Definitions

The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$\left|a_n - L < \varepsilon\right| \qquad \text{whenever} \qquad n > N\,.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim\limits_{x\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence.

Definition

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty\,.$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty\,.$$

Theorem 1

Let $\{a_n\}$ and $\{a_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$.

1. Sum Rule: $\lim\limits_{n\to\infty}(a_n + b_n) = A + B$
2. Difference Rule: $\lim\limits_{n\to\infty}(a_n - b_n) = A - B$
3. Constant Multiple Rule: $\lim\limits_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. Product Rule: $\lim\limits_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. Quotient Rule: $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{A}{B}$ if $B \neq 0$

Theorem 2 - The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ also.

Theorem 3 - The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $x \ge n_0$. Then

$$\lim_{n\to\infty} a_n = L \quad \text{whenever} \quad \lim_{x\to\infty} f(x) = L.$$

Theorem 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Definition

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \ge m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

Definitions

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \cdots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

Theorem 6 - The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

2) $a_n = \frac{1}{n!}$

$a_1 = \frac{1}{1!} = \frac{1}{1} = 1$

$a_2 = \frac{1}{2!} = \frac{1}{(1)(2)} = \frac{1}{2}$

$a_3 = \frac{1}{3!} = \frac{1}{(1)(2)(3)} = \frac{1}{6}$

$a_4 = \frac{1}{4!} = \frac{1}{(1)(2)(3)(4)} = \frac{1}{24}$

4) $a_n = 2 + (-1)^n$

$a_1 = 2 + (-1)^1 = 1$

$a_2 = 2 + (-1)^2 = 3$

$a_3 = 2 + (-1)^3 = 1$

$a_4 = 2 + (-1)^4 = 3$

6) $a_n = \frac{2^n - 1}{2^n}$

$a_1 = \frac{2^1 - 1}{2^1} = \frac{2-1}{2} = \frac{1}{2}$

$a_2 = \frac{2^2 - 1}{2^2} = \frac{4-1}{4} = \frac{3}{4}$

$a_3 = \frac{2^3 - 1}{2^3} = \frac{8-1}{8} = \frac{7}{8}$

$a_4 = \frac{2^4 - 1}{2^4} = \frac{16-1}{16} = \frac{15}{16}$

10) $a_1 = -2$, $a_{n+1} = \frac{n\, a_n}{n+1}$

$a_1 = -2$

$a_2 = \frac{(1)(-2)}{(2)} = -1$

$a_3 = \frac{(2)(-1)}{(3)} = \frac{-2}{3}$

$a_4 = \frac{(3)(\frac{-2}{3})}{(4)} = \frac{-1}{2}$

$a_5 = \frac{(4)(\frac{-1}{2})}{(5)} = \frac{-2}{5}$

$a_6 = \frac{(5)(\frac{-2}{5})}{(6)} = \frac{-2}{6} = \frac{-1}{3}$

$a_7 = \frac{(6)(\frac{-1}{3})}{(7)} = \frac{-2}{7}$

$a_8 = \frac{(7)(\frac{-2}{7})}{(8)} = \frac{-2}{8} = \frac{-1}{4}$

$a_9 = \frac{(8)(\frac{-1}{4})}{(9)} = \frac{-2}{9}$

$a_{10} = \frac{(9)(\frac{-2}{9})}{(10)} = \frac{-2}{10} = \frac{-1}{5}$

12) $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \dfrac{a_{n+1}}{a_n}$

$a_1 = 2$

$a_2 = -1$

$a_3 = \dfrac{(-1)}{(2)} = \dfrac{-1}{2}$

$a_4 = \dfrac{(\frac{-1}{2})}{(-1)} = \dfrac{1}{2}$

$a_5 = \dfrac{(\frac{1}{2})}{(\frac{-1}{2})} = -1$

$a_6 = \dfrac{(-1)}{(\frac{1}{2})} = -2$

$a_7 = \dfrac{(-2)}{(-1)} = 2$

$a_8 = \dfrac{(2)}{(-2)} = -1$

$a_9 = \dfrac{(-1)}{(2)} = \dfrac{-1}{2}$

$a_{10} = \dfrac{(\frac{-1}{2})}{(-1)} = \dfrac{1}{2}$

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18) $\dfrac{-3}{2}, \dfrac{-1}{6}, \dfrac{1}{12}, \dfrac{3}{20}, \dfrac{5}{30}, \ldots \qquad a_n = \dfrac{2n-5}{n(n+1)} \qquad n = 1, 2, 3, \ldots$

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20) $-3, -2, -1, 0, 1, \ldots \qquad a_n = n - 4 \qquad n = 1, 2, 3, \ldots$

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24) $\dfrac{1}{25}, \dfrac{8}{125}, \dfrac{27}{625}, \dfrac{64}{3125}, \dfrac{125}{15625}, \ldots \qquad a_n = \dfrac{n^3}{5^{n+1}} \qquad n = 1, 2, 3, \ldots$

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36) $a_n = \dfrac{n+3}{n^2+5n+6}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{n+3}}{\underset{+\infty}{n^2+5n+6}} = \lim_{n\to\infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n\to\infty} \frac{1}{n+2} = 0 \quad \text{converges}$$

38) $a_n = \dfrac{1-n^3}{70-4n^2}$

diverges

$$\lim_{n\to\infty} \frac{\overset{-\infty}{1-n^3}}{\underset{-\infty}{70-4n^2}} \overset{L}{=} \lim_{n\to\infty} \frac{-3n^2}{-8n} = \lim_{n\to\infty} \frac{3n^2}{8n} \overset{L}{=} \lim_{n\to\infty} \frac{6n}{8} = \lim_{n\to\infty} \frac{3n}{4} = +\infty$$

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42) $a_n = \left(2-\dfrac{1}{2^n}\right)\left(3+\dfrac{1}{2^n}\right)$

converges

$$\lim_{n\to\infty} \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right) = (2-0)(3+0) = 6$$

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46) $a_n = \dfrac{1}{(0.9)^n}$

$$\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{9}{10}\right)^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = +\infty$$ diverges

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48) $a_n = n\pi\cos(n\pi)$

since n are positive integers, $\cos(n\pi) = (-1)^n$

$$\lim_{n\to\infty} n\pi\cos(n\pi) = \lim_{n\to\infty} n\pi(-1)^n \Rightarrow \text{D.N.E.}$$ diverges

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50) $a_n = \dfrac{\sin^2 n}{2^n}$

converges

$0 \le \dfrac{\sin^2 n}{2^n} \le \dfrac{1}{2^n}$ and $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\lim\limits_{n\to\infty} 0 = 0$

so by the Sandwich Theorem for sequences,

$$0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\sin^2 n}{2^n} \le \lim_{n\to\infty} \frac{1}{2^n} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$$

52) $a_n = \frac{3^n}{n^3}$

diverges

$$\lim_{n\to\infty} \frac{\overset{+\infty}{3^n}}{\underset{+\infty}{n^3}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{(\ln 3)3^n}}{\underset{+\infty}{3n^2}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{(\ln 3)^2 3^n}}{\underset{+\infty}{6n}} \overset{L}{=} \lim_{n\to\infty} \frac{(\ln 3)^3 3^n}{6} = +\infty$$

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54) $a_n = \frac{\ln n}{\ln 2n}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{\ln n}}{\underset{+\infty}{\ln 2n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n}(2)} = \lim_{n\to\infty} \left(\frac{1}{n}\right)\left(\frac{2n}{2}\right) = \lim_{n\to\infty} 1 = 1$$ converges

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62) $a_n = (n+4)^{\frac{1}{(n+4)}}$

converges

$$\lim_{n\to\infty} (n+4)^{\frac{1}{(n+4)}} = \lim_{x\to\infty} x^{\frac{1}{x}} = 1$$ by Thm 5 # 2 and let $x = n+4$

$\lim_{x\to\infty} x^{\frac{1}{x}}$

$y = x^{\frac{1}{x}}$

$\ln y = \ln\left(x^{\frac{1}{x}}\right)$

$\ln y = \frac{\ln x}{x}$

$$\lim_{x\to\infty} \frac{\overset{+\infty}{\ln x}}{\underset{+\infty}{x}} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$\ln y = 0 \Rightarrow y = e^0 = 1$

$\therefore \lim_{x\to\infty} x^{\frac{1}{x}} = 1$

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64) $a_n = \ln n - \ln(n+1)$

converges

$$\lim_{n\to\infty} \left(\overset{(+\infty)-(+\infty)}{\ln n - \ln(n+1)}\right) = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln \lim_{n\to\infty} \frac{n}{n+1} = \ln(1) = 0$$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{n}}{\underset{+\infty}{n+1}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1} = \lim_{n\to\infty} 1 = 1$$

68) $a_n = \frac{(-4)^n}{n!}$

$\lim_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Thm 5 # 6 converges

70) $a_n = \frac{n!}{2^n 3^n}$

$\lim_{n\to\infty} \frac{n!}{2^n 3^n} = \lim_{n\to\infty} \frac{n!}{(2\cdot 3)^n} = \lim_{n\to\infty} \frac{n!}{6^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{6^n}{n!}\right)} = \frac{1}{\lim_{n\to\infty}\left(\frac{6^n}{n!}\right)} = \frac{1}{(0)} = +\infty$

by Thm 5 # 6 diverges

72) $a_n = \frac{(n+1)!}{(n+3)!}$

$\lim_{n\to\infty} \frac{(n+1)!}{(n+3)!} = \lim_{n\to\infty} \frac{(n+1)!}{(n+3)(n+2)(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+3)(n+2)} = 0$ converges

78) $a_n = \ln\left(1+\frac{1}{n}\right)^n$

by Thm 5 # 5

$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \ln\left(e^{(1)}\right) = 1$ converges

92) $a_n = \frac{1}{\sqrt{n}} \tan^{-1} n$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \tan^{-1} n = (0)\left(\frac{\pi}{2}\right) = 0$ converges

90) $a_n = (3^n + 5^n)^{\frac{1}{n}}$

$$\lim_{n\to\infty} \overset{(+\infty)^{(0)}}{(3^n + 5^n)^{\frac{1}{n}}} = 5 \quad \text{converges}$$

$y = (3^n + 5^n)^{\frac{1}{n}}$

$\ln y = \ln (3^n + 5^n)^{\frac{1}{n}}$

$\ln y = \dfrac{\ln (3^n + 5^n)}{n}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{\ln(3^n+5^n)}}{\underset{+\infty}{n}} \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{1}{3^n+5^n}\left((\ln 3)3^n + (\ln 5)5^n\right)}{1}$$

$$= \lim_{n\to\infty} \frac{(\ln 3)3^n + (\ln 5)5^n}{3^n + 5^n} = \lim_{n\to\infty} \frac{\frac{(\ln 3)3^n}{5^n} + \frac{(\ln 5)5^n}{5^n}}{\frac{3^n}{5^n} + \frac{5^n}{5^n}}$$

$$= \lim_{n\to\infty} \frac{(\ln 3)\frac{3^n}{5^n} + \ln 5}{\left(\frac{3^n}{5^n}\right) + 1} = \lim_{n\to\infty} \frac{(\ln 3)\left(\frac{3}{5}\right)^n + \ln 5}{\left(\frac{3}{5}\right)^n + 1}$$

$\ln y = \ln 5 \Rightarrow y = 5$

$$= \frac{(\ln 3)(0) + \ln 5}{(0) + 1} = \ln 5 \quad \text{since } 0 < \tfrac{3}{5} < 1,\ \lim_{n\to\infty} \left(\tfrac{3}{5}\right)^n = 0$$

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94) $a_n = \sqrt[n]{n^2 + n}$

$$\lim_{n\to\infty} \sqrt[n]{n^2+n} = \lim_{n\to\infty} \overset{(+\infty)^{(0)}}{(n^2+n)^{\frac{1}{n}}} = 1 \quad \text{converges}$$

$y = (n^2+n)^{\frac{1}{n}}$

$\ln y = \ln (n^2+n)^{\frac{1}{n}}$

$\ln y = \dfrac{\ln(n^2+n)}{n}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{\ln(n^2+n)}}{\underset{+\infty}{n}} \stackrel{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n^2+n}(2n+1)}{1} = \lim_{n\to\infty} \frac{2n+1}{n^2+n}$$

$$\stackrel{L}{=} \lim_{n\to\infty} \frac{2}{2n+1} = 0 \qquad \ln y = 0 \Rightarrow y = e^0 = 1$$

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96) $a_n = \dfrac{(\ln n)^5}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{(\ln n)^5}}{\underset{+\infty}{\sqrt{n}}} \stackrel{L}{=} \lim_{n\to\infty} \frac{5(\ln n)^4\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{\overset{+\infty}{(2)(5)(\ln n)^4}}{\underset{+\infty}{\sqrt{n}}} \stackrel{L}{=} \lim_{n\to\infty} \frac{(2)(5)(4)(\ln n)^3\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\overset{+\infty}{(2)^2(5)(4)(\ln n)^3}}{\underset{+\infty}{\sqrt{n}}} \stackrel{L}{=} \lim_{n\to\infty} \frac{(2)^2(5)(4)(3)(\ln n)^2\left(\frac{1}{n}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{(2)^3(5)(4)(3)(\ln n)^2}{\sqrt{n}} \stackrel{L}{=} \lim_{n\to\infty} \frac{(2)^3(5)(4)(3)(2)(\ln n)\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\overset{+\infty}{(2)^5(5)(4)(3)(\ln n)}}{\underset{+\infty}{\sqrt{n}}} \stackrel{L}{=} \lim_{n\to\infty} \frac{(2)^5(5)(4)(3)\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{(2)^6(5)(4)(3)}{\sqrt{n}} = 0 \quad \text{converges}$$

98) $a_n = \dfrac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}}$

converges

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}\right)\left(\frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}\right) = \lim_{n\to\infty}\frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{(n^2-1)-(n^2+n)} = \lim_{n\to\infty}\frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{-1-n}$$

$$= \lim_{n\to\infty}\frac{\dfrac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2}}}{\dfrac{-1-n}{n}} = \lim_{n\to\infty}\frac{\sqrt{\frac{n^2}{n^2}-\frac{1}{n^2}}+\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}}{\frac{-1}{n}-\frac{n}{n}} = \lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{\frac{-1}{n}-1} = \frac{\sqrt{1-0}+\sqrt{1+0}}{0-1} = -2$$

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102) $a_1 = -1$, $a_{n+1} = \dfrac{a_n+6}{a_n+2}$

since a_n converges, $\lim_{n\to\infty} a_n = L$

and $\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}\left(\dfrac{a_n+6}{a_n+2}\right)$

$$\Downarrow$$

$$L = \frac{L+6}{L+2}$$

$L(L+2) = L+6$

$L^2+2L = L+6$

$L^2+L-6 = 0$

$(L+3)(L-2) = 0$

$L+3=0 \quad | \quad L-2=0$

$L=-3 \quad | \quad L=2$

now, since $a_n > 0$ for $n \geq 2$

$\boxed{\text{we discard } L=-3}$

$\underline{\underline{L=2}}$

104) $a_1 = 0$, $a_{n+1} = \sqrt{8+2a_n}$

since a_n converges, $\lim\limits_{n\to\infty} a_n = L$

and $\lim\limits_{n\to\infty}(a_{n+1}) = \lim\limits_{n\to\infty}\left(\sqrt{8+2a_n}\right)$

$$\Downarrow$$

$$L = \sqrt{8+2L}$$

$$(L)^2 = \left(\sqrt{8+2L}\right)^2$$

$$L^2 = 8+2L$$

$$L^2 - 2L - 8 = 0$$

$$(L+2)(L-4) = 0$$

$L+2=0$ | $L-4=0$
$L=-2$ | $L=4$

now, since $a_n > 0$

for $n \geq 2$

$\underline{\underline{L=4}}$

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106) $a_1 = 3$, $a_{n+1} = 12 - \sqrt{a_n}$

since a_n converges, $\lim\limits_{n\to\infty} a_n = L$

and $\lim\limits_{n\to\infty}(a_{n+1}) = \lim\limits_{n\to\infty}\left(12-\sqrt{a_n}\right)$

$$\Downarrow$$

$$L = 12 - \sqrt{L}$$

$$\sqrt{L} = 12 - L$$

$$(\sqrt{L})^2 = (12-L)^2$$

$$L = 144 - 24L + L^2$$

$$0 = L^2 - 25L + 144$$

$$0 = (L-9)(L-16)$$

$L-9=0$ | $L-16=0$
$L=9$ | $L=16$

now, since

$a_{n+1} = 12 - \sqrt{a_n} < 12$

for $n \geq 1$

$\underline{\underline{L=9}}$

108) $\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots$

$a_1 = \sqrt{1}, \quad a_{n+1} = \sqrt{1+a_n}$

since a_n converges, $\lim_{n\to\infty} a_n = L$

and $\lim_{n\to\infty} (a_{n+1}) = \lim_{n\to\infty} \left(\sqrt{1+a_n}\right)$

$$\Downarrow$$

$$L = \sqrt{1+L}$$

$$(L)^2 = \left(\sqrt{1+L}\right)^2$$

$$L^2 = 1+L$$

$$L^2 - L - 1 = 0$$

$$L = \frac{-(-1) \pm \sqrt{(-1)^2-4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

now, since $a_n > 0$ for $n \geq 1$

$$\underline{\underline{L = \frac{1+\sqrt{5}}{2}}}$$

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122) $a_n = \dfrac{(2n+3)!}{(n+1)!}$

$$a_{n+1} \geq a_n$$

$$\frac{(2(n+1)+3)!}{((n+1)+1)!} > \frac{(2n+3)!}{(n+1)!}$$

$$\frac{(2n+5)!}{(n+2)!} > \frac{(2n+3)!}{(n+1)!}$$

$$\frac{(2n+5)!}{(2n+3)!} > \frac{(n+2)!}{(n+1)!}$$

$$\frac{(2n+5)(2n+4)(2n+3)!}{(2n+3)!} > \frac{(n+2)(n+1)!}{(n+1)!}$$

$$(2n+5)(2n+4) > (n+2)$$

the steps are reversible so the sequence is nondecreasing

the sequence is not bounded since

$$\frac{(2n+3)!}{(n+1)!} = \frac{(2n+3)(2n+2)(2n+1)(2n)(n+(n-1))(n+(n-2))\cdots(n+2)(n+1)!}{(n+1)!}$$

$$= (2n+3)(2n+2)\cdots(n+2)$$

can become as large as we please.

124) $a_n = 2 - \frac{2}{n} - \frac{1}{2^n}$

$$a_{n+1} \geq a_n$$

$$\left(2 - \frac{2}{n+1} - \frac{1}{2^{n+1}}\right) \geq \left(2 - \frac{2}{n} - \frac{1}{2^n}\right)$$

$$\frac{-2}{n+1} - \frac{1}{2^{n+1}} \geq \frac{-2}{n} - \frac{1}{2^n}$$

$$\frac{2}{n} - \frac{2}{n+1} \geq \frac{1}{2^{n+1}} - \frac{1}{2^n}$$

$$\left(\frac{2}{n}\right)\left(\frac{n+1}{n+1}\right) - \left(\frac{2}{n+1}\right)\left(\frac{n}{n}\right) \geq \frac{-1}{2^{n+1}}$$

$$\frac{2(n+1) - 2(n)}{n(n+1)} \geq \frac{-1}{2^{n+1}}$$

$$\frac{2}{n(n+1)} \geq \frac{-1}{2^{n+1}}$$

$$\frac{1}{2^{n+1}} - \frac{1}{2^n} \quad LCD = (2^n)(2^{n+1})$$

$$= \left(\frac{1}{2^{n+1}}\right)\left(\frac{2^n}{2^n}\right) - \left(\frac{1}{2^n}\right)\left(\frac{2^{n+1}}{2^{n+1}}\right)$$

$$= \frac{2^n - 2^{n+1}}{(2^n)(2^{n+1})} = \frac{2^n - (2^n)(2^1)}{(2^n)(2^{n+1})}$$

$$= \frac{2^n(1 - 2^1)}{(2^n)(2^{n+1})} = \frac{-1}{2^{n+1}}$$

the steps are reversible so the sequence is nondecreasing

$$2 - \frac{2}{n} - \frac{1}{2^n} \leq 2$$

the sequence is bounded from above