Definition

Integrals with infinite limits of integration are **improper Integrals of Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{U\to\infty}\int_a^U f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{L\to-\infty}\int_L^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$ where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Definition

Integrals of functions that become infinite at a point within the interval of integration are **improper Integrals of Type II**.

1. If $f(x)$ is continuous on $(a,b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{L\to a^+}\int_L^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{U\to b^+}\int_a^U f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c)\cup(c,b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$ where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Theorem 2

Let f and g be continuous on $[a,\infty)$ with $0\le f(x)\le g(x)$ for all $x\ge a$. Then

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ also converges.

2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

Theorem 2-a
Let f and g be continuous on $(0,a]$ with $0 \le f(x) \le g(x)$ for all $0 < x \le a$. Then

1. If $\int_0^a g(x)\,dx$ converges, then $\int_0^a f(x)\,dx$ also converges.

2. If $\int_0^a f(x)\,dx$ diverges, then $\int_0^a g(x)\,dx$ also diverges.

Reference for Comparison Theorem:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{convergent if} & p > 1 \\ \text{divergent if} & p \le 1 \end{cases} \qquad \int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{divergent if} & p \ge 1 \\ \text{convergent if} & p < 1 \end{cases}$$

Theorem 3
If the positive functions f and g are continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$, and if

$$\lim_{x\to\infty} \frac{\text{smaller}}{\text{larger}} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either *both converge* or *both diverge*.

Theorem 3-a
If the positive functions f and g are continuous on $(0,a]$ with $0 \le f(x) \le g(x)$, and if

$$\lim_{x\to 0} \frac{\text{smaller}}{\text{larger}} = \lim_{x\to 0} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then $\int_0^a f(x)\,dx$ and $\int_0^a g(x)\,dx$ either *both converge* or *both diverge*.

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2) $\int \frac{dx}{x^{1.001}} = \int x^{-1.001} dx = \left[\frac{x^{-0.001}}{-0.001}\right] + C = \frac{-1000}{x^{0.001}} + C$

$\int_1^\infty \frac{dx}{x^{1.001}} = \lim_{U\to\infty} \int_1^U \frac{1}{x^{1.001}} dx = \lim_{U\to\infty} \left[\frac{-1000}{x^{0.001}} + C\right]_1^U$

$= \lim_{U\to\infty} \left\{\left[\frac{-1000}{U^{0.001}} + C\right] - \left[\frac{-1000}{(1)^{0.001}} + C\right]\right\} = [0] - \left[\frac{-1000}{1}\right] = 1000$

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4) $\int \frac{dx}{\sqrt{4-x}} = \int \frac{1}{\sqrt{p}}(-1\,dp) = \int (-1)\, p^{-\frac{1}{2}} dp = (-1)\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$p = 4-x$

$= -2\sqrt{p} + C = -2\sqrt{4-x} + C$

$dp = -1\,dx \Rightarrow -1\,dp = dx$

$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{U\to 4^-} \int_0^U \frac{1}{\sqrt{4-x}} dx = \lim_{U\to 4^-} \left[-2\sqrt{4-x} + C\right]_0^U$

$= \lim_{U\to 4^-} \left\{\left[-2\sqrt{4-U} + C\right] - \left[-2\sqrt{4-(0)} + C\right]\right\} = [0] - [-2(2)] = 4$

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8) $\int \frac{dr}{r^{0.999}} = \int r^{-0.999} dr = \left[\frac{r^{0.001}}{0.001}\right] + C = 1000\, r^{0.001} + C$

$\int_0^1 \frac{dr}{r^{0.999}} = \lim_{L\to 0^+} \int_L^1 \frac{1}{r^{0.999}} dr = \lim_{L\to 0^+} \left[1000\, r^{0.001} + C\right]_L^1$

$= \lim_{L\to 0^+} \left\{\left[1000\,(1)^{0.001} + C\right] - \left[1000\, L^{0.001} + C\right]\right\} = [1000] - [0]$

$= 1000$

6) $\int \frac{dx}{x^{\frac{1}{3}}} = \int x^{-\frac{1}{3}}dx = \left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right] + C = \frac{3}{2}\left(\sqrt[3]{x}\right)^2 + C$

$\int_{-8}^{1} \frac{dx}{x^{\frac{1}{3}}} = \int_{-8}^{0} \frac{1}{x^{\frac{1}{3}}}dx + \int_{0}^{1} \frac{1}{x^{\frac{1}{3}}}dx = [-6] + \left[\frac{3}{2}\right]$

$= \frac{-12}{2} + \frac{3}{2} = \underline{\underline{\frac{-9}{2}}}$

$\int_{-8}^{0} \frac{1}{x^{\frac{1}{3}}}dx = \lim_{U \to 0^-} \int_{-8}^{U} \frac{1}{x^{\frac{1}{3}}}dx = \lim_{U \to 0^-} \left[\frac{3}{2}\left(\sqrt[3]{x}\right)^2 + C\right]_{-8}^{U}$

$= \lim_{U \to 0^-} \left\{ \left[\frac{3}{2}\left(\sqrt[3]{U}\right)^2 + C\right] - \left[\frac{3}{2}\left(\sqrt[3]{(-8)}\right)^2 + C\right] \right\} = [0] - \left[\frac{3}{2}(-2)^2\right] = -6$

$\int_{0}^{1} \frac{1}{x^{\frac{1}{3}}}dx = \lim_{L \to 0^+} \int_{L}^{1} \frac{1}{x^{\frac{1}{3}}}dx = \lim_{L \to 0^+} \left[\frac{3}{2}\left(\sqrt[3]{x}\right)^2 + C\right]$

$= \lim_{L \to 0^+} \left\{ \left[\frac{3}{2}\left(\sqrt[3]{(1)}\right)^2 + C\right] - \left[\frac{3}{2}\left(\sqrt[3]{L}\right)^2 + C\right] \right\} = \left[\frac{3}{2}(1)^2\right] - [0] = \frac{3}{2}$

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10) $\int \frac{2}{x^2+4}dx = \int \frac{2}{x^2+(2)^2}dx = 2\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C = \tan^{-1}\left(\frac{x}{2}\right) + C$

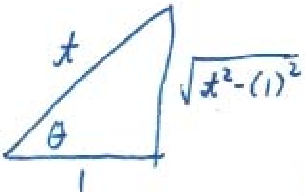
$\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim_{L \to -\infty} \int_{L}^{2} \frac{2}{x^2+4}dx = \lim_{L \to -\infty} \left[\tan^{-1}\left(\frac{x}{2}\right) + C\right]_{L}^{2}$

$= \lim_{L \to -\infty} \left\{ \left[\tan^{-1}\left(\frac{(2)}{2}\right) + C\right] - \left[\tan^{-1}\left(\frac{(-\infty)}{2}\right) + C\right] \right\}$

$= \left[\tan^{-1}(1)\right] - \left[\tan^{-1}(-\infty)\right]$

$= \left[\frac{\pi}{4}\right] - \left[\frac{-\pi}{2}\right] = \frac{\pi}{4} + \frac{\pi}{2} = \underline{\underline{\frac{3\pi}{4}}}$

12) $\int \frac{2\,dt}{t^2-1} = \int \frac{2}{t^2-(1)^2}\,dt = \int \frac{2}{(\sqrt{t^2-(1)^2})^2}\,dt = \int \frac{2}{(\tan\theta)^2}(\sec\theta\tan\theta\,d\theta)$



$\frac{t}{1} = \sec\theta \qquad \frac{\sqrt{t^2-(1)^2}}{1} = \tan\theta$

$t = \sec\theta \qquad \sqrt{t^2-(1)^2} = \tan\theta$

$dt = \sec\theta\tan\theta\,d\theta$

$= \int \frac{2\sec\theta}{\tan\theta}\,d\theta = \int \left(\frac{2}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)d\theta$

$= \int \frac{2}{\sin\theta}\,d\theta = \int 2\csc\theta\,d\theta$

$= 2\left[\ln|\csc\theta - \cot\theta|\right] + C$

$= 2\ln\left|\frac{t}{\sqrt{t^2-(1)^2}} - \frac{1}{\sqrt{t^2-(1)^2}}\right| + C$

$= 2\ln\left|\frac{t-1}{\sqrt{t^2-1}}\right| + C$

$$\int_2^\infty \frac{2\,dt}{t^2-1} = \lim_{U\to\infty}\int_2^U \frac{2}{t^2-1}\,dt = \lim_{U\to\infty}\left[2\ln\left|\frac{t-1}{\sqrt{t^2-1}}\right| + C\right]_2^U$$

$$= \lim_{U\to\infty}\left\{\left[2\ln\left|\frac{U-1}{\sqrt{U^2-1}}\right| + C\right] - \left[2\ln\left|\frac{(2)-1}{\sqrt{(2)^2-1}}\right| + C\right]\right\}$$

$$= \left[2\ln|(1)|\right] - \left[2\ln\left|\frac{1}{\sqrt{3}}\right|\right] = \left[2(0)\right] - \left[2\ln(3^{-\frac{1}{2}})\right]$$

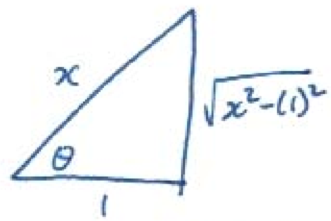
$$= [0] - \left[2\left(-\tfrac{1}{2}\ln 3\right)\right] = \underline{\underline{\ln 3}}$$

$$\lim_{U\to\infty}\frac{\overset{+\infty}{U-1}}{\underset{+\infty}{\sqrt{U^2-1}}} = \lim_{U\to\infty}\frac{\frac{U-1}{\sqrt{U^2}}}{\frac{\sqrt{U^2-1}}{\sqrt{U^2}}} = \lim_{U\to\infty}\frac{\frac{U-1}{U}}{\sqrt{\frac{U^2-1}{U^2}}} = \lim_{U\to\infty}\frac{\frac{U}{U}-\frac{1}{U}}{\sqrt{\frac{U^2}{U^2}-\frac{1}{U^2}}}$$

$$= \lim_{U\to\infty}\frac{1-\frac{1}{U}}{\sqrt{1-\frac{1}{U^2}}} = \frac{1-0}{\sqrt{1-0}} = \frac{1}{\sqrt{1}} = 1$$

18) $$\int \frac{1}{x\sqrt{x^2-(1)^2}}\,dx = \int \frac{1}{(\sec\theta)(\tan\theta)}(\sec\theta\tan\theta\,d\theta)$$

$$= \int 1\,d\theta = [\theta] + C = \sec^{-1}x + C$$



$\frac{x}{1} = \sec\theta$ $\quad \frac{\sqrt{x^2-(1)^2}}{1} = \tan\theta$ $\quad x = \sec\theta$

$\Downarrow$

$x = \sec\theta$ $\quad \sqrt{x^2-(1)^2} = \tan\theta$ $\quad \sec^{-1}x = \theta$

$dx = \sec\theta\tan\theta\,d\theta$

$$\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \left[\frac{\pi}{3}\right] + \left[\frac{\pi}{2} - \frac{\pi}{3}\right] = \underline{\underline{\frac{\pi}{2}}}$$

$$\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{L\to1^+}\int_L^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{L\to1^+}\left[\sec^{-1}x + C\right]_L^2$$

$$= \lim_{L\to1^+}\left\{\left[\sec^{-1}(2)+C\right] - \left[\sec^{-1}(L)+C\right]\right\} = \left[\sec^{-1}(2)\right] - \left[\sec^{-1}(1)\right]$$

$$= \left[\frac{\pi}{3}\right] - [0] = \frac{\pi}{3}$$

$$\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{U\to\infty}\int_2^U \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{U\to\infty}\left[\sec^{-1}x + C\right]_2^U$$

$$= \lim_{U\to\infty}\left\{\left[\sec^{-1}(U)+C\right] - \left[\sec^{-1}(2)+C\right]\right\} = \left[\sec^{-1}(+\infty)\right] - \left[\sec^{-1}(2)\right]$$

$$= \left[\frac{\pi}{2}\right] - \left[\frac{\pi}{3}\right] = \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

20) $\int \frac{16\tan^{-1}x}{1+x^2}\,dx = \int 16\tan^{-1}x\left(\frac{1}{x^2+1}\,dx\right) = \int 16\varphi\,(d\varphi)$

$= 16\left[\frac{\varphi^2}{2}\right] + C$

$= 8\left(\tan^{-1}x\right)^2 + C$

$\varphi = \tan^{-1}x$

$\Downarrow$

$\tan\varphi = x = \frac{x}{1}$

$d\varphi = \frac{1}{x^2+1}\,dx$

$\sec^2\varphi\,\frac{d\varphi}{dx} = 1$

$\frac{d\varphi}{dx} = \frac{1}{\sec^2\varphi} = \frac{1}{\left(\frac{\sqrt{x^2+(1)^2}}{1}\right)^2} = \frac{1}{x^2+1}$

$$\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{U\to\infty}\int_0^U \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{U\to\infty}\left[8\left(\tan^{-1}x\right)^2 + C\right]_0^U$$

$$= \lim_{U\to\infty}\left\{\left[8\left(\tan^{-1}(U)\right)^2 + C\right] - \left[8\left(\tan^{-1}(0)\right)^2 + C\right]\right\}$$

$$= \left[8\left(\tan^{-1}(+\infty)\right)^2\right] - \left[8(0)^2\right] = \left[8\left(\frac{\pi}{2}\right)^2\right] - [0] = 8\left(\frac{\pi^2}{4}\right) = \underline{\underline{2\pi^2}}$$

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22) $\int 2e^{-\theta}\sin\theta\,d\theta = 2\int e^{-\theta}\sin\theta\,d\theta = \left(e^{-\theta}\right)(-\cos\theta) - \int(-\cos\theta)\left(-e^{-\theta}\,d\theta\right)$

$u_1 = e^{-\theta}$ $\quad dv_1 = \sin\theta\,d\theta$

$du_1 = -e^{-\theta}\,d\theta$ $\quad v_1 = -\cos\theta$

$u_2 = e^{-\theta}$ $\quad dv_2 = \cos\theta\,d\theta$

$du_2 = -e^{-\theta}\,d\theta$ $\quad v_2 = \sin\theta$

$\int e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\cos\theta - \int e^{-\theta}\cos\theta\,d\theta$

$\int e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\cos\theta - \left\{\left(e^{-\theta}\right)(\sin\theta) - \int(\sin\theta)\left(-e^{-\theta}\,d\theta\right)\right\}$

$\int e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\cos\theta - e^{-\theta}\sin\theta - \int e^{-\theta}\sin\theta\,d\theta$

$2\int e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\cos\theta - e^{-\theta}\sin\theta$

$\int e^{-\theta}\sin\theta\,d\theta = \frac{-1}{2}e^{-\theta}\cos\theta - \frac{1}{2}e^{-\theta}\sin\theta + C$

$= \frac{-\cos\theta - \sin\theta}{2e^{\theta}} + C$

22) continued

$$\int 2e^{-\theta}\sin\theta\,d\theta = 2\int e^{-\theta}\sin\theta\,d\theta = 2\left(\frac{-\cos\theta-\sin\theta}{2e^{\theta}}\right)+C$$

$$= \frac{-\cos\theta-\sin\theta}{e^{\theta}}+C$$

$$\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta = \lim_{U\to\infty}\int_0^{U} 2e^{-\theta}\sin\theta\,d\theta = \lim_{U\to\infty}\left[\frac{-\cos\theta-\sin\theta}{e^{\theta}}+C\right]_0^{U}$$

$$= \lim_{U\to\infty}\left\{\left[\frac{-\cos(U)-\sin(U)}{e^{(U)}}+C\right]-\left[\frac{-\cos(0)-\sin(0)}{e^{(0)}}+C\right]\right\}$$

$$= \left[0\right]-\left[\frac{-(1)-(0)}{(1)}\right] = [0]-[-1] = \underline{\underline{1}}$$

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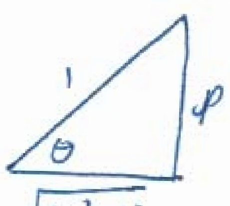
28) $$\int \frac{4r\,dr}{\sqrt{1-r^4}} = \int\frac{2}{\sqrt{1-(r^2)^2}}(2r\,dr) = \int\frac{2}{\sqrt{(1)^2-\rho^2}}\,d\rho$$

$\rho = r^2$

$d\rho = 2r\,dr$

$$= \int\frac{2}{(\cos\theta)}(\cos\theta\,d\theta) = \int 2\,d\theta$$

$$= 2\theta + C = 2(\sin^{-1}\rho)+C = 2\sin^{-1}(r^2)+C$$



$\frac{\rho}{1} = \sin\theta$ $\quad$ $\frac{\sqrt{(1)^2-\rho^2}}{1} = \cos\theta$

$\rho = \sin\theta$ $\quad$ $\sqrt{(1)^2-\rho^2} = \cos\theta$

$d\rho = \cos\theta\,d\theta$

$\rho = \sin\theta \Rightarrow \theta = \sin^{-1}\rho$

$$\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}} = \lim_{U\to 1^-}\int_0^U \frac{4r\,dr}{\sqrt{1-r^2}} = \lim_{U\to 1^-}\left[2\sin^{-1}(r^2)+C\right]_0^U$$

$$= \lim_{U\to 1^-}\left\{\left[2\sin^{-1}(U^2)+C\right]-\left[2\sin^{-1}((0)^2)+C\right]\right\}$$

$$= \left[2\sin^{-1}((1)^2)\right]-\left[2\sin^{-1}(0)\right]$$

$$= \left[2\left(\frac{\pi}{2}\right)\right]-\left[2(0)\right] = \underline{\underline{\pi}}$$

36) $\int \frac{d\theta}{\theta^2-2\theta} = \int \frac{1}{(\theta)^1(\theta-2)^1}d\theta = \int\left(\frac{\left(\frac{-1}{2}\right)}{(\theta)^1} + \frac{\left(\frac{1}{2}\right)}{(\theta-2)^1}\right)d\theta$

$\frac{1}{(\theta)^1(\theta-2)^1} = \frac{A}{(\theta)^1} + \frac{B}{(\theta-2)^1}$

$1 = A(\theta-2) + B(\theta)$

constant term $\quad$ θ-term

$1 = -2A \qquad 0 = A+B$

$A = \frac{-1}{2} \qquad B = -A$

$B = -\left(\frac{-1}{2}\right) = \frac{1}{2}$

$= \frac{-1}{2}\left[\ln|\theta|\right] + \frac{1}{2}\left[\ln|\theta-2|\right] + C$

$= \frac{-1}{2}\ln|\theta| + \frac{1}{2}\ln|\theta-2| + C$

$= \frac{1}{2}\ln\left|\frac{\theta-2}{\theta}\right| + C$

$= \frac{1}{2}\ln\left|1-\frac{2}{\theta}\right| + C$

$$\int_{-1}^{1}\frac{d\theta}{\theta^2-2\theta} = \int_{-1}^{0}\frac{d\theta}{\theta^2-2\theta} + \int_{0}^{1}\frac{d\theta}{\theta^2-2\theta}$$

$$\int_{-1}^{0}\frac{d\theta}{\theta^2-2\theta} = \lim_{U\to0^-}\int_{-1}^{U}\frac{d\theta}{\theta^2-2\theta} = \lim_{U\to0^-}\left[\frac{1}{2}\ln\left|\frac{\theta-2}{\theta}\right| + C\right]_{-1}^{U}$$

$$= \lim_{U\to0^-}\left\{\left[\frac{1}{2}\underbrace{\lim\left|\frac{U-2}{U}\right|}_{\to+\infty} + C\right] - \left[\frac{1}{2}\ln\left|\frac{(-1)-2}{(-1)}\right| + C\right]\right\} = +\infty$$

diverges

Since $\int_{-1}^{0}\frac{d\theta}{\theta^2-2\theta}$ diverges, $\int_{-1}^{1}\frac{d\theta}{\theta^2-2\theta}$ also diverges

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38) $\int \frac{d\theta}{\theta^2-1} = \int \frac{1}{(\theta+1)^1(\theta-1)^1}d\theta = \int\left(\frac{\left(\frac{-1}{2}\right)}{(\theta+1)^1} + \frac{\left(\frac{1}{2}\right)}{(\theta-1)^1}\right)d\theta$

$\frac{1}{(\theta+1)^1(\theta-1)^1} = \frac{A}{(\theta+1)^1} + \frac{B}{(\theta-1)^1}$

$1 = A(\theta-1) + B(\theta+1)$

constant term $\quad$ θ-term

$1 = -A+B \qquad 0 = A+B$

$-A = B$

$1 = (B)+B$

$1 = 2B$

$B = \frac{1}{2}$

$A = -B$

$A = -\left(\frac{1}{2}\right) = \frac{-1}{2}$

$= \frac{-1}{2}\left[\ln|\theta+1|\right] + \frac{1}{2}\left[\ln|\theta-1|\right] + C$

$= -\frac{1}{2}\ln|\theta+1| + \frac{1}{2}\ln|\theta-1| + C$

$= \frac{1}{2}\ln|\theta-1| - \frac{1}{2}\ln|\theta+1| + C$

38) continued

$$\int_0^\infty \frac{d\theta}{\theta^2-1} = \int_0^1 \frac{d\theta}{\theta^2-1} + \int_1^\infty \frac{d\theta}{\theta^2-1} = \int_0^1 \frac{d\theta}{\theta^2-1} + \int_1^2 \frac{d\theta}{\theta^2-1} + \int_2^\infty \frac{d\theta}{\theta^2-1}$$

$$\int_0^1 \frac{d\theta}{\theta^2-1} = \lim_{U\to 1^-} \int_0^U \frac{d\theta}{\theta^2-1} = \lim_{U\to 1^-} \left[\frac{1}{2}\ln|\theta-1| - \frac{1}{2}\ln|\theta+1| + C\right]_0^U$$

$$= \lim_{U\to 1^-} \left\{\left[\frac{1}{2}\underbrace{\ln|U-1|}_{\to -\infty} - \frac{1}{2}\ln|U+1| + C\right] - \left[\frac{1}{2}\ln|(0)-1| - \frac{1}{2}\ln|(0)+1| + C\right]\right\}$$

$$= -\infty$$

diverges

Since $\int_0^1 \frac{d\theta}{\theta^2-1}$ diverges, $\int_0^\infty \frac{d\theta}{\theta^2-1}$ also diverges

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42) $$\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\left(\frac{1}{x}dx\right) = \int \frac{1}{p}\,dp = \ln|p| + C$$

$$= \ln|\ln x| + C$$

$p = \ln x$

$dp = \frac{1}{x}dx$

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{L\to 1^+} \int_L^2 \frac{dx}{x\ln x} = \lim_{L\to 1^+} \Big[\ln|\ln x| + C\Big]_L^2$$

$$= \lim_{L\to 1^+} \left\{\Big[\ln|\ln(2)| + C\Big] - \Big[\underbrace{\ln|\ln L|}_{\to -\infty} + C\Big]\right\}$$

$$= +\infty$$

diverges

46)

$$f(t) = \frac{1}{t - \sin t} \leq g(t) = \frac{1}{t^3} \quad \text{for } (0, 1]$$

since $\int_0^1 \frac{dt}{1-\sin t}$ is not easy to find with techniques learned in earlier sections, we must use the tests of theorem 2 or 3,

$$\int_0^1 g(t)\,dt = \int_0^1 \frac{1}{t^3}\,dt = \int_0^1 t^{-3}\,dt = \lim_{L\to 0^+} \int_L^1 t^{-3}\,dt = \lim_{L\to 0^+} \left[\frac{t^{-2}}{-2} + C\right]_L^1$$

$$= \lim_{L\to 0^+} \left\{ \left[\frac{-1}{2(1)^2} + C\right] - \left[\underbrace{\frac{-1}{2L^2}}_{\to -\infty} + C\right] \right\} = +\infty$$

$\int_0^1 g(t)\,dt = +\infty$ which diverges and we cannot use the Direct Comparison Test.

We should use Limit Comparison Test

$$\lim_{t\to 0} \frac{f(t)}{g(t)} = \lim_{t\to 0} \frac{\left(\frac{1}{t-\sin t}\right)}{\left(\frac{1}{t^3}\right)} = \lim_{t\to 0} \frac{\overset{0}{t^3}}{\underset{0}{t - \sin t}} \overset{L}{=} \lim_{t\to 0} \frac{[3t^2]}{[1-\cos t]}$$

$$= \lim_{t\to 0} \frac{\overset{0}{3t^2}}{1-\cos t} \overset{L}{=} \lim_{t\to 0} \frac{[6t]}{-[-\sin t]} = \lim_{t\to 0} \frac{\overset{0}{6t}}{\underset{0}{\sin t}} \overset{L}{=} \lim_{t\to 0} \frac{[6]}{[\cos t]}$$

$$= \lim_{t\to 0} \frac{6}{\cos t} = \frac{6}{\cos(0)} = \frac{6}{(1)} = 6$$

$\lim_{t\to 0} \frac{f(t)}{g(t)} = 6$ and $\int_0^1 g(t)\,dt = +\infty$, so by Limit Comparison Test $\int_0^1 f(t)\,dt = \int_0^1 \frac{dt}{t - \sin t}$ diverges

50) $\int (-x)\ln(-x)\,dx = (\ln(-x))\left(\frac{-x^2}{2}\right) - \int \left(\frac{-x^2}{2}\right)\left(\frac{1}{x}dx\right)$

$u_1 = \ln(-x) \quad dv_1 = -x\,dx \quad = \frac{-1}{2}x^2\ln(-x) + \int x\,dx$

$du_1 = \left[\frac{1}{-x}(-1)\right]dx \quad v_1 = -\left[\frac{x^2}{2}\right] \quad = \frac{-1}{2}x^2\ln(-x) + \left[\frac{x^2}{2}\right] + C$

$du_1 = \frac{1}{x}dx \quad v_1 = \frac{-x^2}{2} \quad = \frac{-1}{2}x^2\ln(-x) + \frac{1}{2}x^2 + C$

$\int (-x)\ln x\,dx = (\ln x)\left(\frac{-x^2}{2}\right) - \int \left(\frac{-x^2}{2}\right)\left(\frac{1}{x}dx\right)$

$u_1 = \ln x \quad dv_1 = -x\,dx \quad = \frac{-1}{2}x^2\ln x + \int x\,dx$

$du_1 = \frac{1}{x}dx \quad v_1 = \frac{-x^2}{2} \quad = \frac{-1}{2}x^2\ln x + \frac{1}{2}x^2 + C$

$$\int_{-1}^{1} -x\ln|x|\,dx = \int_{-1}^{0}(-x)\ln(-x)\,dx + \int_0^1 (-x)\ln x\,dx$$

$$\int_{-1}^{0}(-x)\ln(-x)\,dx = \lim_{U\to 0^-}\int_{-1}^{U}(-x)\ln(-x)\,dx = \lim_{U\to 0^-}\left[\frac{-1}{2}x^2\ln(-x) + \frac{1}{2}x^2 + C\right]_{-1}^{U}$$

$$= \lim_{U\to 0^-}\left\{\left[\frac{-1}{2}\underbrace{U^2\ln(-U)}_{\to 0} + \frac{1}{2}U^2 + C\right] - \left[\frac{-1}{2}(-1)^2\ln(-(-1)) + \frac{1}{2}(-1)^2 + C\right]\right\}$$

$$= [0+0] - \left[0 + \frac{1}{2}\right] = \frac{-1}{2}$$

$$\lim_{U\to 0^-} \overset{(0)(-\infty)}{U^2\ln(-U)} = \lim_{U\to 0^-}\frac{\overset{-\infty}{\ln(-U)}}{\underset{+\infty}{\frac{1}{U^2}}} \overset{L}{=} \lim_{U\to 0^-}\frac{\left[\frac{1}{-U}(-1)\right]}{\left[-2U^{-3}\right]} = \lim_{U\to 0^-}\frac{\frac{1}{U}}{\frac{-2}{U^3}} = \lim_{U\to 0^-}\left(\frac{1}{U}\right)\left(\frac{U^3}{-2}\right)$$

$$= \lim_{U\to 0^-}\frac{-U^2}{2} = \frac{-(0^-)^2}{2} = 0$$

50) continued

$$\int_0^1 (-x)\ln x\,dx = \lim_{L\to 0^+}\int_L^1 (-x)\ln x\,dx = \lim_{L\to 0^+}\left[\frac{-1}{2}x^2\ln x + \frac{1}{2}x^2 + C\right]_L^1$$

$$= \lim_{L\to 0^+}\left\{\left[\frac{-1}{2}(1)^2\ln(1) + \frac{1}{2}(1)^2 + C\right] - \left[\frac{-1}{2}\underbrace{L^2\ln L}_{\to 0} + \frac{1}{2}L^2 + C\right]\right\}$$

$$= \left[0 + \frac{1}{2}\right] - \left[0 + 0\right] = \frac{1}{2}$$

$$\lim_{L\to 0^+} \overset{(0)(+\infty)}{L^2\ln L} = \lim_{L\to 0^+}\frac{\overset{+\infty}{\ln L}}{\underset{+\infty}{\frac{1}{L^2}}} \overset{L}{=} \lim_{L\to 0^+}\frac{\frac{1}{L}}{\frac{-2}{L^3}} = \lim_{L\to 0^+}\frac{-L^2}{2} = \frac{-(0^+)^2}{2} = 0$$

$$\int_{-1}^1 -x\ln|x|\,dx = \int_{-1}^0 (-x)\ln(-x)\,dx + \int_0^1 (-x)\ln x\,dx$$

$$= \left(\frac{-1}{2}\right) + \left(\frac{1}{2}\right) = 0$$

$\int_{-1}^1 -x\ln|x|\,dx = 0$ and this integral converges

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52) $\displaystyle\int \frac{dx}{\sqrt{x}-1} = ?$ so

$$0 \le f(x) = \frac{1}{\sqrt{x}-1} \le g(x) = \frac{1}{\sqrt{x}} \quad \text{for } [4,\infty)$$

$$\int_4^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{U\to\infty}\int_4^U \frac{1}{\sqrt{x}}\,dx = \lim_{U\to\infty}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C\right]_4^U$$

$$= \lim_{U\to\infty}\left\{\left[2\underset{\downarrow\, +\infty}{\sqrt{U}} + C\right] - \left[2\sqrt{(4)} + C\right]\right\} = +\infty$$

52) continued

we should use Limit Comparison Test

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)}=\lim_{x\to\infty}\frac{\overset{+\infty}{\sqrt{x}}}{\underset{+\infty}{\sqrt{x}-1}}\overset{L}{=}\lim_{x\to\infty}\frac{\left[\frac{1}{2\sqrt{x}}\right]}{\left[\frac{1}{2\sqrt{x}}-0\right]}$$

$$=\lim_{x\to\infty}\frac{2\sqrt{x}}{2\sqrt{x}}=\lim_{x\to\infty}1=1$$

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$ and $\int_4^\infty g(x)\,dx=+\infty$, so by Limit Comparison Test $\int_4^\infty f(x)\,dx=\int_4^\infty\frac{dx}{\sqrt{x}-1}$ diverges

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54) $\int\frac{d\theta}{1+e^\theta}=?$

$$0\le f(\theta)=\frac{1}{1+e^\theta}\le g(\theta)=\frac{1}{e^\theta}\quad\text{for } 0\le\theta<\infty$$

$$\int_0^\infty\frac{1}{e^\theta}\,d\theta=\lim_{v\to\infty}\int_0^v\frac{1}{e^\theta}\,d\theta=\lim_{v\to\infty}\left[-e^{-\theta}+C\right]_0^v$$
$$=\lim_{v\to\infty}\left\{\left[\underset{\searrow 0}{\frac{-1}{e^v}}+C\right]-\left[\frac{-1}{e^{(0)}}+C\right]\right\}=[0]-\left[\frac{-1}{(1)}\right]=1\ \text{converges}$$

$\int_0^\infty g(\theta)\,d\theta=\int_0^\infty\frac{1}{e^\theta}\,d\theta=1$ and by Direct Comparison Test

$\int_0^\infty f(\theta)\,d\theta=\int_0^\infty\frac{d\theta}{1+e^\theta}$ converges

56) $\int \frac{dx}{\sqrt{x^2-1}}$ is a long one.

$$0 \le f(x) = \frac{1}{\sqrt{x^2-1}} \le g(x) = \frac{1}{\sqrt{x^2}} = \frac{1}{x} \quad \text{for } 2 \le x < \infty$$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{U\to\infty} \int_2^U \frac{1}{x}\,dx = \lim_{U\to\infty} \left[\ln|x| + C\right]_2^U$$
$$= \lim_{U\to\infty} \left\{ \left[\underset{\searrow +\infty}{\ln|U|} + C\right] - \left[\ln|(2)| + C\right] \right\} = +\infty$$

we should use Limit Comparison test

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{\frac{x}{\sqrt{x}}}{\frac{\sqrt{x^2-1}}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2-1}{x^2}}}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2}{x^2} - \frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = \frac{1}{\sqrt{1-0}} = 1$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_2^\infty g(x)\,dx = +\infty$, so by Limit Comparison Test $\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges

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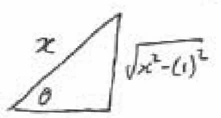
$$\int \frac{dx}{\sqrt{x^2-(1)^2}} = \int \frac{1}{(\tan\theta)} (\sec\theta \tan\theta\, d\theta) = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\left(\frac{x}{1}\right) + \left(\frac{\sqrt{x^2-(1)^2}}{1}\right)\right| + C = \ln\left|x + \sqrt{x^2-1}\right| + C$$

$\frac{x}{1} = \sec\theta \quad \frac{\sqrt{x^2-(1)^2}}{1} = \tan\theta$

$x = \sec\theta \quad \sqrt{x^2-(1)^2} = \tan\theta$

$dx = \sec\theta \tan\theta\, d\theta$

$$\int_2^\infty \frac{dx}{\sqrt{x^2-1}} = \lim_{U\to\infty} \int_2^U \frac{1}{\sqrt{x^2-(1)^2}}\,dx = \lim_{U\to\infty} \left[\ln\left|x + \sqrt{x^2-1}\right| + C\right]_2^U$$
$$= \lim_{U\to\infty} \left\{ \left[\underbrace{\ln\left|U + \sqrt{U^2-1}\right|}_{+\infty} + C\right] - \left[\ln\left|(2) + \sqrt{(2)^2-1}\right| + C\right] \right\} = +\infty$$

diverges

60) $\int \frac{1+\sin x}{x^2}\,dx = ?$

$$0 \le f(x) = \frac{1+\sin x}{x^2} \le g(x) = \frac{2}{x^2} \quad \text{for } \pi \le x < \infty$$

$$\int_\pi^\infty \frac{2}{x^2}\,dx = \lim_{U\to\infty}\int_\pi^U \frac{2}{x^2}\,dx = \lim_{U\to\infty}\left[\frac{-2}{x}+C\right]_\pi^U$$

$$= \lim_{U\to\pi}\left\{\left[\underset{\searrow 0}{\frac{-2}{U}}+C\right]-\left[\frac{-2}{(\pi)}+C\right]\right\} = [0]-\left[\frac{-2}{\pi}\right] = \frac{2}{\pi} \text{ converges}$$

$\int_\pi^\infty g(x)\,dx = \frac{2}{\pi}$ converges so by the Direct Comparison Test and $0 \le f(x) \le g(x)$

$\int_\pi^\infty f(x)\,dx = \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges

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62) $\int \frac{1}{\ln x}\,dx = ?$

$$0 \le f(x) = \frac{1}{x} \le g(x) = \frac{1}{\ln x} \quad \text{for } 2 < x < \infty$$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{U\to\infty}\int_2^U \frac{1}{x}\,dx = \lim_{U\to\infty}\left[\ln|x|+C\right]_2^U$$

$$= \lim_{U\to\infty}\left\{\left[\underset{\searrow +\infty}{\ln|U|}+C\right]-\left[\ln|(2)|+C\right]\right\} = +\infty$$

$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x}\,dx = +\infty$ diverges, so by the Direct Comparison Test and $0 \le f(x) \le g(x)$

$\int_2^\infty g(x)\,dx = \int_2^\infty \frac{1}{\ln x}\,dx$ diverges

66) $\int \frac{1}{e^x - 2^x}\,dx = ?$

$$0 \le f(x) = \frac{1}{e^x - 2^x} \le g(x) = \frac{1}{e^x} \qquad \text{for } 1 \le x < \infty$$

$$\int_1^\infty \frac{1}{e^x}\,dx = \lim_{U\to\infty} \int_1^U \frac{1}{e^x}\,dx = \lim_{U\to\infty} \left[\frac{-1}{e^x} + C\right]_1^U$$

$$= \lim_{U\to\infty} \left\{ \left[\underbrace{\frac{-1}{e^U}}_{\to 0} + C\right] - \left[\frac{-1}{e^{(1)}} + C\right] \right\} = [0] - \left[\frac{-1}{e}\right] = \frac{1}{e} \text{ converges}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\left(\frac{1}{e^x - 2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{e^x}{e^x} - \frac{2^x}{e^x}}$$

$$= \lim_{x\to\infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$$

because $0 < \frac{2}{e} < 1$ and $\lim_{x\to\infty} \left(\frac{2}{e}\right)^x = 0$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{e^x}\,dx = \frac{1}{e}$ converges,

so by Limit Comparison Test

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{e^x - 2^x}\,dx \text{ converges}$$