Definitions

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}.$$

In which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$.

If $|r| \neq 1$, we can determine the convergence or divergence of the Geometric series in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + ar^3 + \cdots + ar^n \\ s_n - rs_n &= a - ar^n \\ s_n(1-r) &= a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so $s_n \to \dfrac{a}{1-r}$ in this case.

On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverge.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $\dfrac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \ge 1$, the series diverges.

Theorem 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

The *n*th-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Theorem 8

If $\sum a_n = A$: and $\sum b_n = B$ are convergent series, then

1. Sum Rule: $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. Difference Rule: $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. Constant Multiple Rule: $\sum ka_n = k\sum a_n = kA$ (any number k)

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

2) $\frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}$

$$s_n=\frac{9}{100}(1)+\frac{9}{100}\left(\frac{1}{100}\right)+\frac{9}{100}\left(\frac{1}{100}\right)^2+\cdots+\frac{9}{100}\left(\frac{1}{100}\right)^{n-1}$$

$$\boxed{a=\frac{9}{100} \quad r=\frac{1}{100}}$$

$$r s_n=\left(\frac{1}{100}\right)s_n=\frac{9}{100}\left(\frac{1}{100}\right)+\frac{9}{100}\left(\frac{1}{100}\right)^2+\frac{9}{100}\left(\frac{1}{100}\right)^3+\cdots+\frac{9}{100}\left(\frac{1}{100}\right)^n$$

$$s_n-r s_n=s_n-\left(\frac{1}{100}\right)s_n=\frac{9}{100}(1)-\frac{9}{100}\left(\frac{1}{100}\right)^n$$

$$s_n\left(1-\frac{1}{100}\right)=\frac{9}{100}\left(1-\left(\frac{1}{100}\right)^n\right)$$

$$s_n=\frac{\frac{9}{100}\left(1-\left(\frac{1}{100}\right)^n\right)}{\left(1-\frac{1}{100}\right)}$$

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\frac{\frac{9}{100}\left(1-\left(\frac{1}{100}\right)^n\right)}{\left(1-\frac{1}{100}\right)}=\frac{\frac{9}{100}(1-0)}{1-\frac{1}{100}}=\frac{\frac{9}{100}}{\frac{99}{100}}=\frac{9}{99}=\frac{1}{11}$$

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6) $\frac{5}{1.2}+\frac{5}{2\cdot3}+\frac{5}{3.4}+\cdots+\frac{5}{n(n+1)}+\cdots$ $\qquad \frac{5}{n(n+1)}=\frac{5}{n}-\frac{5}{n+1}$

$$s_n=\frac{5}{1(1+1)}+\frac{5}{2(2+1)}+\frac{5}{3(3+1)}+\cdots+\frac{5}{n(n+1)}$$

$$s_n=\left(\frac{5}{1}-\frac{5}{(1+1)}\right)+\left(\frac{5}{2}-\frac{5}{(2+1)}\right)+\left(\frac{5}{3}-\frac{5}{(3+1)}\right)+\cdots+\left(\frac{5}{n}-\frac{5}{(n+1)}\right)$$

$$s_n=\left(5-\frac{5}{2}\right)+\left(\frac{5}{2}-\frac{5}{3}\right)+\left(\frac{5}{3}-\frac{5}{4}\right)+\cdots+\left(\frac{5}{n}-\frac{5}{n+1}\right)$$

$$s_n=5-\frac{5}{n+1}$$

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(5-\frac{5}{n+1}\right)=5-0=5$$

8) $\sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \frac{1}{4^6} + \frac{1}{4^7} + \frac{1}{4^8} + \frac{1}{4^9} + \cdots + \frac{1}{4^n} + \cdots$

$s_n = \frac{1}{4^2}(1) + \frac{1}{4^2}\left(\frac{1}{4}\right) + \frac{1}{4^2}\left(\frac{1}{4}\right)^2 + \frac{1}{4^2}\left(\frac{1}{4}\right)^3 + \cdots + \frac{1}{4^2}\left(\frac{1}{4}\right)^{n-1}$

this is a geometric series with $a = \frac{1}{4^2} = \frac{1}{16}$ and $r = \frac{1}{4}$.

since $\left|\frac{1}{4}\right| = |r| < 1$ the sum is $\frac{a}{1-r} = \frac{\left(\frac{1}{16}\right)}{1-\left(\frac{1}{4}\right)} = \frac{\frac{1}{16}}{\frac{3}{4}} = \left(\frac{1}{16}\right)\left(\frac{4}{3}\right) = \frac{1}{12}$

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10) $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = (-1)^0\frac{5}{4^0} + (-1)^1\frac{5}{4^1} + (-1)^2\frac{5}{4^2} + (-1)^3\frac{5}{4^3} + (-1)^4\frac{5}{4^4} + (-1)^5\frac{5}{4^5} + (-1)^6\frac{5}{4^6} + (-1)^7\frac{5}{4^7} + \cdots$

$= 5(1) + 5\left(\frac{-1}{4}\right) + 5\left(\frac{-1}{4}\right)^2 + 5\left(\frac{-1}{4}\right)^3 + 5\left(\frac{-1}{4}\right)^4 + 5\left(\frac{-1}{4}\right)^5 + (5)\left(\frac{-1}{4}\right)^6 + (5)\left(\frac{-1}{4}\right)^7 + \cdots$

$s_n = 5(1) + 5\left(\frac{-1}{4}\right) + 5\left(\frac{-1}{4}\right)^2 + \cdots + 5\left(\frac{-1}{4}\right)^{n-1}$

this is a geometric series with $a = 5$ and $r = \frac{-1}{4}$.

since $\left|\frac{-1}{4}\right| = |r| < 1$ the sum is $\frac{a}{1-r} = \frac{(5)}{1-\left(\frac{-1}{4}\right)} = \frac{5}{\frac{5}{4}} = \left(\frac{5}{1}\right)\left(\frac{4}{5}\right) = 4$

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12) $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = \left(\frac{5}{2^0} - \frac{1}{3^0}\right) + \left(\frac{5}{2^1} - \frac{1}{3^1}\right) + \left(\frac{5}{2^2} - \frac{1}{3^2}\right) + \left(\frac{5}{2^3} - \frac{1}{3^3}\right) + \left(\frac{5}{2^4} - \frac{1}{3^4}\right) + \left(\frac{5}{2^5} - \frac{1}{3^5}\right)$

$+ \left(\frac{5}{2^6} - \frac{1}{3^6}\right) + \left(\frac{5}{2^7} - \frac{1}{3^7}\right) + \cdots$

$= \underbrace{\frac{5}{2^0} + \frac{5}{2^1} + \frac{5}{2^2} + \frac{5}{2^3} + \frac{5}{2^4} + \frac{5}{2^5} + \frac{5}{2^6} + \frac{5}{2^7} + \cdots}_{s_n} - \underbrace{\left(\frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \frac{1}{3^7} + \cdots\right)}_{t_n}$

$s_n = 5(1) + 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \cdots + 5\left(\frac{1}{2}\right)^{n-1}$

this is a geometric series with $a = 5$ and $r = \frac{1}{2}$

since $\left|\frac{1}{2}\right| = |r| < 1$ the sum is $\frac{a}{1-r} = \frac{(5)}{1-\left(\frac{1}{2}\right)} = \frac{5}{\frac{1}{2}} = 10$

12) continued

$t_n = (1) + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^{n-1}$

this is a geometric series with $a = 1$ and $r = \frac{1}{3}$

since $\left|\frac{1}{3}\right| = |r| < 1$ the sum is $\frac{a}{1-r} = \frac{(1)}{1-\left(\frac{1}{3}\right)} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

the original series is a difference of 2 geometric series that converges; therefore, the sum is $(10) - \left(\frac{3}{2}\right) = \frac{23}{2}$

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14) $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right) = \frac{2^{0+1}}{5^0} + \frac{2^{1+1}}{5^1} + \frac{2^{2+1}}{5^2} + \frac{2^{3+1}}{5^3} + \frac{2^{4+1}}{5^4} + \frac{2^{5+1}}{5^5} + \frac{2^{6+1}}{5^6} + \frac{2^{7+1}}{5^7}$

$s_n = 2(1) + 2\left(\frac{2}{5}\right)^1 + 2\left(\frac{2}{5}\right)^2 + \cdots + 2\left(\frac{2}{5}\right)^{n-1}$

this is a geometric series with $a = 2$ and $r = \frac{2}{5}$

since $\left|\frac{2}{5}\right| = |r| < 1$ the sum is $\frac{a}{1-r} = \frac{(2)}{1-\left(\frac{2}{5}\right)} = \frac{2}{\frac{3}{5}} = \frac{10}{3}$

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16) $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

this series is geometric with $a = 1$ and $r = -3$

since $|-3| = |r| > 1$ the series diverge

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18) $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

$\left(\frac{-2}{3}\right)^2(1) + \left(\frac{-2}{3}\right)^2\left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)^2\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2\left(\frac{-2}{3}\right)^3 + \cdots + \left(\frac{-2}{3}\right)^2\left(\frac{-2}{3}\right)^{n-1} + \cdots$

this series is geometric with $a = \left(\frac{-2}{3}\right)^2 = \frac{4}{9}$ and $r = \frac{-2}{3}$

since $\left|\frac{-2}{3}\right| = |r| < 1$ the series converges to $\frac{a}{1-r} = \frac{\left(\frac{4}{9}\right)}{1-\left(\frac{-2}{3}\right)} = \frac{\frac{4}{9}}{\frac{5}{3}} = \frac{4}{15}$

20) $\left(\frac{1}{3}\right)^{-2}-\left(\frac{1}{3}\right)^{-1}+(1)-\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^{2}-\cdots$

$\left(\frac{1}{3}\right)^{-2}(1)+\left(\frac{1}{3}\right)^{-2}\left(\frac{-1}{3}\right)^{1}+\left(\frac{1}{3}\right)^{-2}\left(\frac{-1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{-2}\left(\frac{-1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{-2}\left(\frac{-1}{3}\right)^{4}+\cdots+\left(\frac{1}{3}\right)^{-2}\left(\frac{-1}{3}\right)^{n-1}+\cdots$

this series is geometric with $a=\left(\frac{1}{3}\right)^{-2}$ and $r=\frac{-1}{3}$

since $\left|\frac{-1}{3}\right|=|r|<1$ the series converges to $\frac{a}{1-r}=\frac{\left(\frac{1}{3}\right)^{-2}}{1-\left(\frac{-1}{3}\right)}=\frac{9}{\frac{4}{3}}=\frac{27}{4}$

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22) $\frac{9}{4}-\frac{27}{8}+\frac{81}{16}-\frac{243}{32}+\frac{729}{64}-\cdots$

$\frac{9}{4}(1)+\frac{9}{4}\left(\frac{-3}{2}\right)+\frac{9}{4}\left(\frac{-3}{2}\right)^{2}+\frac{9}{4}\left(\frac{-3}{2}\right)^{3}+\frac{9}{4}\left(\frac{-3}{2}\right)^{4}+\cdots+\frac{9}{4}\left(\frac{-3}{2}\right)^{n-1}+\cdots$

this series is geometric with $a=\frac{9}{4}$ and $r=\frac{-3}{2}$

since $\left|\frac{-3}{2}\right|=|r|>1$ the series diverge

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24) $0.\overline{234}=0.234234234\ldots$

$0.\overline{234}=\sum_{n=0}^{\infty}\frac{234}{1000}\left(\frac{1}{1000}\right)^{n}$ $\quad a=\frac{234}{1000}$ $\quad r=\frac{1}{1000}$

$=\frac{a}{1-r}=\frac{\left(\frac{234}{1000}\right)}{1-\left(\frac{1}{1000}\right)}=\frac{\frac{234}{1000}}{\frac{999}{1000}}=\frac{234}{999}$

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26) $0.\bar{d}=0.dddd\ldots$, where d is a digit

$0.\bar{d}=\sum_{n=0}^{\infty}\frac{d}{10}\left(\frac{1}{10}\right)^{n}$ $\quad a=\frac{d}{10}$ $\quad r=\frac{1}{10}$

$=\frac{a}{1-r}=\frac{\left(\frac{d}{10}\right)}{1-\left(\frac{1}{10}\right)}=\frac{\frac{d}{10}}{\frac{9}{10}}=\frac{d}{9}$

30) $3.\overline{142857} = 3.142857142857\ldots$

$$3.\overline{142857} = 3 + 0.\overline{142857} = 3 + \sum_{n=0}^{\infty} \left(\frac{142857}{1000000}\right)\left(\frac{1}{1000000}\right)^n \quad a = \frac{142857}{1000000},\ r = \frac{1}{1000000}$$

$$= 3 + \frac{a}{1-r} = 3 + \frac{\left(\frac{142857}{1000000}\right)}{1-\left(\frac{1}{1000000}\right)} = 3 + \frac{\frac{142857}{1000000}}{\frac{999999}{1000000}} = 3 + \frac{142857}{999999}$$

$$= \frac{2999997}{999999} + \frac{142857}{999999} = \frac{3142854}{999999} = \frac{116402}{37,037}$$

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32) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

$$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{\overset{+\infty}{n^2+n}}{\underset{+\infty}{n^2+5n+6}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{2n+1}}{\underset{+\infty}{2n+5}} \overset{L}{=} \lim_{n\to\infty} \frac{2}{2} =$$

$$= \lim_{n\to\infty} 1 = 1 \neq 0$$

since $\displaystyle\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} \neq 0$, the series diverge

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34) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{n}}{\underset{+\infty}{n^2+3}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{2n} = 0$$

since $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+3} = 0$, the test is inconclusive

36) $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{e^n}}{\underset{+\infty}{e^n+n}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{e^n}}{\underset{+\infty}{e^n+1}} \overset{L}{=} \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} 1 = 1$$

since $\lim_{n\to\infty} \frac{e^n}{e^n+n} \neq 0$, the series diverge

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38) $\sum_{n=0}^{\infty} \cos n\pi$

$\lim_{n\to\infty} \cos n\pi$ D.N.E.

since $\lim_{n\to\infty} \cos n\pi$ does not exist, the series diverge

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40) $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

$$s_n = \left(\frac{3}{1^2} - \frac{3}{(1+1)^2}\right) + \left(\frac{3}{2^2} - \frac{3}{(2+1)^2}\right) + \left(\frac{3}{3^2} - \frac{3}{(3+1)^2}\right) + \cdots + \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$$

$$= \left(\frac{3}{1^2} - \frac{3}{2^2}\right) + \left(\frac{3}{2^2} - \frac{3}{3^2}\right) + \left(\frac{3}{3^2} - \frac{3}{4^2}\right) + \cdots + \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right) = 3 - \frac{3}{(n+1)^2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(3 - \frac{3}{(n+1)^2}\right) = 3 - 0 = 3$$

the series converges to 3.

42) $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

$s_n = (\tan(1) - \tan(1-1)) + (\tan(2) - \tan(2-1)) + \dots + (\tan(n) - \tan(n-1))$

$= (\tan(1) - \tan(0)) + (\tan(2) - \tan(1)) + \dots + (\tan(n) - \tan(n-1))$

$= \tan(n) - \tan(0) = \tan(n) - 0 = \tan(n)$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \tan(n)$ D.N.E. so the series diverge

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44) $\sum_{n=1}^{\infty} (\sqrt{n+4} - \sqrt{n+3})$

$s_n = (\sqrt{1+4} - \sqrt{1+3}) + (\sqrt{2+4} - \sqrt{2+3}) + (\sqrt{3+4} - \sqrt{3+3}) + \dots + (\sqrt{n+4} - \sqrt{n+3})$

$= (\sqrt{5} - \sqrt{4}) + (\sqrt{6} - \sqrt{5}) + (\sqrt{7} - \sqrt{6}) + \dots + (\sqrt{n+4} - \sqrt{n+3})$

$= \sqrt{n+4} - \sqrt{4} = \sqrt{n+4} - 2$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\sqrt{n+4} - 2) = +\infty$ so the series diverge

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46) $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left(\frac{(3)}{(2n-1)^1} + \frac{(-3)}{(2n+1)^1} \right)$

$\frac{6}{(2n-1)^1(2n+1)^1} = \frac{A}{(2n-1)^1} + \frac{B}{(2n+1)^1}$ $\quad = \sum_{n=1}^{\infty} \left(\frac{3}{2n-1} - \frac{3}{2n+1} \right)$

$6 = A(2n+1) + B(2n-1)$

constant term

$6 = A - B$

$6 = A + A$

$A = 3$

n-term

$0 = 2A + 2B$

$2A = -2B$

$A = -B$

$B = -A = -3$

46) continued

$$s_n = \left(\frac{3}{2(1)-1} - \frac{3}{2(1)+1}\right) + \left(\frac{3}{2(2)-1} - \frac{3}{2(2)+1}\right) + \left(\frac{3}{2(3)-1} - \frac{3}{2(3)+1}\right) + \cdots + \left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$$

$$= \left(\frac{3}{1} - \frac{3}{3}\right) + \left(\frac{3}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) + \cdots + \left(\frac{3}{2n-1} - \frac{3}{2n+1}\right)$$

$$= 3 - \frac{3}{2n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3 - \frac{3}{2n+1}\right) = 3 - 0 = \underline{\underline{3}}$$

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48) $$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{(0)}{(n)^1} + \frac{(1)}{(n)^2} + \frac{(0)}{(n+1)^1} + \frac{(-1)}{(n+1)^2}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{(n)^1} + \frac{B}{(n)^2} + \frac{C}{(n+1)^1} + \frac{D}{(n+1)^2}$$

$$2n+1 = A\left((n)(n+1)^2\right) + B\left((n+1)^2\right) + C\left(n^2(n+1)\right) + D\left(n^2\right)$$

$$2n+1 = A\left(n^3+2n^2+n\right) + B\left(n^2+2n+1\right) + C\left(n^3+n^2\right) + D\left(n^2\right)$$

constant term: $1 = B$

n-term: $2 = A + 2B$; $2 = A + 2(1)$; $A = 0$

n^2-term: $0 = 2A + B + C + D$; $0 = 2(0) + (1) + (0) + D$; $D = -1$

n^3-term: $0 = A + C$; $C = 0$

$$s_n = \left(\frac{1}{1^2} - \frac{1}{(1+1)^2}\right) + \left(\frac{1}{2^2} - \frac{1}{(2+1)^2}\right) + \left(\frac{1}{3^2} - \frac{1}{(3+1)^2}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)^2}\right) = 1 - 0 = \underline{\underline{1}}$$

50) $\sum_{n=1}^{\infty}\left(\frac{1}{2^{\frac{1}{n}}}-\frac{1}{2^{\frac{1}{n+1}}}\right)$

$$s_n=\left(\frac{1}{2^{\frac{1}{1}}}-\frac{1}{2^{\frac{1}{1+1}}}\right)+\left(\frac{1}{2^{\frac{1}{2}}}-\frac{1}{2^{\frac{1}{2+1}}}\right)+\left(\frac{1}{2^{\frac{1}{3}}}-\frac{1}{2^{\frac{1}{3+1}}}\right)+\cdots+\left(\frac{1}{2^{\frac{1}{n}}}-\frac{1}{2^{\frac{1}{n+1}}}\right)$$

$$=\left(\frac{1}{2}+\frac{1}{2^{\frac{1}{2}}}\right)+\left(\frac{1}{2^{\frac{1}{2}}}-\frac{1}{2^{\frac{1}{3}}}\right)+\left(\frac{1}{2^{\frac{1}{3}}}-\frac{1}{2^{\frac{1}{4}}}\right)+\cdots+\left(\frac{1}{2^{\frac{1}{n}}}-\frac{1}{2^{\frac{1}{n+1}}}\right)$$

$$=\frac{1}{2}-\frac{1}{2^{\frac{1}{n+1}}}$$

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2^{\frac{1}{n+1}}}\right)=\frac{1}{2}-\frac{1}{2^0}=\frac{1}{2}-\frac{1}{1}=\underline{\underline{\frac{-1}{2}}}$$

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52) $\sum_{n=1}^{\infty}\left(\tan^{-1}(n)-\tan^{-1}(n+1)\right)$

$$s_n=\left(\tan^{-1}(1)-\tan^{-1}(1+1)\right)+\left(\tan^{-1}(2)-\tan^{-1}(2+1)\right)+\cdots+\left(\tan^{-1}(n)-\tan^{-1}(n+1)\right)$$

$$=\left(\tan^{-1}(1)-\tan^{-1}(2)\right)+\left(\tan^{-1}(2)-\tan^{-1}(3)\right)+\cdots+\left(\tan^{-1}(n)-\tan^{-1}(n+1)\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(n+1)=\frac{\pi}{4}-\tan^{-1}(n+1)$$

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(\frac{\pi}{4}-\tan^{-1}(n+1)\right)=\frac{\pi}{4}-\left(\frac{\pi}{2}\right)=\underline{\underline{\frac{-\pi}{4}}}$$

---

54) $\sum_{n=0}^{\infty}(\sqrt{2})^n=(\sqrt{2})^0+(\sqrt{2})^1+(\sqrt{2})^2+\cdots+(\sqrt{2})^n+\cdots$

$$=1+(\sqrt{2})+(\sqrt{2})^2+\cdots+(\sqrt{2})^n$$

this is a geometric series with $a=1$ and $r=\sqrt{2}$

since $|\sqrt{2}|=|r|>1$, the series diverges

56) $\sum_{n=1}^{\infty} (-1)^{n+1} n = (-1)^{1+1}(1) + (-1)^{2+1}(2) + (-1)^{3+1}(3) + (-1)^{4+1}(4) + \cdots + (-1)^{n+1} n + \cdots$

$= 1 - 2 + 3 - 4 + \cdots$

since $\lim_{n\to\infty} (-1)^{n+1} n \neq 0$, the series diverges by nth term test

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58) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \frac{\cos(0)\pi}{5^0} + \frac{\cos(1)\pi}{5^1} + \frac{\cos(2)\pi}{5^2} + \cdots + \frac{\cos n\pi}{5^n} + \cdots$

$= \frac{(1)}{(1)} + \frac{(-1)}{5} + \frac{(1)}{5^2} + \cdots + \frac{(-1)^n}{5^n} + \cdots$ $\quad$ $\cos n\pi = (-1)^n$ when n is positive integer

$= 1 + \left(\frac{-1}{5}\right)^1 + \left(\frac{-1}{5}\right)^2 + \cdots + \left(\frac{-1}{5}\right)^n + \cdots$

this is a geometric series with $a = 1$ and $r = \frac{-1}{5}$

since $\left|\frac{-1}{5}\right| = |r| < 1$, the series converges to $\frac{a}{1-r} = \frac{(1)}{1-\left(\frac{-1}{5}\right)} = \frac{1}{\frac{6}{5}} = \frac{5}{6}$

---

60) $\sum_{n=1}^{\infty} \ln \frac{1}{3^n} = \ln \frac{1}{3^1} + \ln \frac{1}{3^2} + \ln \frac{1}{3^3} + \cdots + \ln \frac{1}{3^n} + \cdots$

$\lim_{n\to\infty} \ln \frac{1}{3^n} = \ln \lim_{n\to\infty} \frac{1}{3^n} = \ln(0^+) = -\infty$

since $\lim_{n\to\infty} \frac{1}{3^n} \neq 0$, the series diverges by nth term test

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64) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{(-1)} \neq 0$ {see Thm 5 # 5, sec. 10.1}

since $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n \neq 0$, the series diverge by nth term test

66) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n)(n)\cdots(n)}{(1)(2)\cdots(n)} > \lim_{n\to\infty} n = \infty$$

so by Direct Comparison Thm $\lim_{n\to\infty} \frac{n^n}{n!} = \infty \neq 0$

so this series diverges by nth term test

68) $\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+4^n}$

$$\lim_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} = \lim_{n\to\infty} \frac{\frac{2^n+4^n}{4^n}}{\frac{3^n+4^n}{4^n}} = \lim_{n\to\infty} \frac{\frac{2^n}{4^n}+\frac{4^n}{4^n}}{\frac{3^n}{4^n}+\frac{4^n}{4^n}} = \lim_{n\to\infty} \frac{(\frac{2}{4})^n+1}{(\frac{3}{4})^n+1}$$

$$= \lim_{n\to\infty} \frac{(\frac{1}{2})^n+1}{(\frac{3}{4})^n+1} = \frac{0+1}{0+1} = 1$$

$\lim_{n\to\infty} \frac{2^n+4^n}{3^n+4^n} \neq 0$ so this series diverges by nth term test

70) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln \lim_{n\to\infty} \frac{n}{2n+1} \overset{L}{=} \ln \lim_{n\to\infty} \frac{1}{2} = \ln\left(\frac{1}{2}\right)$$

$\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) \neq 0$ so this series diverges

by nth term test

72) $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}} = \frac{e^{(0)\pi}}{\pi^{(0)e}} + \frac{e^{(1)\pi}}{\pi^{(1)e}} + \frac{e^{(2)\pi}}{\pi^{(2)e}} + \cdots + \frac{e^{n\pi}}{\pi^{ne}} + \cdots$

$$= 1 + \left(\frac{e^{\pi}}{\pi^{e}}\right)^1 + \left(\frac{e^{\pi}}{\pi^{e}}\right)^2 + \cdots + \left(\frac{e^{\pi}}{\pi^{e}}\right)^n + \cdots$$

this is a geometric series with $a=1$ and $r = \frac{e^{\pi}}{\pi^{e}}$

since $\left|\frac{e^{\pi}}{\pi^{e}}\right| = |r| > 1$, the series diverges

---

74) $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

$s_n = \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2-1}\right)\right) + \left(\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3-1}\right)\right) + \cdots + \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

$= \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{1}\right)\right) + \left(\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{2}\right)\right) + \cdots + \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

$= \sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{1}\right) = \sin\left(\frac{\pi}{n}\right) - \sin(\pi) = \sin\left(\frac{\pi}{n}\right) - (0) = \sin\left(\frac{\pi}{n}\right)$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$ this series converges

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76) $\sum_{n=0}^{\infty} \left(\ln(4e^n - 1) - \ln(2e^n + 1)\right)$

$\lim_{n\to\infty} \left(\ln(4e^n - 1) - \ln(2e^n + 1)\right) = \lim_{n\to\infty} \ln\left(\frac{4e^n - 1}{2e^n + 1}\right) = \ln\left(\lim_{n\to\infty} \frac{\overset{+\infty}{4e^n - 1}}{\underset{+\infty}{2e^n + 1}}\right)$

$\overset{L}{=} \ln\left(\lim_{n\to\infty} \frac{4e^n}{2e^n}\right) = \ln\left(\lim_{n\to\infty} 2\right) = \ln 2$

since $\lim_{n\to\infty} \left(\ln(4e^n - 1) - \ln(2e^n + 1)\right) \neq 0$ this series diverges

by the nth term test

78) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = (-1)^0 x^{2(0)} + (-1)^1 x^{2(1)} + (-1)^2 x^{2(2)} + \cdots + (-1)^n x^{2n} + \cdots$

$= 1 - x^2 + x^4 + \cdots + (-1)^n x^{2n} + \cdots$

$= 1 + (-x^2)^1 + (-x^2)^2 + \cdots + (-x^2)^n + \cdots$

this is a geometric series with $a = 1$ and $r = -x^2$

it will converge to $\frac{a}{1-r} = \frac{(1)}{1-(-x^2)} = \frac{1}{1+x^2}$ as long as

$|-x^2| = |r| < 1 \Rightarrow |x| < \pm\sqrt{1} = 1$, $|x| < 1$

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80) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

$= \frac{(-1)^0}{2}\left(\frac{1}{3+\sin x}\right)^0 + \frac{(-1)^1}{2}\left(\frac{1}{3+\sin x}\right)^1 + \frac{(-1)^2}{2}\left(\frac{1}{3+\sin x}\right)^2 + \cdots + \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n + \cdots$

$= \frac{1}{2}(1) - \frac{1}{2}\left(\frac{1}{3+\sin x}\right) + \frac{1}{2}\left(\frac{1}{3+\sin x}\right)^2 + \cdots + \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n + \cdots$

$= \frac{1}{2}(1) + \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^1 + \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^2 + \cdots + \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^n + \cdots$

this is a geometric series with $a = \frac{1}{2}$ and $r = \frac{-1}{3+\sin x}$

it will converge to $\frac{a}{1-r} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{-1}{3+\sin x}\right)} = \frac{\frac{1}{2}}{\frac{4+\sin x}{3+\sin x}} = \frac{3+\sin x}{2(4+\sin x)}$

as long as

$\left|\frac{-1}{3+\sin x}\right| = |r| < 1$

this will be true for all x

because $\frac{1}{4} = \frac{1}{3+(1)} \le \frac{1}{3+\sin x} \le \frac{1}{3+(-1)} = \frac{1}{2}$

which insures that for all x

$|r| = \left|\frac{1}{3+\sin x}\right| < 1$

82) $\sum_{n=0}^{\infty}(-1)^n x^{-2n} = \frac{(-1)^0}{x^{2(0)}} + \frac{(-1)^1}{x^{2(1)}} + \frac{(-1)^2}{x^{2(2)}} + \cdots + \frac{(-1)^n}{x^{2n}} + \cdots$

$= 1 - \frac{1}{x^2} + \frac{1}{x^4} + \cdots + \frac{(-1)^n}{x^{2n}} + \cdots$

$= 1 + \left(\frac{-1}{x^2}\right)^1 + \left(\frac{-1}{x^2}\right)^2 + \cdots + \left(\frac{-1}{x^2}\right)^n + \cdots$

this is a geometric series with $a=1$ and $r=\frac{-1}{x^2}$

it will converge to $\frac{a}{1-r} = \frac{(1)}{1-\left(\frac{-1}{x^2}\right)} = \frac{1}{\frac{x^2+1}{x^2}} = \frac{x^2}{x^2+1}$

as long as $\left|\frac{-1}{x^2}\right| = |r| < 1 \Rightarrow \underline{\underline{|x|>1}}$

---

84) $\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n(x-3)^n = \left(\frac{-1}{2}\right)^0(x-3)^0 + \left(\frac{-1}{2}\right)^1(x-3)^1 + \left(\frac{-1}{2}\right)^2(x-3)^2 + \cdots + \left(\frac{-1}{2}\right)^n(x-3)^n + \cdots$

$= 1 - \frac{x-3}{2} + \frac{(x-3)^2}{2^2} + \cdots + \frac{(-1)^n(x-3)^n}{2^n} + \cdots$

$= 1 + \frac{(-(x-3))}{2} + \frac{(-(x-3))^2}{2^2} + \cdots + \frac{(-(x-3))^n}{2^n} + \cdots$

$= 1 + \left(\frac{3-x}{2}\right) + \left(\frac{3-x}{2}\right)^2 + \cdots + \left(\frac{3-x}{2}\right)^n + \cdots$

this is a geometric series with $a=1$ and $r=\frac{3-x}{2}$

it will converge to $\frac{a}{1-r} = \frac{(1)}{1-\left(\frac{3-x}{2}\right)} = \frac{1}{\frac{2-3+x}{2}} = \frac{1}{\frac{x-1}{2}} = \frac{2}{x-1}$

as long as $\left|\frac{3-x}{2}\right| < 1$

$-1 < \frac{3-x}{2} < 1$

$-2 < 3-x < 2$

$-5 < -x < -1$

$5 > x > 1$

$\underline{\underline{1 < x < 5}}$

86) $\sum_{n=0}^{\infty} (\ln x)^n = (\ln x)^0 + (\ln x)^1 + (\ln x)^2 + (\ln x)^3 + \cdots + (\ln x)^n + \cdots$

$= 1 + (\ln x)^1 + (\ln x)^2 + (\ln x)^3 + \cdots + (\ln x)^n + \cdots$

this is a geometric series with $a = 1$ and $r = \ln x$

it will converge to $\frac{a}{1-r} = \frac{(1)}{1-(\ln x)} = \frac{1}{1-\ln x}$

as long as $|\ln x| = |r| < 1 \Rightarrow$ $-1 < \ln x < 1$
$e^{-1} < x < e^1$

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98) $1 + e^b + e^{2b} + e^{3b} + \cdots = 9$

$1 + (e^b)^1 + (e^b)^2 + (e^b)^3 + \cdots + (e^b)^n + \cdots = 9$

this is a geometric series with $a = 1$ and $r = e^b$

it will converge because the series is equal to 9

and $\frac{a}{1-r} = \frac{(1)}{1-(e^b)} = \frac{1}{1-e^b}$ also $\frac{a}{1-r} = \frac{1}{1-e^b} = 9$

$\frac{1}{9} = 1 - e^b$

$e^b = 1 - \frac{1}{9} = \frac{8}{9}$

$\Downarrow$

$\underline{b = \ln\left(\frac{8}{9}\right)}$