

for $n\geq 1$ $\qquad 0<\frac{|\sin n|}{n^2}\leq\frac{1}{n^2}$

$$0<\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}\leq\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series and it converges because $p=2>1$

since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and $0<\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{|\sin n|}{n^2}\leq\sum_{n=1}^{\infty}\frac{1}{n^2}$

by the Direct Comparison Test

$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$ converges absolutely

24) $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$ $\quad a_n = \frac{(-2)^{n+1}}{n+5^n} \quad |a_n| = \frac{2^{n+1}}{n+5^n}$

for $n \geq 1$ $\quad 0 < \frac{2^{n+1}}{n+5^n} < \frac{2^{n+1}}{5^n} = \frac{(2^n)(2^1)}{5^n} = 2\left(\frac{2}{5}\right)^n$

$$0 < \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} < \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$$

$\sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n = \sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{2}{5}\right)^{n-1}$ is a geometric series with $a = \frac{4}{5}$ and $r = \frac{2}{5}$

this series converges because $|r| = \left|\frac{2}{5}\right| < 1$

since $\sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$ converges and $0 < \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} < \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$

by the Direct Comparison Test $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$ converges absolutely

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26) $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt[n]{10}\right)$ $\quad a_n = (-1)^{n+1}\left(\sqrt[n]{10}\right) \quad |a_n| = \sqrt[n]{10} = 10^{\frac{1}{n}}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 10^{\frac{1}{n}} = 10^0 = 1 \neq 0$

by nth-Term Test for Divergence, $\sum_{n=1}^{\infty} |a_n|$ diverges

so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt[n]{10}\right)$ diverges

28) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ $\quad a_n = (-1)^{n+1} \frac{1}{n \ln n} \quad |a_n| = \frac{1}{n \ln n} = u_n$

① for $n \geq 2$ $\quad u_n = \frac{1}{n \ln n} > 0$

② for $x \geq 2$ $\quad f(x) = \frac{1}{x \ln x}$ $\quad \frac{df}{dx} = \frac{(x \ln x)[0] - (1)\left[(x)\left[\frac{1}{x}\right] + (\ln x)[1]\right]}{(x \ln x)^2}$

$$= \frac{-\{1 + \ln x\}}{(x \ln x)^2} < 0$$

which implies $f(x)$ is decreasing

this means $u_n > u_{n+1} > 0$ for $n \geq 2$

③ $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n \ln n} = 0$

by Alternating Series Test, $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ converges

but $\int \frac{1}{x \ln x} dx = \int \frac{1}{p} dp = \ln|p| + C$

$p = \ln x$ $\qquad = \ln|\ln x| + C$

$dp = \frac{1}{x} dx$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{U \to \infty} \int_2^{U} \frac{1}{x \ln x} dx = \lim_{U \to \infty} \left[\ln|\ln x| + C\right]_2^U$$

$$= \lim_{U \to \infty} \left\{ \left[\underbrace{\ln|\ln U|}_{+\infty} + C\right] - \left[\ln|\ln(2)| + C\right] \right\} = +\infty$$

$\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges. By Integral test $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} u_n$ diverges.

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ converges conditionally

30) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n-\ln n}$ $\quad a_n = (-1)^n \frac{\ln n}{n-\ln n}$ $\quad |a_n| = \frac{\ln n}{n-\ln n} = u_n$

1. for $n > e$ $\quad u_n = \frac{\ln n}{n-\ln n} > 0$

2. for $x > e$ $\quad f(x) = \frac{\ln x}{x - \ln x}$

$$\frac{df}{dx} = \frac{(x-\ln x)\left[\frac{1}{x}\right] - (\ln x)\left[1-\frac{1}{x}\right]}{(x-\ln x)^2} = \frac{\left(1-\frac{\ln x}{x}\right)-\left(\ln x - \frac{\ln x}{x}\right)}{(x-\ln x)^2}$$

$$= \frac{1-\ln x}{(x-\ln x)^2} < 0$$ which implies $f(x)$ is decreasing

this means $u_n > u_{n+1} > 0$ for $n > e$

3. $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\overset{+\infty}{\ln n}}{\underset{+\infty}{n-\ln n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x}} = \lim_{n\to\infty}\left(\frac{\frac{1}{x}}{1-\frac{1}{x}}\right)\left(\frac{\frac{x}{1}}{\frac{x}{1}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{x-1} = 0$$

by Alternating Series Test, $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n-\ln n}$ converges

but for $n \geq 1$

$n - \ln n < n$

$\frac{1}{n-\ln n} > \frac{1}{n}$

$\frac{\ln n}{n-\ln n} > \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n} > \sum_{n=1}^{\infty} \frac{1}{n}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series and it diverges because $p = 1 \leq 1$

by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n} = \sum_{n=1}^{\infty} |a_n|$ diverges

Therefore $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n-\ln n}$ converges conditionaly

32) $\sum_{n=1}^{\infty} (-5)^{-n}$ $\qquad a_n = (-5)^{-n} \qquad |a_n| = 5^{-n} = \frac{1}{5^n} = \left(\frac{1}{5}\right)^n$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^{n-1}$ is a geometric series with $a = \frac{1}{5}$ and $r = \frac{1}{5}$. It converges because $|r| = \left|\frac{1}{5}\right| < 1$

since $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-5)^{-n}$ converges absolutely

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34) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ $\qquad a_n = \frac{(-1)^{n-1}}{n^2+2n+1} \qquad |a_n| = \frac{1}{n^2+2n+1}$

for $n = 1$

$$n^2+2n+1 > n^2$$

$$\Downarrow$$

$$0 < \frac{1}{n^2+2n+1} < \frac{1}{n^2}$$

$$0 < \sum_{n=1}^{\infty} \frac{1}{n^2+2n+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series and it converges because $p = 2 > 1$

since $0 < \sum_{n=1}^{\infty} \frac{1}{n^2+2n+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by Direct Comparison Test $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2+2n+1}$ converges

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ converges absolutely

36) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ for $n \geq 1$ $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$a_n = \frac{(-1)^n}{n}$ $\quad |a_n| = \frac{1}{n} = u_n$

1 for $n \geq 1$ $\quad u_n = \frac{1}{n} > 0$

2 and $\quad u_n = \frac{1}{n} > u_{n+1} = \frac{1}{n+1} > 0$

3 $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

by Alternating Series Test, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ converges

but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series and it diverges because $p = 1 \leq 1$

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ converges conditionally

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38) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$ $\quad a_n = \frac{(-1)^{n+1}(n!)^2}{(2n)!}$ $\quad |a_n| = \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} \right| = \lim_{n\to\infty} \left(\frac{((n+1)!)^2}{(2n+2)!} \right) \left(\frac{(2n)!}{(n!)^2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} \right) \left(\frac{(2n)!}{(n!)^2} \right) = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}}{\frac{4n^2}{n^2} + \frac{6n}{n^2} + \frac{2}{n^2}} = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} = \frac{1+0+0}{4+0+0} = \frac{1}{4} < 1$$

by Ratio Test $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$ converges absolutely

40) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$ $\quad a_n = (-1)^n \frac{(n!)^2 3^n}{(2n+1)!} \quad |a_n| = \frac{(n!)^2 3^n}{(2n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2 3^{n+1}}{(2(n+1)+1)!}}{\frac{(n!)^2 3^n}{(2n+1)!}} \right| = \lim_{n\to\infty} \left(\frac{((n+1)!)^2 3^{n+1}}{(2n+3)!}\right)\left(\frac{(2n+1)!}{(n!)^2 3^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{((n+1)n!)^2 (3^n)(3^1)}{(2n+3)(2n+2)(2n+1)!}\right)\left(\frac{(2n+1)!}{(n!)^2 3^n}\right) = \lim_{n\to\infty} \frac{(n+1)^2 (3)}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{3n^2+6n+3}{4n^2+10n+6} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^2}+\frac{6n}{n^2}+\frac{3}{n^2}}{\frac{4n^2}{n^2}+\frac{10n}{n^2}+\frac{6}{n^2}} = \lim_{n\to\infty} \frac{3+\frac{6}{n}+\frac{3}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}}$$

$$= \frac{3+0+0}{4+0+0} = \frac{3}{4} < 1$$

by the Ratio Test, $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$ converges absolutely

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42) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$ $\quad a_n = (-1)^n\left(\sqrt{n^2+n} - n\right) \quad |a_n| = \left(\sqrt{n^2+n} - n\right)$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \left(\frac{\sqrt{n^2+n} - n}{1}\right)\left(\frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}\right)$$

$$= \lim_{n\to\infty} \frac{(n^2+n)-(n^2)}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^2}}}{\frac{\sqrt{n^2+n} + n}{\sqrt{n^2}}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+n}}{\sqrt{n^2}} + \frac{n}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}} + \frac{n}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2} \neq 0$$

by the nth-Term Test for Divergence,
$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$ diverges

44) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ $\quad a_n = \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ $\quad |a_n| = \frac{1}{\sqrt{n}+\sqrt{n+1}} = u_n$

① for $n \geq 1$ $\quad u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} > 0$

② and $\quad u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} > u_{n+1} = \frac{1}{\sqrt{n+1}+\sqrt{n+2}} > 0$

③ $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}} = 0$

by Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges

but for $n \geq 1$ $\quad \sqrt{n}+\sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}+\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}+\sqrt{n+1}}}_{|a_n|} < \sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt{n}}}_{|b_n|} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series and it diverges because $p = \frac{1}{2} \leq 1$

$$\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)\left(\frac{\sqrt{n}}{1}\right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n}}{\sqrt{n}}+\frac{\sqrt{n+1}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{\frac{n}{n}+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+\sqrt{1+0}} = \frac{1}{2} \neq 0$$

since $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \frac{1}{2} \neq 0$ and $\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

by Limit Comparison Test, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges conditionaly

46) $\sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n = \sum_{n=1}^{\infty}(-1)^n \frac{1}{\sinh n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{\left(\frac{e^n-e^{-n}}{2}\right)} = \sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{e^n-e^{-n}}\right)$

$a_n = (-1)^n\left(\frac{2}{e^n-e^{-n}}\right) \qquad |a_n| = \frac{2}{e^n-e^{-n}}$

for $n \geq 1 \quad e^n - e^{-n} < e^n \Rightarrow \frac{1}{e^n-e^{-n}} > \frac{1}{e^n} \Rightarrow \frac{2}{e^n-e^{-n}} > \frac{2}{e^n}$

$$\sum_{n=1}^{\infty}\underbrace{\frac{2}{e^n-e^{-n}}}_{|a_n|} > \sum_{n=1}^{\infty}\underbrace{\frac{2}{e^n}}_{|b_n|}$$

$\sum_{n=1}^{\infty}\frac{2}{e^n} = \sum_{n=1}^{\infty}\frac{2}{e}\left(\frac{1}{e}\right)^{n-1}$ this is a geometric series with $a = \frac{2}{e}$ and $r = \frac{1}{e}$

and this is convergent because $|r| = \left|\frac{1}{e}\right| < 1$

$$\lim_{n\to\infty}\frac{|a_n|}{|b_n|} = \lim_{n\to\infty}\frac{\frac{2}{e^n-e^{-n}}}{\frac{2}{e^n}} = \lim_{n\to\infty}\left(\frac{2}{e^n-e^{-n}}\right)\left(\frac{e^n}{2}\right) = \lim_{n\to\infty}\frac{e^n}{e^n-e^{-n}}$$

$$= \lim_{n\to\infty}\left(\frac{\frac{e^n}{1}}{\frac{e^n}{1}-\frac{1}{e^n}}\right)\left(\frac{\frac{1}{e^n}}{\frac{1}{e^n}}\right) = \lim_{n\to\infty}\frac{1}{1-\frac{1}{e^{2n}}} = \frac{1}{1-0} = 1 \neq 0$$

since $\lim_{n\to\infty}\frac{|a_n|}{|b_n|} = 1 \neq 0$ and $\sum_{n=1}^{\infty}|b_n| = \sum_{n=1}^{\infty}\frac{2}{e^n}$ converges

by Limit Comparison Test, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{2}{e^n-e^{-n}}$ converges

Therefore, $\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{e^n-e^{-n}}\right) = \sum_{n=1}^{\infty}(-1)^n \operatorname{csch} n$

converges absolutely

48) $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}+\cdots=\sum_{n=1}^{\infty} a_n$

$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\frac{1}{64}+\cdots=\sum_{n=1}^{\infty} |a_n|$

$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{1}{n^2}$ this is p-series

and it converges because $p=2>1$ {absolute convergence}

$\sum_{n=1}^{\infty} a_n$ converges because absolute convergence implies convergence.

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50) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$

5th term will be when $n=5$

$|\text{error}|<\left|(-1)^{(5)+1}\frac{1}{10^{(5)}}\right|=\left|(-1)^6\frac{1}{100000}\right|$

$=\frac{1}{100000}=0.00001$

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52) $\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n,\ 0<t<1$

5th term will be when $n=4$

$|\text{error}|<\left|(-1)^{(4)}t^{(4)}\right|=t^4<1$

because $0<t<1$

64) $\sum_{n=2}^{\infty} \frac{3}{10+n^{4/3}}$

for $n \geq 2$ $\quad 10+n^{4/3} > n^{\frac{4}{3}} \Rightarrow \frac{1}{10+n^{4/3}} < \frac{1}{n^{4/3}}$

$$\Rightarrow \frac{3}{10+n^{4/3}} < \frac{3}{n^{4/3}} \Rightarrow 0 < \sum_{n=2}^{\infty} \frac{3}{10+n^{4/3}} < \sum_{n=2}^{\infty} \frac{3}{n^{4/3}}$$

$\sum_{n=2}^{\infty} \frac{3}{n^{4/3}}$ is a part of p-series that converges because $p = \frac{4}{3} > 1$ so $\sum_{n=2}^{\infty} \frac{3}{n^{4/3}}$ converges

since $0 < \sum_{n=2}^{\infty} \frac{3}{10+n^{4/3}} < \sum_{n=2}^{\infty} \frac{3}{n^{4/3}}$ and $\sum_{n=2}^{\infty} \frac{3}{n^{4/3}}$ converges by Direct Comparison Test, $\sum_{n=2}^{\infty} \frac{3}{10+n^{4/3}}$ converges

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68) $\sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n \quad a_n = \frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n \quad |a_n| = \frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)+1}{((n+1)+2)!}\left(\frac{3}{2}\right)^{n+1}}{\frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty}\left(\frac{n+2}{(n+3)!}\left(\frac{3}{2}\right)^{n+1}\right)\left(\frac{(n+2)!}{n+1}\left(\frac{2}{3}\right)^n\right)$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{(n+3)(n+2)!}\left(\frac{3}{2}\right)^n\left(\frac{3}{2}\right)^1\right)\left(\frac{(n+2)!}{n+1}\left(\frac{2}{3}\right)^n\right) = \lim_{n\to\infty}\left(\frac{n+2}{(n+3)(n+1)}\right)\left(\frac{3}{2}\right)$$

$$= \lim_{n\to\infty} \frac{\overset{+\infty}{3n+6}}{\underset{+\infty}{2n^2+8n+6}} \overset{L}{=} \lim_{n\to\infty} \frac{3}{4n+8} = 0 < 1$$

by Ratio Test, $\sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^n$ converges

76) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^3 \qquad a_n=\left(\frac{\ln n}{n}\right)^3$

for $n \geq 2 \quad \ln n > 1 \Rightarrow \dfrac{\ln n}{n} > \dfrac{1}{n} \Rightarrow \left(\dfrac{\ln n}{n}\right)^3 > \dfrac{1}{n^3}$

$$\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^3 > \sum_{n=2}^{\infty}\frac{1}{n^3}$$

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3}$ is a p-series; because $p=3>1$, it converges.

$b_n = \dfrac{1}{n^3}$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\left(\frac{\ln n}{n}\right)^3}{\frac{1}{n^3}}=\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)^3\left(\frac{n^3}{1}\right)=\lim_{n\to\infty}(\ln n)^3=+\infty$$

since $\displaystyle\sum_{n=2}^{\infty} b_n=\sum_{n=2}^{\infty}\frac{1}{n^3}$ converges and $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=+\infty$

the Limit Comparison Test is inconclusive

so we need to use a different $\displaystyle\sum_{n=2}^{\infty} b_n$

let $\displaystyle\sum_{n=2}^{\infty} b_n=\sum_{n=2}^{\infty}\frac{1}{n^2}$ which is a convergent p-series because $p=2>1$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\left(\frac{\ln n}{n}\right)^3}{\frac{1}{n^2}}=\lim_{n\to\infty}\left(\frac{\ln n}{n}\right)^3\left(\frac{n^2}{1}\right)=\lim_{n\to\infty}\frac{\overset{+\infty}{(\ln n)^3}}{\underset{+\infty}{n}}$$

$$\overset{L}{=}\lim_{n\to\infty}\frac{3(\ln n)^2\left(\frac{1}{n}\right)}{1}=\lim_{n\to\infty}\frac{3\overset{+\infty}{(\ln n)^2}}{\underset{+\infty}{n}}\overset{L}{=}\lim_{n\to\infty}\frac{6(\ln n)\left(\frac{1}{n}\right)}{1}=\lim_{n\to\infty}\frac{\overset{+\infty}{6(\ln n)}}{\underset{+\infty}{n}}$$

$$\overset{L}{=}\lim_{n\to\infty}\frac{6\left(\frac{1}{n}\right)}{1}=\lim_{n\to\infty}\frac{6}{n}=0$$

since $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=0$ and $\displaystyle\sum_{n=2}^{\infty} b_n=\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges

by Limit Comparison Test, $\displaystyle\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^3$ converges