Theorem 10 - Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.

2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Theorem 11 - Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

2) $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

for $n \geq 1$

$n^4+2 \geq n^4$

$\Downarrow$

$\frac{1}{n^4+2} \leq \frac{1}{n^4}$

$\Downarrow$

$\frac{n}{n^4+2} \leq \frac{n}{n^4}$

$\Downarrow$

$\frac{n-1}{n^4+2} \leq \frac{n}{n^4+2} \leq \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{n-1}{n^4+2} \leq \sum_{n=1}^{\infty} \frac{n}{n^4+2} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$ have nonnegative terms for $n \geq 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series because $p=3>1$

So by Comparison Test $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ converges.

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6) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$

for $n \geq 1$

$n3^n \geq 3^n$

$\Downarrow$

$\frac{1}{n3^n} \leq \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{n3^n} \leq \sum_{n=1}^{\infty} \frac{1}{3^n}$ have nonnegative terms for $n \geq 1$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} + \cdots$$

$$= \frac{1}{3}(1) + \frac{1}{3}\left(\frac{1}{3}\right)^1 + \frac{1}{3}\left(\frac{1}{3}\right)^2 + \cdots + \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} + \cdots$$

this is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$

since $|r| = \left|\frac{1}{3}\right| < 1$ $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges

So by Comparison Test $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ converges.

8) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

for $n \ge 1$

$1 \le \sqrt{n}$

$\Downarrow$

$2 \le 2\sqrt{n}$

$\Downarrow$

$3 = 2+1 \le 2\sqrt{n}+1$

$\Downarrow$

$3 \le 3n \le n(2\sqrt{n}+1)$

$\Downarrow$

$3 \le 2n\sqrt{n} + n$

$\Downarrow$

$n^2+3 \le n^2+2n\sqrt{n}+n$

$\Downarrow$

$$1 \le \frac{n^2+2n\sqrt{n}+n}{n^2+3}$$

$\Downarrow$

$$1 \le \frac{n(n+2\sqrt{n}+1)}{n^2+3}$$

$\Downarrow$

$$\frac{1}{n} \le \frac{n+2\sqrt{n}+1}{n^2+3}$$

$\Downarrow$

$$\frac{1}{n} \le \frac{(\sqrt{n}+1)^2}{n^2+3}$$

$\Downarrow$

$$\sqrt{\frac{1}{n}} \le \sqrt{\frac{(\sqrt{n}+1)^2}{n^2+3}}$$

$\Downarrow$

$$\frac{\sqrt{1}}{\sqrt{n}} \le \frac{\sqrt{(\sqrt{n}+1)^2}}{\sqrt{n^2+3}}$$

$\Downarrow$

$$\frac{1}{\sqrt{n}} \le \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \le \sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ have nonnegative terms for $n \ge 1$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series because $p = \frac{1}{2} \le 1$

So by Comparison Test $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges

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10) $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^2+2}}$ compare with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series because $p = \frac{1}{2} \le 1$

also both series have positive terms for $n \ge 1$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{\frac{n+1}{n^2+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{n(n+1)}{n^2+2}} = \lim_{n\to\infty}\sqrt{\frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty}\left(\frac{\overset{+\infty}{n^2+n}}{\underset{+\infty}{n^2+2}}\right)} \overset{L}{=} \sqrt{\lim_{n\to\infty}\left(\frac{\overset{+\infty}{2n+1}}{\underset{+\infty}{2n}}\right)}$$

$$\overset{L}{=} \sqrt{\lim_{n\to\infty}\left(\frac{2}{2}\right)} = \sqrt{\lim_{n\to\infty} 1} = \sqrt{1} = 1 > 0$$

$\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^2+2}}$ diverges by Limit Comparison Test

14) $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ compare with $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ is a convergent geometric series because $|r| = |\frac{2}{5}| < 1$.

also both series have positive terms for $n \geq 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{\left(\frac{2}{5}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+3}{5n+4}\right)^n \left(\frac{5}{2}\right)^n = \lim_{n\to\infty} \left(\left(\frac{2n+3}{5n+4}\right)\left(\frac{5}{2}\right)\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n = \left(e^{\frac{7}{10}}\right) > 0$$

$$y = \left(\frac{10n+15}{10n+8}\right)^n$$

$$\ln y = \ln\left(\frac{10n+15}{10n+8}\right)^n$$

$$\ln y = n \ln\left(\frac{10n+15}{10n+8}\right)$$

$$\ln y = \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{\overset{0}{\ln\left(\frac{10n+15}{10n+8}\right)}}{\underset{0}{\frac{1}{n}}} = \lim_{n\to\infty} \frac{\ln(10n+15) - \ln(10n+8)}{\frac{1}{n}}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{\frac{10(10n+8) - 10(10n+15)}{(10n+15)(10n+8)}}{\frac{-1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{100n+80-100n-150}{(10n+15)(10n+8)}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \left(\frac{-70}{(10n+15)(10n+8)}\right)\left(\frac{n^2}{-1}\right)$$

$$= \lim_{n\to\infty} \frac{\overset{+\infty}{70n^2}}{\underset{+\infty}{100n^2+230n+120}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{140n}}{\underset{+\infty}{200n+230}}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{140}{200} = \frac{140}{200} = \frac{14}{20} = \frac{7}{10}$$

$$\ln y = \frac{7}{10}$$

$$\Downarrow$$

$$y = e^{\frac{7}{10}}$$

by Limit Comparison Test, since $\lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{\left(\frac{2}{5}\right)^n} = e^{\frac{7}{10}} > 0$

and $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ is a convergent geometric series,

$\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ converges

16) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series since $p = 2 > 1$.

also both series have positive terms for $n \geq 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(\frac{-2}{n^3}\right)}{\left(\frac{-2}{n^3}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}}$$

$$= \frac{1}{1+0} = 1 > 0$$

by Limit Comparison Test, since $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ converges

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18) $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

for $n \geq 1$

$n + \sqrt{n} < n + n + n = 3n$

$\Downarrow$

$\frac{1}{n} < \frac{3}{n+\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a diverging p-series because $p = 1 \leq 1$

$\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges

compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series with $p = 1 \leq 1$

also both series have positive terms for $n \geq 1$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} \overset{L}{=} \lim_{n\to\infty} \frac{3}{1+\frac{1}{2\sqrt{n}}}$$

$$= \frac{3}{1+0} = 3 > 0$$

by Limit Comparison Test, since $\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = 3 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a diverging p-series,

$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges

22) $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

for $n \geq 1$

$$\frac{n+1}{n^2\sqrt{n}} \geq \frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series because $p = \frac{3}{2} > 1$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \left(\frac{n+1}{n^2\sqrt{n}}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(\frac{n}{n} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1 > 0$$

since $\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series.

by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ converges

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24) $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$

for $n \geq 3$

$$\frac{5n^3-3n}{n^2(n-2)(n^2+5)} \geq \frac{n^3}{n^2(n)(n^2)} = \frac{1}{n^2}$$

$$\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)} \geq \sum_{n=3}^{\infty} \frac{1}{n^2}$$

$\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent p-series because $p = 2 > 1$

$$\lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n-2)(n^2+5)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{5n^3-3n}{n^2(n-2)(n^2+5)}\right)\left(\frac{n^2}{1}\right)$$

$$= \lim_{n\to\infty} \frac{\overset{+\infty}{5n^3-3n}}{\underset{+\infty}{n^3-2n^2+5n-10}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{15n^2-3}}{\underset{+\infty}{3n^2-4n+5}}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{30n}}{\underset{+\infty}{6n-4}} \overset{L}{=} \lim_{n\to\infty} \frac{30}{6} = \frac{30}{6} = 5 > 0$$

since $\lim_{n\to\infty} \frac{\frac{5n^3-3n}{n^2(n-2)(n^2+5)}}{\frac{1}{n^2}} = 5 > 0$ and $\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent p-series, by the Limit Comparison Test, $\sum_{n=3}^{\infty} \frac{5n^3-3n}{n^2(n-2)(n^2+5)}$ converges

26) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series because $p = \frac{3}{2} > 1$

for $n \geq 1$

$0 \leq \frac{1}{\sqrt{n^3+2}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

$0 \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ converges

we want the start the ratios this way so we can end up

$$\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{1}{\sqrt{n^3+2}}} = \lim_{n\to\infty} \frac{\sqrt{n^3+2}}{\sqrt{n^3}} = \lim_{n\to\infty} \sqrt{\frac{n^3+2}{n^3}} = \lim_{n\to\infty} \sqrt{\frac{n^3}{n^3}+\frac{2}{n^3}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n^3}}$$

$$= \sqrt{1+0} = 1 > 0$$

since $\lim_{n\to\infty} \frac{\frac{1}{n^{3/2}}}{\frac{1}{\sqrt{n^3+2}}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$ converges

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30) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ this is similar to exercise 28

use the Limit Comparison Test

$\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ is a convergent p-series because $p = \frac{5}{4} > 1$

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \left(\frac{(\ln n)^2}{n^{3/2}}\right)\left(\frac{n^{5/4}}{1}\right) = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{1/4}} \overset{L}{=} \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{\frac{1}{4}n^{-3/4}} = \lim_{n\to\infty} \frac{8 \ln n}{n^{1/4}}$$

$$\overset{L}{=} \lim_{n\to\infty} \frac{8\left(\frac{1}{n}\right)}{\frac{1}{4}n^{-3/4}} = \lim_{n\to\infty} \frac{(4)(8)}{n^{1/4}} = 0$$

since $\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ is a convergent p-series by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ converges

34) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

for $n \geq 1$

$0 \leq \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$0 \leq \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series because $p = \frac{3}{2} > 1$

by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges

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36) $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \left(\frac{n}{n^2 \cdot 2^n} + \frac{2^n}{n^2 2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right)$

for $n \geq 1$

$0 \leq \frac{1}{n2^n} + \frac{1}{n^2} \leq \frac{1}{2^n} + \frac{1}{n^2} = \left(\frac{1}{2}\right)^n + \frac{1}{n^2}$

$0 \leq \sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right) \leq \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \frac{1}{n^2}\right)$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series with $a = 1$, $r = \frac{1}{2}$ $(|r| = |\frac{1}{2}| < 1)$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series because $p = 2 > 1$

$\sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^n + \frac{1}{n^2}\right)$ converges

by Direct Comparison Test, $\sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ converges

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38) $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ use the n th-term test

$$\lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} = \lim_{n\to\infty} \left(\frac{3^{n-1}}{3^n} + \frac{1}{3^n}\right) = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3^n}\right) = \frac{1}{3} + 0 = \frac{1}{3} \neq 0$$

since $\lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} = \frac{1}{3} \neq 0$, by the nth-term test $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ diverges

40) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ use Limit Comparison Test and $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is a convergent geometric series $|r| = \left|\frac{3}{4}\right| < 1$

$$\lim_{n\to\infty} \frac{\left(\frac{2^n+3^n}{3^n+4^n}\right)}{\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty} \left(\frac{2^n+3^n}{3^n+4^n}\right)\left(\frac{4^n}{3^n}\right) = \lim_{n\to\infty} \left(\frac{8^n+12^n}{9^n+12^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{\frac{8^n+12^n}{1}}{\frac{9^n+12^n}{1}}\right)\left(\frac{\frac{1}{12^n}}{\frac{1}{12^n}}\right) = \lim_{n\to\infty} \frac{\frac{8^n}{12^n}+\frac{12^n}{12^n}}{\frac{9^n}{12^n}+\frac{12^n}{12^n}} = \lim_{n\to\infty} \frac{\left(\frac{8}{12}\right)^n+1}{\left(\frac{9}{12}\right)^n+1} = \frac{0+1}{0+1} = 1 > 0$$

since $\lim_{n\to\infty} \frac{\frac{2^n+3^n}{3^n+4^n}}{\left(\frac{3}{4}\right)^n} = 1 > 0$ and $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is convergent

by Limit Comparison Test $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges

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42) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ since for $n \geq 1$, $\sqrt{n} > \ln n \Rightarrow 1 > \frac{\ln n}{\sqrt{n}}$

$\Downarrow$

$\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a convergent geometric series $|r| = \left|\frac{1}{e}\right| < 1$ $\qquad \frac{1}{e^n} > \frac{\ln n}{\sqrt{n}\, e^n}$

$$\lim_{n\to\infty} \frac{\frac{\ln n}{\sqrt{n}\, e^n}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{\overset{+\infty}{\ln n}}{\underset{+\infty}{\sqrt{n}}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

since $\lim_{n\to\infty} \frac{\frac{\ln n}{\sqrt{n}\, e^n}}{\frac{1}{e^n}} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is convergent

by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$ converges

44) $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$ $\qquad \frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)(n)(n-1)!} = \frac{1}{(n+2)(n+1)(n)} \le \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series because $p = 3 > 1$ $\qquad \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!} \le \sum_{n=1}^{\infty} \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{\frac{(n-1)!}{(n+2)!}}{\frac{1}{n^3}} = \lim_{n\to\infty} \left(\frac{(n-1)!}{(n+2)(n+1)(n)(n-1)!}\right)\left(\frac{n^3}{1}\right) = \lim_{n\to\infty} \frac{n^2}{(n+2)(n+1)} = \lim_{n\to\infty} \frac{\overset{+\infty}{n^2}}{\underset{+\infty}{n^2+3n+2}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{3n}{n^2}+\frac{2}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}+\frac{2}{n^2}} = \frac{1}{1+0+0} = 1 > 0$$

since $\lim_{n\to\infty} \frac{\frac{(n-1)!}{(n+2)!}}{\frac{1}{n^3}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent

by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$ converges

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46) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ $\qquad \tan\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\cos\left(\frac{1}{n}\right)}$

compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent harmonic series

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \left(\frac{1}{\cos\left(\frac{1}{n}\right)}\right)\left(\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right) = \left(\frac{1}{\cos(0)}\right)(1) = 1 > 0$$

since $\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

by Limit Comparison Test, $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ diverges

48) $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$

for $n \geq 1$

$\sec^{-1} n < \frac{\pi}{2}$

$0 \leq \frac{\sec^{-1} n}{n^{1.3}} < \frac{\frac{\pi}{2}}{n^{1.3}}$

$0 \leq \sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}} < \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.3}}$

$\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.3}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$ is a convergent p-series because $p = 1.3 > 1$

by Direct Comparison Test, $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{1.3}}$ converges

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50) $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$ compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series because $p = 2 > 1$

$$\lim_{n\to\infty} \frac{\frac{\tanh n}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{\tanh n}{n^2}\right)\left(\frac{n^2}{1}\right) = \lim_{n\to\infty} \left(\frac{\sinh n}{\cosh n}\right) = \lim_{n\to\infty} \frac{\frac{e^n - e^{-n}}{2}}{\frac{e^n + e^{-n}}{2}}$$

$$= \lim_{n\to\infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n\to\infty} \left(\frac{\frac{e^n}{1} - \frac{1}{e^n}}{\frac{e^n}{1} + \frac{1}{e^n}}\right)\left(\frac{\frac{1}{e^n}}{\frac{1}{e^n}}\right) = \lim_{n\to\infty} \frac{1 - \frac{1}{e^{2n}}}{1 + \frac{1}{e^{2n}}} = \frac{1-0}{1+0} = 1 > 0$$

since $\lim_{n\to\infty} \frac{\frac{\tanh n}{n^2}}{\frac{1}{n^2}} = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$ converges

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52) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

for $n \geq 1$

$\frac{\sqrt[n]{n}}{n^2} \geq \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series because $p = 2 > 1$

52) continued

$$y = \sqrt[n]{n} = n^{\frac{1}{n}}$$

$$\ln y = \ln\left(n^{\frac{1}{n}}\right) \qquad \lim_{n\to\infty} \frac{\ln n}{n} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$\ln y = \frac{\ln n}{n} \qquad = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\ln y = 0 \Rightarrow y = e^0 = 1$$

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \sqrt[n]{n} = 1 > 0$$

since $\lim\limits_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = 1 > 0$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges

by Limit Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges

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54) $\sum\limits_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$

$$1^2+2^2+3^2+\cdots+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

see formula in section 5.2 (pg 313)

$$0 \le \frac{1}{1+2^2+3^2+\cdots+n^2} = \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)(2n+1)} \le \frac{6}{n^3}$$

$$0 \le \sum_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2} \le \sum_{n=1}^{\infty} \frac{6}{n^3}$$

$\sum\limits_{n=1}^{\infty} \frac{6}{n^3}$ is convergent p-series because $p = 3 > 1$

by Direct Comparison Test, $\sum\limits_{n=1}^{\infty} \frac{1}{1+2^2+3^2+\cdots+n^2}$

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56) $\sum\limits_{n=2}^{\infty} \frac{(\ln n)^2}{n}$ compare with $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ which is a divergent p-series because $p = 1 \le 1$

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n}}{\frac{1}{n}} = \lim_{n\to\infty} (\ln n)^2 = +\infty$$

since $\lim\limits_{n\to\infty} \frac{\frac{(\ln n)^2}{n}}{\frac{1}{n}} = +\infty$ and $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ diverges

by Limit Comparison Test, $\sum\limits_{n=2}^{\infty} \frac{(\ln n)^2}{n}$ diverges