MATH 21200 section 8.5

2) $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-1)^1(x-2)^1} = \frac{A}{(x-1)^1} + \frac{B}{(x-2)^1}$

$5x-7 = A(x-2) + B(x-1)$

constant term

$-7 = -2A - B$

$-7 = -2A + (A-5)$

$-2 = -A$

$A = 2$

x-term

$5 = A + B$

$5 - A = B$

$(A-5) = -B$

$B = 5-(2) = 3$

$\frac{5x-7}{x^2-3x+2} = \frac{(2)}{(x-1)^1} + \frac{(3)}{(x-2)^1}$

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4) $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{(x-1)^1} + \frac{B}{(x-1)^2}$

$2x+2 = A(x-1) + B$

constant term

$2 = -A + B$

$2 = -(2) + B$

$B = 4$

x-term

$2 = A$

$\frac{2x+2}{x^2-2x+1} = \frac{(2)}{(x-1)^1} + \frac{(4)}{(x-1)^2}$

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6) $\frac{z}{z^3-z^2-6z} = \frac{z}{z(z^2-z-6)} = \frac{z}{(z)^1(z+2)^1(z-3)^1} = \frac{A}{(z)^1} + \frac{B}{(z+2)^1} + \frac{C}{(z-3)^1}$

$z = A(z+2)(z-3) + B(z(z-3)) + C(z(z+2))$

$z = A(z^2-z-6) + B(z^2-3z) + C(z^2+2z)$

constant term

$0 = -6A$

$A = 0$

z-term

$1 = -A - 3B + 2C$

$1 = -3B + 2C$

$1 = 3(C) + 2C$

$1 = 5C \quad C = \frac{1}{5}$

$z^2$-term

$0 = A + B + C$

$0 = B + C$

$-B = C$

$B = -C = \frac{-1}{5}$

$\frac{z}{z^3-z^2-6z} = \frac{(0)}{(z)^1} + \frac{(-\frac{1}{5})}{(z+2)^1} + \frac{(\frac{1}{5})}{(z-3)^1}$

8) $\dfrac{t^4+9}{t^4+9t^2} = 1 + \dfrac{(-9t^2+9)}{t^4+9t^2} = 1 + \dfrac{(-9t^2+9)}{(t)^2(t^2+9)^1}$

$$
\begin{array}{r}
1 \\
t^4+0t^3+9t^2+0t+0 \,\overline{\big)\, t^4+0t^3+0t^2+0t+9} \\
-(t^4+0t^3+9t^2+0t+0) \\
\hline
-9t^2+9
\end{array}
$$

$$\frac{(-9t^2+9)}{(t)^2(t^2+9)^1} = \frac{A}{(t)^1} + \frac{B}{(t)^2} + \frac{(Ct+D)}{(t^2+9)^1}$$

$$-9t^2+9 = A(t(t^2+9)) + B(t^2+9) + (Ct+D)(t)^2$$
$$-9t^2+9 = A(t^3+9t) + B(t^2+9) + C(t^3) + D(t^2)$$

| constant term | $t$-term | $t^2$-term | $t^3$ term |
|---|---|---|---|
| $9 = 9B$ | $0 = 9A$ | $-9 = B + D$ | $0 = A + C$ |
| $B = 1$ | $A = 0$ | $-9 = (1) + D$ | $C = -A$ |
| | | $-10 = D$ | $C = 0$ |

$$\frac{(-9t^2+9)}{(t)^2(t^2+9)^1} = \frac{(0)}{(t)^1} + \frac{(1)}{(t)^2} + \frac{((0)t+(-10))}{(t^2+9)^1} = \frac{1}{t^2} + \frac{(-10)}{t^2+9}$$

$$\frac{t^4+9}{t^4+9t^2} = 1 + \left(\frac{1}{t^2}\right) + \left(\frac{-10}{t^2+9}\right)$$

12) $\dfrac{2x+1}{x^2-7x+12} = \dfrac{2x+1}{(x-3)^1(x-4)^1} = \dfrac{A}{(x-3)^1} + \dfrac{B}{(x-4)^1}$

$2x+1 = A(x-4) + B(x-3)$

constant term

$1 = -4A - 3B$

$1 = -4A - 3(2-A)$

$1 = -A - 6$

$7 = -A$

$A = -7$

x-term

$2 = A + B$

$B = (2-A)$

$B = 2-(-7)$

$B = 9$

$\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx = \int\left(\frac{(-7)}{(x-3)^1} + \frac{(9)}{(x-4)^1}\right)dx$

$= -7\left[\ln|x-3|\right] + 9\left[\ln|x-4|\right] + C$

$= 9\ln|x-4| - 7\ln|x-3| + C$

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14) $\dfrac{y+4}{y^2+y} = \dfrac{y+4}{(y)^1(y+1)^1} = \dfrac{A}{(y)^1} + \dfrac{B}{(y+1)^1}$

$y+4 = A(y+1) + B(y)$

constant term

$4 = A$

y-term

$1 = A + B$

$1 = (4) + B$

$B = -3$

$\displaystyle\int \frac{y+4}{y^2+y}\,dy = \int\left(\frac{(4)}{(y)^1} + \frac{(-3)}{(y+1)^1}\right)dy$

$= 4\left[\ln|y|\right] + (-3)\left[\ln|y+1|\right] + C$

$= 4\ln|y| - 3\ln|y+1| + C$

$\displaystyle\int_{\frac{1}{2}}^{1} \frac{y+4}{y^2+y}\,dy = \Big[4\ln|y| - 3\ln|y+1| + C\Big]_{\frac{1}{2}}^{1} = \Big[4\ln|(1)| - 3\ln|(1)+1| + C\Big] - \Big[4\ln|(\tfrac{1}{2})| - 3\ln|(\tfrac{1}{2})+1| + C\Big]$

$= \Big[4(0) - 3\ln(2)\Big] - \Big[4\ln(\tfrac{1}{2}) - 3\ln(\tfrac{3}{2})\Big] = \Big[-3\ln(2)\Big] - \Big[4\{\ln(1) - \ln(2)\} - 3\{\ln(3) - \ln(2)\}\Big]$

$= \Big[-3\ln 2\Big] - \Big[\{4(0) - 4\ln 2\} - \{3\ln 3 - 3\ln 2\}\Big] = \Big[-3\ln 2\Big] - \Big[-4\ln 2 - 3\ln 3 + 3\ln 2\Big]$

$= -3\ln 2 + 4\ln 2 + 3\ln 3 - 3\ln 2 = 3\ln 3 - 2\ln 2 = \ln(3^3) - \ln(2^2) = \ln 27 - \ln 4 = \ln\left(\tfrac{27}{4}\right)$

18) $$\frac{x^3}{x^2-2x+1} = x+2+\frac{(3x-2)}{x^2-2x+1}$$

$$
\begin{array}{r}
x+2 \phantom{x+0} \\
x^2-2x+1 \overline{\smash{\big)}\, x^3+0x^2+0x+0} \\
-(x^3-2x^2+x) \phantom{+0} \\
\hline
+2x^2-x+0 \\
-(2x^2-4x+2) \\
\hline
+3x-2
\end{array}
$$

$$\frac{(3x-2)}{x^2-2x+1} = \frac{3x-2}{(x-1)^2} = \frac{A}{(x-1)^1} + \frac{B}{(x-1)^2}$$

$$3x-2 = A(x-1)+B$$

constant term

$-2 = -A+B$

$-2 = -(3)+B$

$1 = B$

$x$ - term

$3 = A$

$$\int \frac{x^3}{x^2-2x+1}\,dx = \int \left(x+2+\frac{(3)}{(x-1)^1}+\frac{(1)}{(x-1)^2}\right)dx$$

$$= \left[\frac{x^2}{2}\right] + 2\left[x\right] + 3\left[\ln|x-1|\right] + \left[\frac{(x-1)^{-1}}{-1}\right] + C$$

$$= \frac{1}{2}x^2 + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$$

$$\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx = \left[\frac{1}{2}x^2 + 2x + 3\ln|x-1| - \frac{1}{x-1} + C\right]_{-1}^{0}$$

$$= \left[\frac{1}{2}(0)^2 + 2(0) + 3\ln|(0)-1| - \frac{1}{(0)-1} + C\right] - \left[\frac{1}{2}(-1)^2 + 2(-1) + 3\ln|(-1)-1| - \frac{1}{(-1)-1} + C\right]$$

$$= \left[0+0+0-\frac{1}{-1}\right] - \left[\frac{1}{2} - 2 + 3\ln|-2| - \frac{1}{(-2)}\right]$$

$$= [1] - \left[\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right] = [1] - [-1 + 3\ln 2 = 2 - 3\ln 2$$

20)
$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)^1(x+1)^2} = \frac{A}{(x-1)^1} + \frac{B}{(x+1)^1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B((x-1)(x+1)) + C(x-1)$$
$$x^2 = A(x^2+2x+1) + B(x^2-1) + C(x-1)$$

| constant term | x - term | $x^2$ - term |
|---|---|---|
| $0 = A - B - C$ | $0 = 2A + C$ | $1 = A + B$ |
| $0 = A - B + (2A)$ | $-C = 2A$ | $1 = A + (3A)$ |
| $0 = 3A - B$ | $C = -2A$ | $1 = 4A$ |
| $B = 3A$ | $C = -2\left(\frac{1}{4}\right) = \frac{-1}{2}$ | $A = \frac{1}{4}$ |
| $B = 3\left(\frac{1}{4}\right) = \frac{3}{4}$ | | |

$$\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)} = \int \left( \frac{\left(\frac{1}{4}\right)}{(x-1)^1} + \frac{\left(\frac{3}{4}\right)}{(x+1)^1} + \frac{\left(\frac{-1}{2}\right)}{(x+1)^2} \right) dx$$

$$= \frac{1}{4}\left[\ln|x-1|\right] + \frac{3}{4}\left[\ln|x+1|\right] + \left(\frac{-1}{2}\right)\left[\frac{(x+1)^{-1}}{-1}\right] + C$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$$

22) $\dfrac{3t^2+t+4}{t^3+t} = \dfrac{3t^2+t+4}{(t)^1(t^2+1)^1} = \dfrac{A}{(t)^1} + \dfrac{(Bt+C)}{(t^2+1)^1}$

$3t^2+t+4 = A(t^2+1) + (Bt+C)(t)$

$3t^2+t+4 = A(t^2+1) + B(t^2) + C(t)$

constant term $\quad$ $t$-term $\quad$ $t^2$-term

$4 = A$ $\quad$ $1 = C$ $\quad$ $3 = A+B$

$3 = (4)+B$

$B = -1$

$\displaystyle\int \frac{3t^2+t+4}{t^3+t}\,dt = \int\left(\frac{(4)}{(t)^1} + \frac{(-1)t+(1)}{(t^2+1)^1}\right)dt = \int\left(\frac{4}{t} - \frac{t}{t^2+1} + \frac{1}{t^2+1}\right)dt$

$= 4\left[\ln|t|\right] - \left[\frac{1}{2}\ln|t^2+1|\right] + \left[\frac{1}{1}\tan^{-1}\left(\frac{t}{1}\right)\right] + C$

$= 4\ln|t| - \frac{1}{2}\ln|t^2+1| + \tan^{-1}t + C$

$\displaystyle\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = \left[4\ln|t| - \frac{1}{2}\ln|t^2+1| + \tan^{-1}t + C\right]_1^{\sqrt{3}}$

$= \left[4\ln|(\sqrt{3})| - \frac{1}{2}\ln|(\sqrt{3})^2+1| + \tan^{-1}(\sqrt{3}) + C\right] - \left[4\ln|(1)| - \frac{1}{2}\ln|(1)^2+1| + \tan^{-1}(1) + C\right]$

$= \left[4\ln(3^{\frac{1}{2}}) - \frac{1}{2}\ln|4| + \left(\frac{\pi}{3}\right)\right] - \left[0 - \frac{1}{2}\ln|2| + \left(\frac{\pi}{4}\right)\right]$

$= \left[4\left(\frac{1}{2}\ln 3\right) - \frac{1}{2}\ln(2^2) + \frac{\pi}{3}\right] - \left[\frac{-1}{2}\ln(2) + \frac{\pi}{4}\right]$

$= \left[2\ln 3 - \ln 2 + \frac{\pi}{3}\right] - \left[\frac{-1}{2}\ln(2) + \frac{\pi}{4}\right] = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$

$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{4\pi}{12} - \frac{3\pi}{12} = 2\ln 3 - \frac{1}{2}\ln 2 - \frac{\pi}{12} = \ln(3^2) - \ln(\sqrt{2}) - \frac{\pi}{12}$

$= \ln\left(\frac{9}{\sqrt{2}}\right) - \frac{\pi}{12}$

24) $$\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{(Ax+B)}{(4x^2+1)^1} + \frac{(Cx+D)}{(4x^2+1)^2}$$

$$8x^2+8x+2 = (Ax+B)(4x^2+1) + (Cx+D)$$

$$8x^2+8x+2 = A(4x^3+x)+B(4x^2+1) + C(x) + D$$

| constant term | $x$-term | $x^2$-term | $x^3$-term |
| --- | --- | --- | --- |
| $2 = B + D$ | $8 = A + C$ | $8 = 4B$ | $0 = 4A$ |
| $2 = (2) + D$ | $8 = (0) + C$ | $B = 2$ | $A = 0$ |
| $D = 0$ | $C = 8$ | | |

$$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = \int \left(\frac{(0)x+(2)}{(4x^2+1)^1} + \frac{(8)x+(0)}{(4x^2+1)^2}\right)dx$$

$$= \int \left(\frac{2}{4x^2+1} + \frac{8x}{(4x^2+1)^2}\,dx\right.$$

$$= \int \frac{2}{(2x)^2+(1)^2}\,dx + \int \frac{8x}{(4x^2+1)^2}\,dx$$

$$= \left[\tan^{-1}\left(\frac{2x}{1}\right)\right] + \left[\frac{-1}{(4x^2+1)}\right] + C$$

$$= \underline{\underline{\tan^{-1}(2x) - \frac{1}{4x^2+1} + C}}$$

$$\int \frac{8x}{(4x^2+1)^2}\,dx = \int \frac{1}{(4x^2+1)^2}(8x\,dx)$$

$p = 4x^2+1$

$dp = 8x\,dx$

$$= \int \frac{1}{p^2}\,dp$$

$$= \int p^{-2}\,dp$$

$$= \left[\frac{p^{-1}}{-1}\right] + C$$

$$= \frac{-1}{p} + C$$

$$= \frac{-1}{(4x^2+1)} + C$$

$$\int \frac{2}{(2x)^2+(1)^2}\,dx = \int \frac{1}{(2x)^2+(1)^2}(2\,dx) = \int \frac{1}{q^2+(1)^2}\,dq =$$

$q = 2x$

$dq = 2\,dx$

$$= \frac{1}{1}\tan^{-1}\left(\frac{q}{1}\right) + C = \tan^{-1}\left(\frac{2x}{1}\right) + C$$

26) $\dfrac{s^4+81}{s(s^2+9)^2} = \dfrac{s^4+81}{(s)^1(s^2+9)^2} = \dfrac{A}{(s)^1} + \dfrac{(Bs+C)}{(s^2+9)^1} + \dfrac{(Ds+E)}{(s^2+9)^2}$

$s^4+81 = A(s^2+9)^2 + (Bs+C)(s(s^2+9)) + (Ds+E)(s)$

$s^4+81 = A(s^4+18s^2+81) + B(s^4+9s^2) + C(s^3+9s) + D(s^2) + E(s)$

| constant term | s-term | $s^2$-term | $s^3$-term | $s^4$-term |
|---|---|---|---|---|
| $81 = 81A$ | $0 = 9C + E$ | $0 = 18A + 9B + D$ | $0 = C$ | $1 = A + B$ |
| $A = 1$ | $0 = 9(0) + E$ | $0 = 18(1) + 9(0) + D$ | | $1 = (1) + B$ |
| | $E = 0$ | $D = -18$ | | $B = 0$ |

$\displaystyle\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int\left(\frac{(1)}{(s)^1} + \frac{(0)s+(0)}{(s^2+9)^1} + \frac{(-18)s+(0)}{(s^2+9)^2}\right)ds$

$\displaystyle = \int\left(\frac{1}{s} - \frac{18s}{(s^2+9)^2}\right)ds$

$\displaystyle = \left[\ln|s|\right] - \left[\frac{-9}{s^2+9}\right] + C$

$\displaystyle = \ln|s| + \frac{9}{s^2+9} + C$

$\displaystyle\int \frac{18s}{(s^2+9)^2}\,ds = \int \frac{9}{(s^2+9)^2}(2s\,ds)$

$p = s^2+9$

$dp = 2s\,ds$

$\displaystyle = \int \frac{9}{p^2}\,dp$

$\displaystyle = \int 9p^{-2}\,dp$

$\displaystyle = 9\left[\frac{p^{-1}}{-1}\right] + C$

$\displaystyle = \frac{-9}{p} + C$

$\displaystyle = \frac{-9}{s^2+9} + C$

28) $\frac{1}{x^4+x} = \frac{1}{x(x^3+1)} = \frac{1}{(x)^1(x+1)^1(x^2-x+1)^1} = \frac{A}{(x)^1} + \frac{B}{(x+1)^1} + \frac{(Cx+D)}{(x^2-x+1)^1}$

$\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$

$1 = A((x+1)(x^2-x+1)) + B(x(x^2-x+1)) + (Cx+D)(x(x+1))$

$1 = A(x^3+1) + B(x^3-x^2+x) + C(x^3+x^2) + D(x^2+x)$

constant term | $x$-term | $x^2$-term | $x^3$-term

$1 = A$

$0 = B+D$

$D = -B$

$D = -(\frac{-1}{3})$

$D = \frac{1}{3}$

$0 = -B+C+D$

$0 = -B+(-B-1)+D$

$1 = -2B+D$

$1 = -2B+(-B)$

$1 = -3B$

$B = \frac{-1}{3}$

$0 = A+B+C$

$0 = (1)+B+C$

$C = -B-1$

$C = -(\frac{-1}{3})-1 = \frac{1}{3}-1 = \frac{-2}{3}$

$$\int \frac{1}{x^4+x}\,dx = \int\left(\frac{(1)}{(x)^1} + \frac{(\frac{-1}{3})}{(x+1)^1} + \frac{(\frac{-2}{3})x+(\frac{1}{3})}{(x^2-x+1)^1}\right)dx$$

$$= \int\left(\frac{1}{x} - \frac{\frac{1}{3}}{x+1} - \frac{1}{3}\left(\frac{2x-1}{x^2-x+1}\right)\right)dx$$

$$= \ln|x| - \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x^2-x+1| + C$$

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$$\int \frac{2x-1}{x^2-x+1}\,dx = \int \frac{1}{x^2-x+1}(2x-1\,dx) = \int \frac{1}{p}\,dp$$

$p = x^2-x+1$

$dp = 2x-1\,dx$

$$= \ln|p| + C$$

$$= \ln|x^2-x+1| + C$$

30) $\dfrac{x^2+x}{x^4-3x^2-4} = \dfrac{x^2+x}{(x^2+1)(x^2-4)} = \dfrac{x^2+x}{(x+2)^1(x-2)^1(x^2+1)^1} = \dfrac{A}{(x+2)^1} + \dfrac{B}{(x-2)^1} + \dfrac{(Cx+D)}{(x^2+1)^1}$

$$x^2+x = A\big((x-2)(x^2+1)\big) + B\big((x+2)(x^2+1)\big) + (Cx+D)\big((x+2)(x-2)\big)$$

$$x^2+x = A(x^3-2x^2+x-2) + B(x^3+2x^2+x+2) + C(x^3-4x) + D(x^2-4)$$

constant term

$0 = -2A + 2B - 4D$

$\underline{4 = -8A + 8B + 4D}$

$4 = -10A + 10B$

$\underline{2 = 10A + 10B}$

$6 = 20B$

$B = \frac{6}{20} = \frac{3}{10}$

$x$-term

$1 = A + B - 4C$

$1 = (-C) - 4C$

$1 = -5C$

$C = \frac{-1}{5}$

$x^2$-term

$1 = -2A + 2B + D$

$4 = -8A + 8B + 4D$

$1 = -2\left(\frac{-1}{10}\right) + 2\left(\frac{3}{10}\right) + D$

$1 = \frac{1}{5} + \frac{3}{5} + D$

$1 = \frac{4}{5} + D$

$D = 1 - \frac{4}{5} = \frac{1}{5}$

$x^3$-term

$0 = A + B + C$

$-C = A + B$

$-\left(\frac{-1}{5}\right) = A + B$

$\frac{1}{5} = A + B$

$2 = 10A + 10B$

$2 = 10A + 10\left(\frac{3}{10}\right)$

$2 = 10A + 3$

$-1 = 10A$ $\quad A = \frac{-1}{10}$

$$\int \frac{x^2+x}{x^4-3x^2-4}\,dx = \int \left( \frac{\left(\frac{-1}{10}\right)}{(x+2)^1} + \frac{\left(\frac{3}{10}\right)}{(x-2)^1} + \frac{\left(\frac{-1}{5}\right)x + \left(\frac{1}{5}\right)}{(x^2+1)^1} \right) dx$$

$p = x^2+1$

$dp = 2x\,dx$

$\frac{1}{2}\,dp = x\,dx$

$$= \int \left( \frac{\frac{3}{10}}{x-2} - \frac{\frac{1}{10}}{x+2} - \frac{1}{5}\left(\frac{x}{x^2+1}\right) + \frac{1}{5}\left(\frac{1}{x^2+(1)^2}\right) \right) dx$$

$$= \frac{3}{10}\Big[\ln|x-2|\Big] - \frac{1}{10}\Big[\ln|x+2|\Big] - \frac{1}{5}\Big[\frac{1}{2}\ln|x^2+1|\Big] + \frac{1}{5}\Big[\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\Big] + C$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln(x^2+1) + \frac{1}{5}\tan^{-1}x + C$$

32) $$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{(A\theta+B)}{(\theta^2+1)^1} + \frac{(C\theta+D)}{(\theta^2+1)^2} + \frac{(E\theta+F)}{(\theta^2+1)^3}$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1 = (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + (E\theta+F)$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1 = A(\theta^5+2\theta^3+\theta)+B(\theta^4+2\theta^2+1)+C(\theta^3+\theta)+D(\theta^2+1)+E\theta+F$$

| constant term | $\theta$-term | $\theta^2$-term | $\theta^3$-term | $\theta^4$-term | $\theta^5$-term |
|---|---|---|---|---|---|
| $1=B+F$ | $-3=A+C+E$ | $2=2B+D$ | $-4=2A+C$ | $1=B$ | $0=A$ |
| $1=(1)+F$ | $-3=(0)+(-4)+E$ | $2=2(1)+D$ | $-4=2(0)+C$ | | |
| $F=0$ | $E=1$ | $D=0$ | $C=-4$ | | |

$$\int\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}d\theta = \int\left(\frac{(0)\theta+(1)}{(\theta^2+1)^1} + \frac{(-4)\theta+(0)}{(\theta^2+1)^2} + \frac{(1)\theta+(0)}{(\theta^2+1)^3}\right)d\theta$$

$$= \int\frac{1}{\theta^2+(1)^2}d\theta - 4\int\frac{1}{(\theta^2+1)^2}(\theta d\theta) + \int\frac{1}{(\theta^2+1)^3}(\theta d\theta)$$

$p=\theta^2+1$

$dp=2\theta d\theta$

$\frac{1}{2}dp=\theta d\theta$

$$= \int\frac{1}{\theta^2+(1)^2}d\theta - 4\int\frac{1}{p^2}\left(\frac{1}{2}dp\right) + \int\frac{1}{p^3}\left(\frac{1}{2}dp\right)$$

$$= \left[\frac{1}{1}\tan^{-1}\left(\frac{\theta}{1}\right)\right] - 2\left[\frac{p^{-1}}{-1}\right] + \frac{1}{2}\left[\frac{p^{-2}}{-2}\right] + C$$

$$= \tan^{-1}\theta + \frac{2}{\theta^2+1} - \frac{1}{4(\theta^2+1)^2} + C$$

34) $\dfrac{x^4}{x^2-1} = x^2+1+\dfrac{(+1)}{x^2-1} = x^2+1+\left(\dfrac{(-\frac{1}{2})}{(x+1)^1}+\dfrac{(\frac{1}{2})}{(x-1)^1}\right)$

$$\begin{array}{r} x^2+1 \\ x^2+0x-1\,\overline{)\,x^4+0x^3+0x^2+0x+0} \\ \underline{-(x^4+0x^3-x^2)} \\ +x^2+0x+0 \\ \underline{-(x^2+0x-1)} \\ +1 \end{array}$$

$\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)^1(x-1)^1} = \dfrac{A}{(x+1)^1}+\dfrac{B}{(x-1)^1}$

$1 = A(x-1)+B(x+1)$

| constant term | $x$-term |
|---|---|
| $1=-A+B$ | $0=A+B$ |
| $1=(B)+B$ | $-A=B$ |
| $1=2B$ | $A=-B$ |
| $B=\frac{1}{2}$ | $A=-(\frac{1}{2})=\frac{-1}{2}$ |

$$\int\frac{x^4}{x^2-1}\,dx = \int\left(x^2+1-\frac{1}{2(x+1)}+\frac{1}{2(x-1)}\right)dx$$

$$=\left[\frac{x^3}{3}\right]+\left[x\right]-\frac{1}{2}\left[\ln|x+1|\right]+\frac{1}{2}\left[\ln|x-1|\right]+C$$

$$=\frac{1}{3}x^3+x-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C$$

$$=\frac{1}{3}x^3+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$$

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36) $\dfrac{16x^3}{4x^2-4x+1} = 4x+4+\dfrac{(12x-4)}{4x^2-4x+1} = 4x+4+\left(\dfrac{(6)}{(2x-1)^1}+\dfrac{(2)}{(2x-1)^2}\right)$

$$\begin{array}{r} 4x+4 \\ 4x^2-4x+1\,\overline{)\,16x^3+0x^2+0x+0} \\ \underline{-(16x^3-16x^2+4x)} \\ +16x^2-4x+0 \\ \underline{-(16x^2-16x+4)} \\ +12x-4 \end{array}$$

$\dfrac{12x-4}{4x^2-4x+1} = \dfrac{12x-4}{(2x-1)^2} = \dfrac{A}{(2x-1)^1}+\dfrac{B}{(2x-1)^2}$

$12x-4 = A(2x-1)+B$

| constant term | $x$-term |
|---|---|
| $-4=-A+B$ | $12=2A$ |
| $-4=-(6)+B$ | $A=6$ |
| $B=2$ | |

36) continued

$$\int \frac{16x^3}{4x^2-4x+1}\,dx = \int \left(4x+4+\frac{6}{2x-1}+\frac{2}{(2x-1)^2}\right)dx$$

$p = 2x-1$

$dp = 2\,dx$

$$= \int 4x\,dx + \int 4\,dx + 3\int \frac{2}{2x-1}\,dx + \int \frac{2}{(2x+1)^2}\,dx$$

$$= \int 4x\,dx + \int 4\,dx + 3\int \frac{1}{p}\,dp + \int \frac{1}{p^2}\,dp$$

$$= 4\left[\frac{x^2}{2}\right] + 4[x] + 3[\ln|p|] + \left[\frac{p^{-1}}{-1}\right] + C$$

$$= 2x^2 + 4x + 3\ln|2x-1| - \frac{1}{2x-1} + C$$

---

40) $$\int \frac{e^{4t}+2e^{2t}-e^{t}}{e^{2t}+1}\,dt = \int \frac{(e^t)^4+2(e^t)^2-(e^t)}{(e^t)^2+1}\,dt$$

$p = e^t$

$dp = e^t\,dt$

$$= \int \frac{(e^t)^3+2(e^t)-1}{(e^t)^2+1}\,(e^t\,dt)$$

$$= \int \frac{p^3+2p-1}{p^2+1}\,dp$$

$$\begin{array}{r|l} & p \\ p^2+0p+1 & p^3+0p^2+2p-1 \\ & -(p^3+0p^2+p) \\ \hline & +p-1 \end{array}$$

$$= \int \left(p + \frac{(p-1)}{p^2+1}\right)dp$$

$$= \int \left(p + \frac{p}{p^2+1} - \frac{1}{p^2+(1)^2}\right)dp$$

$$= \left[\frac{p^2}{2}\right] + \left[\frac{1}{2}\ln|p^2+1|\right] - \left[\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right] + C$$

$$= \frac{1}{2}(e^t)^2 + \frac{1}{2}\ln|(e^t)^2+1| - \tan^{-1}(e^t) + C$$

$$= \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$$

42) $\displaystyle\int \frac{\sin\theta \, d\theta}{\cos^2\theta + \cos\theta - 2} = \int \frac{1}{(\cos\theta)^2 + (\cos\theta) - 2} (\sin\theta \, d\theta)$

$p = \cos\theta$

$dp = -\sin\theta \, d\theta$

$-1 dp = \sin\theta \, d\theta$

$$= \int \frac{1}{p^2 + p - 2}(-1dp) = \int \frac{-1}{(p+2)^1(p-1)^1} dp$$

$$= \int \left( \frac{\left(\frac{1}{3}\right)}{(p+2)^1} + \frac{\left(\frac{-1}{3}\right)}{(p-1)} \right) dp$$

$$\frac{-1}{(p+2)^1(p-1)^1} = \frac{A}{(p+2)^1} + \frac{B}{(p-1)^1}$$

$-1 = A(p-1) + B(p+2)$

constant term $\quad$ $p$-term

$-1 = -A + 2B \qquad 0 = A + B$

$-1 = (B) + 2B \qquad B = -A$

$-1 = 3B \qquad A = -B$

$B = \frac{-1}{3} \qquad A = -\left(\frac{-1}{3}\right) = \frac{1}{3}$

$$= \frac{1}{3}\left[\ln|p+2|\right] - \frac{1}{3}\left[\ln|p-1|\right] + C$$

$$= \frac{1}{3}\ln\left|\frac{p+2}{p-1}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C$$

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56) $\displaystyle\int \frac{x+2}{x^3 - 2x^2 - 3x} dx = \int \frac{x+2}{(x)^1(x+1)^1(x-3)^1} dx$

$$\frac{x+2}{(x)^1(x+1)^1(x-3)^1} = \frac{A}{(x)^1} + \frac{B}{(x+1)^1} + \frac{C}{(x-3)^1}$$

$x + 2 = A\big((x+1)(x-3)\big) + B\big(x(x-3)\big) + C\big(x(x+1)\big)$

$x + 2 = A(x^2 - 2x - 3) + B(x^2 - 3x) + C(x^2 + x)$

constant term $\quad$ $x$-term $\quad$ $x^2$-term

$2 = -3A \qquad 1 = -2A - 3B + C \qquad 0 = A + B + C$

$A = \frac{-2}{3} \qquad 1 = -2A - 3B + (-A - B) \qquad C = -A - B$

$1 = -3A - 4B \qquad C = -\left(\frac{-2}{3}\right) - \left(\frac{1}{4}\right)$

$1 = (2) - 4B \qquad = \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12}$

$-1 = -4B \qquad B = \frac{1}{4} \qquad C = \frac{5}{12}$

$$= \int \left( \frac{\left(\frac{-2}{3}\right)}{(x)^1} + \frac{\left(\frac{1}{4}\right)}{(x+1)^1} + \frac{\left(\frac{5}{12}\right)}{(x-3)^1} \right) dx$$

$$= \frac{-2}{3}\left[\ln|x|\right] + \frac{1}{4}\left[\ln|x+1|\right] + \frac{5}{12}\left[\ln|x-3|\right] + C$$

$$= \frac{-2}{3}\ln|x| + \frac{1}{4}\ln|x+1| + \frac{5}{12}\ln|x-3| + C$$

60) $\int \frac{x^4-1}{x^5-5x+1}\,dx = \int \frac{1}{x^5-5x+1}\left((x^4-1)dx\right)$

$p = x^5-5x+1$

$dp = 5x^4-5\,dx$

$dp = 5(x^4-1)dx$

$\frac{1}{5}dp = (x^4-1)dx$

$= \int \frac{1}{p}\left(\frac{1}{5}dp\right)$

$= \frac{1}{5}\ln|p| + C$

$= \frac{1}{5}\ln|x^5-5x+1| + C$

---

64) $\int \frac{x}{x+\sqrt{x^2+2}}\,dx = \int \frac{x}{\sqrt{x^2+2}+x}\,dx$

$= \int \left(\frac{x}{\sqrt{x^2+2}+x}\right)\left(\frac{\sqrt{x^2+2}-x}{\sqrt{x^2+2}-x}\right)dx = \int \frac{x\sqrt{x^2+2}-x^2}{(x^2+2)-x^2}\,dx$

$= \int \frac{x\sqrt{x^2+2}-x^2}{2}\,dx = \frac{1}{2}\int x\sqrt{x^2+2}\,dx - \frac{1}{2}\int x^2 dx$

$= \frac{1}{2}\left[\frac{1}{3}\left(\sqrt{x^2+2}\right)^3\right] - \frac{1}{2}\left[\frac{x^3}{3}\right] + C$

$= \frac{1}{6}\left(\sqrt{x^2+2}\right)^3 - \frac{1}{6}x^3 + C$

$\int x\sqrt{x^2+2}\,dx = \int \sqrt{x^2+2}\,(x\,dx)$

$p = x^2+2$

$dp = 2x\,dx$

$\frac{1}{2}dp = x\,dx$

$= \int \sqrt{p}\left(\frac{1}{2}dp\right)$

$= \frac{1}{2}\int p^{\frac{1}{2}}dp$

$= \frac{1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{1}{3}\left(\sqrt{p}\right)^3 + C$

$= \frac{1}{3}\left(\sqrt{x^2+2}\right)^3 + C$