Definition

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$ converges.

Theorem 12 - The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 13 - The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho .$$

Then **(a)** the series *converges absolutely* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

Theorem 14 - The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho .$$

Then **(a)** the series *converges absolutely* if $\rho < 1$,

(b) the series *diverges* if $\rho > 1$ or ρ is infinite,

(c) the test is *inconclusive* if $\rho = 1$.

2) $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$ $\quad a_n = (-1)^n \frac{n+2}{3^n}$ $\quad a_{n+1} = (-1)^{(n+1)} \frac{(n+1)+2}{3^{(n+1)}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{(n+1)} \frac{(n+1)+2}{3^{(n+1)}}}{(-1)^n \frac{n+2}{3^n}} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)+2}{3^{(n+1)}}}{\frac{n+2}{3^n}}$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)+2}{3^{(n+1)}}\right)\left(\frac{3^n}{n+2}\right) = \lim_{n\to\infty} \left(\frac{n+3}{(3^n)(3^1)}\right)\left(\frac{3^n}{n+2}\right) = \lim_{n\to\infty} \frac{n+3}{3(n+2)}$$

$$= \lim_{n\to\infty} \frac{n+3}{3n+6} \stackrel{L}{=} \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1$$

since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1$, by the Ratio Test the series converges absolutely

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4) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n\,3^{n-1}}$ $\quad a_n = \frac{2^{(n+1)}}{n\,3^{(n-1)}}$ $\quad a_{n+1} = \frac{2^{(n+1)+1}}{(n+1)\,3^{((n+1)-1)}} = \frac{2^{(n+2)}}{(n+1)\,3^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{2^{(n+2)}}{(n+1)\,3^n}}{\frac{2^{(n+1)}}{n\,3^{(n-1)}}} \right| = \lim_{n\to\infty} \left(\frac{2^{(n+2)}}{(n+1)\,3^n}\right)\left(\frac{n\,3^{(n-1)}}{2^{(n+1)}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(2^{(n+1)})(2^1)}{(n+1)(3^{(n-1)})(3^1)}\right)\left(\frac{n\,3^{(n-1)}}{2^{(n+1)}}\right) = \lim_{n\to\infty} \frac{(2)(n)}{(n+1)(3)} = \lim_{n\to\infty} \frac{2n}{3n+3}$$

$$\stackrel{L}{=} \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3} < 1$$

since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} < 1$, by the Ratio Test the series converges absolutely

6) $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ $\quad a_n = \frac{3^{(n+2)}}{\ln n} \quad a_{n+1} = \frac{3^{((n+1)+2)}}{\ln(n+1)} = \frac{3^{(n+3)}}{\ln(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{(n+3)}}{\ln(n+1)}}{\frac{3^{(n+2)}}{\ln n}}\right| = \lim_{n\to\infty}\left(\frac{3^{(n+3)}}{\ln(n+1)}\right)\left(\frac{\ln n}{3^{(n+2)}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(3^{(n+2)})(3^1)}{\ln(n+1)}\right)\left(\frac{\ln n}{3^{(n+2)}}\right) = \lim_{n\to\infty}\frac{3\ln n}{\ln(n+1)} \overset{L}{=} \lim_{n\to\infty}\frac{\frac{3}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{3(n+1)}{n}$$

$$= \lim_{n\to\infty}\frac{3n+3}{n} \overset{L}{=} \lim_{n\to\infty}\frac{3}{1} = 3 > 1$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3 > 1$, by the Ratio Test the series diverges

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8) $\sum_{n=1}^{\infty} \frac{n\,5^n}{(2n+3)\ln(n+1)}$ $\quad a_n = \frac{n\,5^n}{(2n+3)\ln(n+1)} \quad a_{n+1} = \frac{(n+1)\,5^{(n+1)}}{(2(n+1)+3)\ln((n+1)+1)} = \frac{(n+1)\,5^{(n+1)}}{(2n+5)\ln(n+2)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)\,5^{(n+1)}}{(2n+5)\ln(n+2)}}{\frac{n\,5^n}{(2n+3)\ln(n+1)}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)\,5^{(n+1)}}{(2n+5)\ln(n+2)}\right)\left(\frac{(2n+3)\ln(n+1)}{n\,5^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\,5^n(5^1)}{(2n+5)\ln(n+2)}\right)\left(\frac{(2n+3)\ln(n+1)}{n\,5^n}\right) = \lim_{n\to\infty}\left(\frac{5(n+1)(2n+3)}{n(2n+5)}\right)\left(\frac{\ln(n+1)}{\ln(n+2)}\right)$$

$$= \left\{\lim_{n\to\infty}\frac{10n^2+25n+15}{2n^2+10n}\right\}\left\{\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n+2)}\right\} = \{5\}\{1\} = 5 > 1$$

$$\lim_{n\to\infty}\frac{10n^2+25n+15}{2n^2+10n} \overset{L}{=} \lim_{n\to\infty}\frac{20n+25}{4n} \overset{L}{=} \lim_{n\to\infty}\frac{20}{4} = \frac{20}{4} = 5$$

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n+2)} \overset{L}{=} \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n+2}} = \lim_{n\to\infty}\frac{n+2}{n+1} \overset{L}{=} \lim_{n\to\infty}\frac{1}{1} = \frac{1}{1} = 1$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5 > 1$, by the Ratio Test the series diverges

10) $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ $\quad a_n = \frac{4^n}{(3n)^n} = \left(\frac{4}{3n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4}{3n}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4}{3n}\right)^n} = \lim_{n\to\infty} \frac{4}{3n} = 0 < 1$$

since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$, by the Root Test the series converges absolutely

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12) $\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$ $\quad a_n = \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{(n+1)}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(-\ln\left(e^2+\frac{1}{n}\right)\right)^{(n+1)}\right|} = \lim_{n\to\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{\frac{(n+1)}{n}}$$

$$= \lim_{n\to\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{\left(1+\frac{1}{n}\right)} = \left(\ln\left(e^2+0\right)\right)^{(1+0)} = \left(\ln\left(e^2\right)\right)^1 = 2 > 1$$

since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 2 > 1$, by the Root Test the series diverges

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14) $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$ $\quad a_n = \sin^n\left(\frac{1}{\sqrt{n}}\right) = \left(\sin\left(\frac{1}{\sqrt{n}}\right)\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\sin\left(\frac{1}{\sqrt{n}}\right)\right)^n\right|} = \lim_{n\to\infty} \left(\sin\left(\frac{1}{\sqrt{n}}\right)\right) = \sin(0) = 0 < 1$$

since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$, by the Ratio Test the series converges absolutely

16) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$ $\quad a_n = \frac{(-1)^n}{n^{(1+n)}}$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{n^{(1+n)}}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{n^{\frac{(1+n)}{n}}} = \lim_{n\to\infty} \frac{1}{n^{(\frac{1}{n}+1)}}$

$= \lim_{n\to\infty} \frac{1}{(n^{\frac{1}{n}})(n^1)} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})(n)} = 0 < 1$

since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$, by the Root Test the series converges absolutely

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20) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$ $\quad a_n = \left(\frac{n-2}{n}\right)^n = \left(\frac{n}{n} - \frac{2}{n}\right)^n = \left(1 - \frac{2}{n}\right)^n = \left(1 + \frac{(-2)}{n}\right)^n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{(-2)}{n}\right)^n = e^{(-2)} = \frac{1}{e^2} \neq 0$ {sec. 10.1 Thm 5 #5}

since $\lim_{n\to\infty} a_n = \frac{1}{e^2} \neq 0$, by the nth-Term Test for Divergence this series diverges.

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24) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \frac{(-2)^1}{3^1} + \frac{(-2)^2}{3^2} + \frac{(-2)^3}{3^3} + \dots$

$= \frac{-2}{3}(1) + \frac{-2}{3}\left(\frac{-2}{3}\right)^1 + \frac{-2}{3}\left(\frac{-2}{3}\right)^2 + \dots = \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right)^{n-1}$

this is a geometric series with $a = \frac{-2}{3}$ and $r = \frac{-2}{3}$

since $|r| = \left|\frac{-2}{3}\right| < 1$ this series converges to $\frac{a}{1-r}$

$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)\left(\frac{-2}{3}\right)^{n-1} = \frac{\left(\frac{-2}{3}\right)}{1-\left(\frac{-2}{3}\right)} = \frac{\frac{-2}{3}}{\frac{5}{3}} = \frac{-2}{5}$

26) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n \qquad a_n = \left(1 - \frac{1}{3n}\right)^n = \left(1 + \frac{\left(\frac{-1}{3}\right)}{n}\right)^n$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{\left(\frac{-1}{3}\right)}{n}\right)^n = e^{\left(\frac{-1}{3}\right)} = \frac{1}{\sqrt[3]{e}} \neq 0$ {sec 10.1 Thm 5 #5}

since $\lim_{n \to \infty} a_n = \frac{1}{\sqrt[3]{a}} \neq 0$, by the nth-Term Test for Divergence this series diverges.

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32) $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n} \qquad a_n = \frac{n \ln n}{(-2)^n} \qquad a_{n+1} = \frac{(n+1)\ln(n+1)}{(-2)^{(n+1)}}$

$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left|\frac{\frac{(n+1)\ln(n+1)}{(-2)^{(n+1)}}}{\frac{n \ln n}{(-2)^n}}\right| = \lim_{n \to \infty} \left(\frac{(n+1)\ln(n+1)}{2^{(n+1)}}\right)\left(\frac{2^n}{n \ln n}\right)$

$= \lim_{n \to \infty} \left(\frac{(n+1)\ln(n+1)}{(2^n)(2^1)}\right)\left(\frac{2^n}{n \ln n}\right) = \lim_{n \to \infty} \left(\frac{n+1}{2n}\right)\left(\frac{\ln(n+1)}{\ln n}\right)$

$= \left\{\lim_{n \to \infty} \frac{n+1}{2n}\right\}\left\{\lim_{n \to \infty} \frac{\ln(n+1)}{\ln n}\right\} = \left\{\frac{1}{2}\right\}\left\{1\right\} = \frac{1}{2} < 1$

$\lim_{n \to \infty} \frac{n+1}{2n} \overset{L}{=} \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$

$\lim_{n \to \infty} \frac{\ln(n+1)}{\ln n} \overset{L}{=} \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{n+1} \overset{L}{=} \lim_{n \to \infty} \frac{1}{1} = \frac{1}{1} = 1$

since $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$, by the Ratio Test this series converges absolutely

34) $\sum_{n=1}^{\infty} e^{-n}(n^3)$ $\quad a_n = \frac{n^3}{e^n} \quad a_{n+1} = \frac{(n+1)^3}{e^{(n+1)}} = \frac{n^3+3n^2+3n+1}{(e^n)(e^1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^3}{e^{(n+1)}}}{\frac{n^3}{e^n}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^3}{e^{(n+1)}}\right)\left(\frac{e^n}{n^3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^3+3n^2+3n+1}{(e^n)(e^1)}\right)\left(\frac{e^n}{n^3}\right) = \lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{en^3}$$

$$= \lim_{n\to\infty}\left(\frac{n^3}{en^3}+\frac{3n^2}{en^3}+\frac{3n}{en^3}+\frac{1}{en^3}\right) = \lim_{n\to\infty}\left(\frac{1}{e}+\frac{3}{en}+\frac{3}{en^2}+\frac{1}{en^3}\right) = \frac{1}{e}+0+0+0 = \frac{1}{e} < 1$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1$, by the Ratio Test this series converges absolutely

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36) $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$ $\quad a_n = \frac{n2^n(n+1)!}{3^n n!} \quad a_{n+1} = \frac{(n+1)2^{(n+1)}((n+1)+1)!}{3^{(n+1)}(n+1)!} = \frac{(n+1)2^{(n+1)}(n+2)!}{3^{(n+1)}(n+1)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)2^{(n+1)}(n+2)!}{3^{(n+1)}(n+1)!}}{\frac{n2^n(n+1)!}{3^n n!}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)2^{(n+1)}(n+2)!}{3^{(n+1)}(n+1)!}\right)\left(\frac{3^n n!}{n2^n(n+1)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)(2^n)(2^1)(n+2)(n+1)!}{(3^n)(3^1)(n+1)n!}\right)\left(\frac{3^n n!}{n(2^n)(n+1)!}\right) = \lim_{n\to\infty}\frac{2(n+1)(n+2)}{3n(n+1)}$$

$$= \lim_{n\to\infty}\frac{2n^2+6n+4}{3n^2+3n} \overset{L}{=} \lim_{n\to\infty}\frac{4n+6}{6n+3} \overset{L}{=} \lim_{n\to\infty}\frac{4}{6} = \frac{4}{6} = \frac{2}{3} < 1$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} < 1$, by the Ratio Test this series converges absolutely

38) $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$ $\quad a_n = \frac{n!}{(-n)^n} \quad a_{n+1} = \frac{(n+1)!}{(-(n+1))^{(n+1)}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(-(n+1))^{(n+1)}}}{\frac{n!}{(-n)^n}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{(n+1)}}\right)\left(\frac{n^n}{n!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\,n!}{((n+1)^n)((n+1)^1)}\right)\left(\frac{n^n}{n!}\right) = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(\frac{n}{n}+\frac{1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e^1} = \frac{1}{e} < 1$$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1$ {sec. 10.1 Thm 5 #5}

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1$, by the Ratio Test this series converges absolutely

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40) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{\frac{n}{2}}}$ $\quad a_n = \frac{n}{(\ln n)^{\frac{n}{2}}}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{n}{(\ln n)^{\frac{n}{2}}}\right|} = \lim_{n\to\infty}\frac{\sqrt[n]{n}}{(\ln n)^{\frac{1}{2}}} = \frac{\lim_{n\to\infty}\sqrt[n]{n}}{\lim_{n\to\infty}\sqrt{\ln n}} = \frac{1}{\lim_{n\to\infty}\sqrt{\ln n}}$$

$$= 0 < 1$$

$\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$

$y = n^{\frac{1}{n}}$

$\ln y = \ln\left(n^{\frac{1}{n}}\right)$

$\ln y = \frac{\ln n}{n}$

$\lim_{n\to\infty}\frac{\ln n}{n} \overset{L}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{1}$

$= \lim_{n\to\infty}\frac{1}{n} = 0$

$\ln y = 0 \Rightarrow y = e^0 = 1$

since $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 0 < 1$, by the Root Test this series converges absolutely

42) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$ $\qquad a_n = \frac{(-3)^n}{n^3 2^n} \qquad a_{n+1} = \frac{(-3)^{(n+1)}}{(n+1)^3 2^{(n+1)}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-3)^{(n+1)}}{(n+1)^3 2^{(n+1)}}}{\frac{(-3)^n}{n^3 2^n}}\right| = \lim_{n\to\infty}\left(\frac{3^{(n+1)}}{(n+1)^3 2^{(n+1)}}\right)\left(\frac{n^3 2^n}{3^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(3^n)(3^1)}{(n+1)^3(2^n)(2^1)}\right)\left(\frac{n^3(2^n)}{(3^n)}\right) = \lim_{n\to\infty}\frac{3n^3}{2(n+1)^3} = \lim_{n\to\infty}\frac{3n^3}{2n^3+6n^2+6n+2}$$

$$\stackrel{L}{=} \lim_{n\to\infty}\frac{9n^2}{6n^2+12n+6} \stackrel{L}{=} \lim_{n\to\infty}\frac{18n}{12n+12} \stackrel{L}{=} \lim_{n\to\infty}\frac{18}{12} = \frac{18}{12} = \frac{3}{2} > 1$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{2} > 1$, by the Ratio Test this series diverges

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44) $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$ $\qquad a_n = \frac{(2n+3)(2^n+3)}{3^n+2} \qquad a_{n+1} = \frac{(2(n+1)+3)(2^{(n+1)}+3)}{3^{(n+1)}+2} = \frac{(2n+5)(2^{(n+1)}+3)}{3^{(n+1)}+2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2n+5)(2^{(n+1)}+3)}{3^{(n+1)}+2}}{\frac{(2n+3)(2^n+3)}{3^n+2}}\right| = \lim_{n\to\infty}\left(\frac{(2n+5)(2^{(n+1)}+3)}{3^{(n+1)}+2}\right)\left(\frac{3^n+2}{(2n+3)(2^n+3)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n+5}{2n+3}\right)\left(\frac{((2^n)(2^1)+3)(3^n+2)}{((3^n)(3^1)+2)(2^n+3)}\right) = \lim_{n\to\infty}\left(\frac{2n+5}{2n+3}\right)\left(\frac{(2)(2^n)(3^n)+(2)(2^n)(2)+(3)(3^n)+6}{(3)(3^n)(2^n)+(3)(3^n)(3)+(2)(2^n)+6}\right)$$

$$= \left\{\lim_{n\to\infty}\frac{2n+5}{2n+3}\right\}\left\{\lim_{n\to\infty}\frac{2(6^n)+4(2^n)+3(3^n)+6}{3(6^n)+9(3^n)+2(2^n)+6}\right\} = \{1\}\left\{\frac{2}{3}\right\} = \frac{2}{3} < 1$$

$$\lim_{n\to\infty}\frac{2n+5}{2n+3} \stackrel{L}{=} \lim_{n\to\infty}\frac{2}{2} = \frac{2}{2} = 1$$

$$\lim_{n\to\infty}\frac{2(6^n)+4(2^n)+3(3^n)+6}{3(6^n)+9(3^n)+2(2^n)+6} = \lim_{n\to\infty}\frac{\frac{2(6^n)}{6^n}+\frac{4(2^n)}{6^n}+\frac{3(3^n)}{6^n}+\frac{6}{6^n}}{\frac{3(6^n)}{6^n}+\frac{9(3^n)}{6^n}+\frac{2(2^n)}{6^n}+\frac{6}{6^n}}$$

$$= \lim_{n\to\infty}\frac{2+\frac{4}{3^n}+\frac{3}{2^n}+\frac{6}{6^n}}{3+\frac{9}{2^n}+\frac{2}{3^n}+\frac{6}{6^n}} = \frac{2+0+0+0}{3+0+0+0} = \frac{2}{3}$$

since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3} < 1$, by the Ratio Test this series converges absolutely

46) $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

for $n \geq 3$, $n^{(2^n)} > n^{(n^2)} > 3^{n^2}$

so $\frac{2^{n^2}}{n^{(2^n)}} < \frac{2^{n^2}}{n^{(n^2)}} < \frac{2^{n^2}}{3^{n^2}} = \left(\frac{2}{3}\right)^{n^2} < \left(\frac{2}{3}\right)^n \Rightarrow 0 < \frac{2^{n^2}}{n^{(2^n)}} < \left(\frac{2}{3}\right)^n$

since $\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^3 \left(\frac{2}{3}\right)^{n-1}$ which is a geometric series with $a = \left(\frac{2}{3}\right)^3$ and $r = \frac{2}{3}$

this geometric series converges because $|r| = \left|\frac{2}{3}\right| < 1$

and $0 < \sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}} < \sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^3 \left(\frac{2}{3}\right)^{n-1}$, by the Direct Comparison Test $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$ converges

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60) $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$ $\qquad a_n = \frac{(n!)^n}{n^{(n^2)}} = \frac{(n!)^n}{(n^n)^n} = \left(\frac{n!}{n^n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(n!)^n}{n^{(n^2)}}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n!}{n^n}\right)^n} = \lim_{n\to\infty} \frac{n!}{n^n} = 0 < 1$

$\lim_{n\to\infty} \frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right) \leq \lim_{n\to\infty} \frac{1}{n} = 0$

by Direct Comparison Test $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0 < 1$, by the Root Test this series converges absolutely.