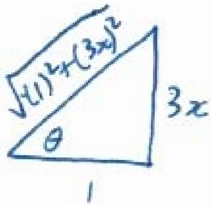
MATH 21200 section 8.4

2) $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}} = \int \frac{3}{\sqrt{(1)^2+(3x)^2}}\,dx = \int \frac{3}{(\sec\theta)}\left(\tfrac{1}{3}\sec^2\theta\,d\theta\right)$



$\frac{3x}{1} = \tan\theta$ $\quad \frac{\sqrt{(1)^2+(3x)^2}}{1} = \sec\theta$

$3x = \tan\theta$ $\quad \sqrt{(1)^2+(3x)^2} = \sec\theta$

$x = \frac{1}{3}\tan\theta$

$dx = \frac{1}{3}\sec^2\theta\,d\theta$

$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\left(\frac{\sqrt{(1)^2+(3x)^2}}{1}\right) + \left(\frac{3x}{1}\right)\right| + C$

$= \ln\left|3x + \sqrt{1+9x^2}\right| + C$

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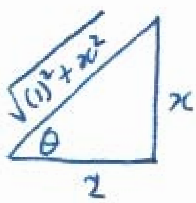
4) $\displaystyle\int \frac{dx}{8+2x^2} = \int \frac{1}{2(4+x^2)}\,dx = \int \frac{1}{2\left(\sqrt{(2)^2+x^2}\right)^2}\,dx = \int \frac{1}{2(2\sec\theta)^2}\left(2\sec^2\theta\,d\theta\right)$



$= \int \frac{1}{4}\,d\theta = \frac{1}{4}\theta + C = \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C$

$\frac{x}{2} = \tan\theta$ $\quad \frac{\sqrt{(2)^2+x^2}}{2} = \sec\theta$

$x = 2\tan\theta$ $\quad \sqrt{(2)^2+x^2} = 2\sec\theta$

$dx = 2\sec^2\theta\,d\theta$

$\frac{x}{2} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$\displaystyle\int_0^2 \frac{dx}{8+2x^2} = \left[\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C\right]_0^2$

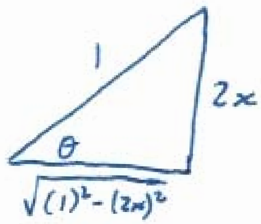
$= \left[\frac{1}{4}\tan^{-1}\left(\frac{(2)}{2}\right) + C\right] - \left[\frac{1}{4}\tan^{-1}\left(\frac{(0)}{2}\right) + C\right]$

$= \left[\frac{1}{4}\tan^{-1}(1)\right] - \left[\frac{1}{4}\tan^{-1}(0)\right]$

$= \left[\frac{1}{4}\left(\frac{\pi}{4}\right)\right] - \left[\frac{1}{4}(0)\right]$

$= \frac{\pi}{16}$

6) $\int \frac{2\,dx}{\sqrt{1-4x^2}} = \int \frac{2}{\sqrt{(1)^2-(2x)^2}}\,dx = \int \frac{2}{(\cos\theta)}\left(\frac{1}{2}\cos\theta\,d\theta\right) = \int 1\,d\theta$



$= \theta + C = \sin^{-1}\left(\frac{2x}{1}\right) + C = \sin^{-1}(2x) + C$

$\int_0^{\frac{1}{2\sqrt{2}}} \frac{2\,dx}{\sqrt{1-4x^2}} = \left[\sin^{-1}(2x) + C\right]_0^{\frac{1}{2\sqrt{2}}}$

$= \left[\sin^{-1}\left(2\left(\frac{1}{2\sqrt{2}}\right)\right) + C\right] - \left[\sin^{-1}(2(0)) + C\right]$

$= \left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] - \left[\sin^{-1}(0)\right]$

$= \left[\left(\frac{\pi}{4}\right)\right] - [0] = \frac{\pi}{4}$

$\frac{2x}{1} = \sin\theta$ $\qquad \frac{\sqrt{(1)^2-(2x)^2}}{1} = \cos\theta$

$2x = \sin\theta$ $\qquad \sqrt{(1)^2-(2x)^2} = \cos\theta$

$x = \frac{1}{2}\sin\theta$

$dx = \frac{1}{2}\cos\theta\,d\theta$

$\frac{2x}{1} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{2x}{1}\right)$

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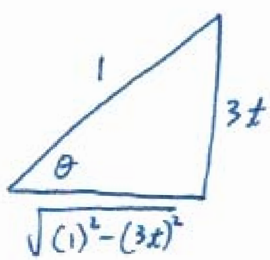
8) $\int \sqrt{1-9t^2}\,dt = \int \sqrt{(1)^2-(3t)^2}\,dt = \int (\cos\theta)\left(\frac{1}{3}\cos\theta\,d\theta\right)$



$= \int \frac{1}{3}\cos^2\theta\,d\theta = \int \frac{1}{3}\left\{\frac{1}{2}\left(1+\cos(2\theta)\right)\right\}d\theta$

$= \int \left(\frac{1}{6} + \frac{1}{6}\cos(2\theta)\right)d\theta$

$= \frac{1}{6}[\theta] + \frac{1}{6}\left[\frac{1}{2}\sin(2\theta)\right] + C$

$= \frac{1}{6}\theta + \frac{1}{6}\left\{\sin\theta\cos\theta\right\} + C$

$= \frac{1}{6}\left(\sin^{-1}\left(\frac{3t}{1}\right)\right) + \frac{1}{6}\left\{\left(\frac{3t}{1}\right)\left(\frac{\sqrt{(1)^2-(3t)^2}}{1}\right)\right\} + C$

$= \frac{1}{6}\sin^{-1}(3t) + \frac{1}{2}t\sqrt{1-9t^2} + C$

$\frac{3t}{1} = \sin\theta$ $\qquad \frac{\sqrt{(1)^2-(3t)^2}}{1} = \cos\theta$

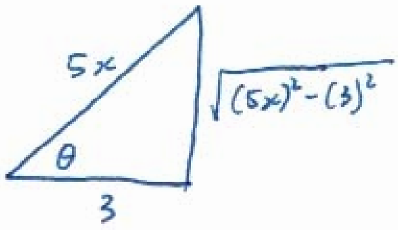
$3t = \sin\theta$ $\qquad \sqrt{(1)^2-(3t)^2} = \cos\theta$

$t = \frac{1}{3}\sin\theta$

$dt = \frac{1}{3}\cos\theta\,d\theta$

$\frac{3t}{1} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{3t}{1}\right)$

10) $x > \frac{3}{5}$, $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5}{\sqrt{(5x)^2-(3)^2}}\,dx = \int \frac{5}{(3\tan\theta)}\left(\tfrac{3}{5}\sec\theta\tan\theta\,d\theta\right)$



$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\left(\frac{5x}{3}\right) + \left(\frac{\sqrt{(5x)^2-(3)^2}}{3}\right)\right| + C$

$= \ln\left|\frac{5x+\sqrt{25x^2-9}}{3}\right| + C$

$\frac{5x}{3} = \sec\theta \quad \frac{\sqrt{(5x)^2-(3)^2}}{3} = \tan\theta$

$x = \frac{3}{5}\sec\theta \quad \sqrt{(5x)^2-(3)^2} = 3\tan\theta$

$dx = \frac{3}{5}\sec\theta\tan\theta\,d\theta$

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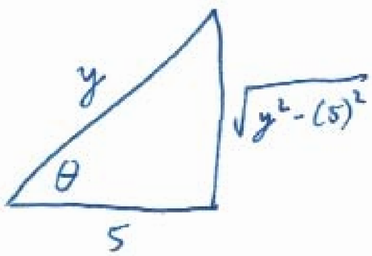
12) $y > 5$, $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{\sqrt{y^2-(5)^2}}{y^3}\,dy = \int \frac{(5\tan\theta)}{(5\sec\theta)^3}\left(5\sec\theta\tan\theta\,d\theta\right)$



$= \int \frac{\tan^2\theta}{5\sec^2\theta}\,d\theta = \int \frac{1}{5}(\tan^2\theta)(\cos^2\theta)\,d\theta$

$= \int \frac{1}{5}\left(\frac{\sin^2\theta}{\cos^2\theta}\right)(\cos^2\theta)\,d\theta = \int \frac{1}{5}\sin^2\theta\,d\theta$

$= \int \frac{1}{5}\left\{\frac{1}{2}(1-\cos(2\theta))\right\}d\theta$

$= \int \left(\frac{1}{10} - \frac{1}{10}\cos(2\theta)\right)d\theta$

$= \frac{1}{10}[\theta] - \frac{1}{10}\left[\frac{1}{2}\sin(2\theta)\right] + C$

$= \frac{1}{10}[\theta] - \frac{1}{10}[\sin\theta\cos\theta] + C$

$= \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right)\right] - \frac{1}{10}\left[\left(\frac{\sqrt{y^2-(5)^2}}{y}\right)\left(\frac{5}{y}\right)\right] + C$

$= \frac{1}{10}\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{2y^2} + C$

$\frac{y}{5} = \sec\theta \quad \frac{\sqrt{y^2-(5)^2}}{5} = \tan\theta$

$y = 5\sec\theta \quad \sqrt{y^2-(5)^2} = 5\tan\theta$

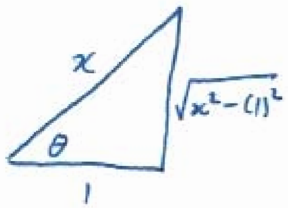
$dy = 5\sec\theta\tan\theta\,d\theta$

$\frac{y}{5} = \sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{y}{5}\right)$

from triangle above

$\sin\theta = \frac{\sqrt{y^2-(5)^2}}{y}$, $\cos\theta = \frac{5}{y}$

14) $x>1$, $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2}{x^3\sqrt{x^2-(1)^2}}\,dx = \int \frac{2}{(\sec\theta)^3(\tan\theta)}(\sec\theta\tan\theta\,d\theta)$



$\frac{x}{1} = \sec\theta \quad \frac{\sqrt{x^2-(1)^2}}{1} = \tan\theta$

$x = \sec\theta \quad \sqrt{x^2-(1)^2} = \tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\frac{x}{1} = \sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{1}\right)$

$$= \int \frac{2}{\sec^2\theta}\,d\theta = \int 2\cos^2\theta\,d\theta$$

$$= \int 2\left\{\tfrac{1}{2}(1+\cos(2\theta))\right\}d\theta$$

$$= \int (1+\cos(2\theta))\,d\theta = [\theta] + \left[\tfrac{1}{2}\sin(2\theta)\right] + C$$

$$= \theta + [\sin\theta\cos\theta] + C$$

$$= \left(\sec^{-1}\left(\frac{x}{1}\right)\right) + \left(\frac{\sqrt{x^2-(1)^2}}{x}\right)\left(\frac{1}{x}\right) + C$$

$$= \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$$

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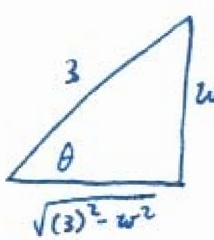
20) $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{\sqrt{(3)^2-w^2}}{w^2}\,dw = \int \frac{(3\cos\theta)}{(3\sin\theta)^2}(3\cos\theta\,d\theta)$



$\frac{w}{3} = \sin\theta \quad \frac{\sqrt{(3)^2-w^2}}{3} = \cos\theta$

$w = 3\sin\theta \quad \sqrt{(3)^2-w^2} = 3\cos\theta$

$dw = 3\cos\theta\,d\theta$

$\frac{w}{3} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{w}{3}\right)$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

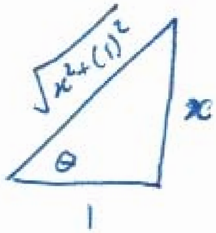
$$= \int \csc^2\theta\,d\theta - \int 1\,d\theta$$

$$= [-\cot\theta] - [\theta] + C$$

$$= -\left(\frac{\sqrt{(3)^2-w^2}}{w}\right) - \left(\sin^{-1}\left(\frac{w}{3}\right)\right) + C$$

$$= \frac{-\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$$

18) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{1}{x^2\sqrt{x^2+(1)^2}}\,dx = \int \frac{1}{(\tan\theta)^2(\sec\theta)}(\sec^2\theta\, d\theta)$



$\dfrac{x}{1} = \tan\theta \quad \dfrac{\sqrt{x^2+(1)^2}}{1} = \sec\theta$

$x = \tan\theta \quad \sqrt{x^2+(1)^2} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \int \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^2}\,d\theta$$

$p = \sin\theta$
$dp = \cos\theta\, d\theta$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta}(\cos\theta\, d\theta)$$

$$= \int \frac{1}{p^2}(dp) = \int p^{-2}\,dp = \left[\frac{p^{-1}}{-1}\right] + C = \frac{-1}{p} + C$$

$$= \frac{-1}{\sin\theta} + C = -\csc\theta + C = -\left(\frac{\sqrt{x^2+(1)^2}}{x}\right) + C$$

$$= \frac{-\sqrt{x^2+1}}{x} + C$$

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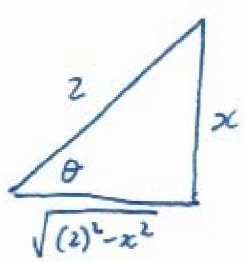
24) $\displaystyle\int \frac{dx}{(4-x^2)^{3/2}} = \int \frac{1}{\left(\sqrt{(2)^2-x^2}\right)^3}\,dx = \int \frac{1}{(2\cos\theta)^3}(2\cos\theta\, d\theta)$



$$= \int \frac{1}{4\cos^2\theta}\,d\theta = \int \frac{1}{4}\sec^2\theta\, d\theta = \frac{1}{4}\left[\tan\theta\right] + C$$

$$= \frac{1}{4}\left(\frac{x}{\sqrt{(2)^2-x^2}}\right) + C = \frac{x}{4\sqrt{4-x^2}} + C$$

$\dfrac{x}{2} = \sin\theta \quad \dfrac{\sqrt{(2)^2-x^2}}{2} = \cos\theta$

$x = 2\sin\theta \quad \sqrt{(2)^2-x^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

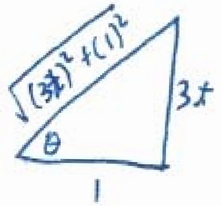
$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \left[\frac{x}{4\sqrt{4-x^2}} + C\right]_0^1$$

$$= \left[\frac{(1)}{4\sqrt{4-(1)^2}} + C\right] - \left[\frac{(0)}{4\sqrt{4-(0)^2}} + C\right]$$

$$= \left[\frac{1}{4\sqrt{3}}\right] - \left[0\right]$$

$$= \frac{1}{4\sqrt{3}}$$

30) $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2} = \int \frac{6}{\left(\sqrt{(3t)^2+(1)^2}\right)^4}\,dt = \int \frac{6}{(\sec\theta)^4}\left(\frac{1}{3}\sec^2\theta\,d\theta\right)$



$\displaystyle = \int \frac{2}{\sec^2\theta}\,d\theta = \int 2\cos^2\theta\,d\theta$

$\displaystyle = \int 2\left\{\frac{1}{2}(1+\cos(2\theta))\right\}d\theta = \int (1+\cos(2\theta))\,d\theta$

$\displaystyle \frac{3t}{1} = \tan\theta \quad \frac{\sqrt{(3t)^2+(1)^2}}{1} = \sec\theta$

$\displaystyle = \int 1\,d\theta + \int \cos(2\theta)\,d\theta = [\theta] + \left[\frac{1}{2}\sin(2\theta)\right] + C$

$\displaystyle t = \frac{1}{3}\tan\theta \quad \sqrt{(3t)^2+(1)^2} = \sec\theta$

$= \theta + \sin\theta\cos\theta + C$

$\displaystyle dt = \frac{1}{3}\sec^2\theta\,d\theta$

$\displaystyle = \left(\tan^{-1}\left(\frac{3t}{1}\right)\right) + \left(\frac{3t}{\sqrt{(3t)^2+(1)^2}}\right)\left(\frac{1}{\sqrt{(3t)^2+(1)^2}}\right) + C$

$\displaystyle \frac{3t}{1} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{3t}{1}\right)$

$\displaystyle = \tan^{-1}(3t) + \frac{3t}{9t^2+1} + C$

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32) $\displaystyle\int \frac{x\,dx}{25+4x^2} = \int \frac{1}{25+4x^2}(x\,dx) = \int \frac{1}{p}\left(\frac{1}{8}\,dp\right)$

$p = 25+4x^2$

$\displaystyle = \frac{1}{8}\left[\ln|p|\right] + C = \frac{1}{8}\ln|25+4x^2| + C$

$dp = 8x\,dx$

$\displaystyle \frac{1}{8}dp = x\,dx$

$\displaystyle = \frac{1}{8}\ln(25+4x^2) + C$

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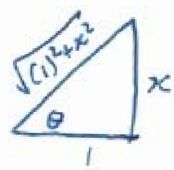
40) $\displaystyle\int \frac{dx}{1+x^2} = \int \frac{1}{\left(\sqrt{(1)^2+x^2}\right)^2}\,dx = \int \frac{1}{(\sec\theta)^2}(\sec^2\theta\,d\theta) = \int 1\,d\theta$



$\displaystyle = [\theta] + C = \left(\tan^{-1}\left(\frac{x}{1}\right)\right) + C$

$= \tan^{-1}x + C$

$\displaystyle \frac{x}{1} = \tan\theta \quad \frac{\sqrt{(1)^2+x^2}}{1} = \sec\theta$

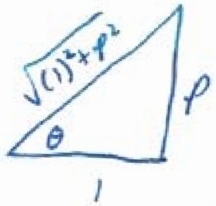
$x = \tan\theta \quad \sqrt{(1)^2+x^2} = \sec\theta$

$dx = \sec^2\theta\,d\theta$

$\displaystyle \frac{x}{1} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{1}\right)$

36) $\displaystyle\int \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \int \frac{1}{(1+(e^t)^2)^{3/2}}(e^t\,dt) = \int \frac{1}{(1+p^2)^{\frac{3}{2}}}\,dp = \int \frac{1}{(\sqrt{(1)^2+p^2})^3}\,dp$

$p = e^t \qquad dp = e^t\,dt$



$\frac{p}{1} = \tan\theta \qquad \frac{\sqrt{(1)^2+p^2}}{1} = \sec\theta$

$p = \tan\theta \qquad \sqrt{(1)^2+p^2} = \sec\theta$

$dp = \sec^2\theta\,d\theta$

$$= \int \frac{1}{(\sec\theta)^3}(\sec^2\theta\,d\theta) = \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \cos\theta\,d\theta = [\sin\theta] + C$$

$$= \left(\frac{p}{\sqrt{(1)^2+p^2}}\right) + C$$

$$= \frac{(e^t)}{\sqrt{(1)^2+(e^t)^2}} + C = \frac{e^t}{\sqrt{1+e^{2t}}} + C$$

$$\int_{\ln(\frac{3}{4})}^{\ln(\frac{4}{3})} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \left[\frac{e^t}{\sqrt{1+e^{2t}}} + C\right]_{\ln(\frac{3}{4})}^{\ln(\frac{4}{3})}$$

$$= \left[\frac{e^{(\ln(\frac{4}{3}))}}{\sqrt{1+e^{2(\ln(\frac{4}{3}))}}} + C\right] - \left[\frac{e^{(\ln(\frac{3}{4}))}}{\sqrt{1+e^{2(\ln(\frac{3}{4}))}}} + C\right]$$

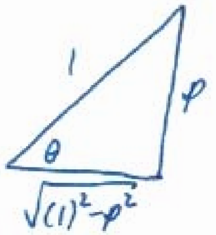
$$= \left[\frac{(\frac{4}{3})}{\sqrt{1+(e^{(\ln(\frac{4}{3}))})^2}}\right] - \left[\frac{(\frac{3}{4})}{\sqrt{1+(e^{(\ln(\frac{3}{4}))})^2}}\right] = \left[\frac{\frac{4}{3}}{\sqrt{1+(\frac{4}{3})^2}}\right] - \left[\frac{\frac{3}{4}}{\sqrt{1+(\frac{3}{4})^2}}\right]$$

$$= \left[\frac{\frac{4}{3}}{\sqrt{1+\frac{16}{9}}}\right] - \left[\frac{\frac{3}{4}}{\sqrt{1+\frac{9}{16}}}\right] = \left[\frac{\frac{4}{3}}{\sqrt{\frac{9}{9}+\frac{16}{9}}}\right] - \left[\frac{\frac{3}{4}}{\sqrt{\frac{16}{16}+\frac{9}{16}}}\right]$$

$$= \left[\frac{\frac{4}{3}}{\sqrt{\frac{25}{9}}}\right] - \left[\frac{\frac{3}{4}}{\sqrt{\frac{25}{16}}}\right] = \left[\frac{\frac{4}{3}}{\frac{5}{3}}\right] - \left[\frac{\frac{3}{4}}{\frac{5}{4}}\right] = \left[\frac{4}{5}\right] - \left[\frac{3}{5}\right] = \underline{\underline{\frac{1}{5}}}$$

44) $\int \frac{\sqrt{1-(\ln x)^2}}{x \ln x} dx = \int \frac{\sqrt{1-(\ln x)^2}}{\ln x}\left(\frac{1}{x} dx\right) = \int \frac{\sqrt{1-p^2}}{p} dp = \int \frac{\sqrt{(1)^2-p^2}}{p} dp$

$p = \ln x \quad dp = \frac{1}{x} dx$



$\frac{p}{1} = \sin\theta \quad \frac{\sqrt{(1)^2-p^2}}{1} = \cos\theta$

$p = \sin\theta \quad \sqrt{(1)^2-p^2} = \cos\theta$

$dp = \cos\theta\, d\theta$

$= \int \frac{(\cos\theta)}{(\sin\theta)}(\cos\theta\, d\theta) = \int \frac{\cos^2\theta}{\sin\theta} d\theta$

$= \int \frac{(1-\sin^2\theta)}{\sin\theta} d\theta = \int \left(\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}\right) d\theta$

$= \int \csc\theta\, d\theta - \int \sin\theta\, d\theta$

$= \left[\ln|\csc\theta - \cot\theta|\right] - \left[-\cos\theta\right] + C$

$= \ln|\csc\theta - \cot\theta| + \cos\theta + C$

$= \ln\left|\left(\frac{1}{p}\right) - \left(\frac{\sqrt{(1)^2-p^2}}{p}\right)\right| + \left(\frac{\sqrt{(1)^2-p^2}}{1}\right) + C$

$= \ln\left|\frac{1-\sqrt{1-p^2}}{p}\right| + \sqrt{1-p^2} + C$

$= \ln\left|\frac{1-\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C$

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46) $\int \sqrt{\frac{x}{1-x^3}} dx = \int \sqrt{\frac{x}{1-(x^{\frac{3}{2}})^2}} dx = \int \frac{\sqrt{x}}{\sqrt{1-(x^{\frac{3}{2}})^2}} dx$

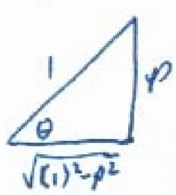
$p = x^{\frac{3}{2}}$

$dp = \frac{3}{2} x^{\frac{1}{2}} dx = \frac{3}{2}\sqrt{x}\, dx$

$\frac{2}{3} dp = \sqrt{x}\, dx$

$= \int \frac{1}{\sqrt{1-(x^{3/2})^2}}(\sqrt{x}\, dx) = \int \frac{1}{\sqrt{1-p^2}}\left(\frac{2}{3} dp\right)$

$= \int \frac{2}{3\sqrt{(1)^2-p^2}} dp = \int \frac{2}{3(\cos\theta)}(\cos\theta\, d\theta)$



$\frac{p}{1} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{p}{1}\right)$

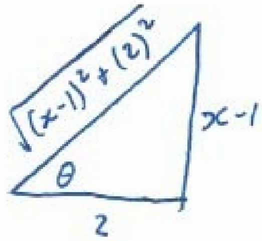
$p = \sin\theta \quad \frac{\sqrt{(1)^2-p^2}}{1} = \cos\theta$

$dp = \cos\theta\, d\theta \quad \sqrt{(1)^2-p^2} = \cos\theta$

$= \int \frac{2}{3} d\theta = \frac{2}{3}[\theta] + C = \frac{2}{3}\left(\sin^{-1}\left(\frac{p}{1}\right)\right) + C$

$= \frac{2}{3}\sin^{-1}(x^{3/2}) + C = \frac{2}{3}\sin^{-1}(\sqrt{x})^3 + C$

50) $$\int \frac{1}{\sqrt{x^2-2x+5}}\,dx = \int \frac{1}{\sqrt{(x^2-2x+1)+4}}\,dx = \int \frac{1}{\sqrt{(x-1)^2+(2)^2}}\,dx$$



$$= \int \frac{1}{(2\sec\theta)}(2\sec^2\theta\,d\theta) = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$\frac{x-1}{2} = \tan\theta$

$x = 1 + 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\frac{\sqrt{(x-1)^2+(2)^2}}{2} = \sec\theta$

$\sqrt{(x-1)^2+(2)^2} = 2\sec\theta$

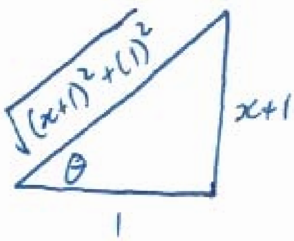
$$= \ln\left|\left(\frac{\sqrt{(x-1)^2+(2)^2}}{2}\right) + \left(\frac{x-1}{2}\right)\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2-2x+5}}{2} + \frac{x-1}{2}\right| + C$$

$$= \ln\left|\frac{x-1+\sqrt{x^2-2x+5}}{2}\right| + C$$

---

52) $$\int \frac{\sqrt{x^2+2x+2}}{x^2+2x+1}\,dx = \int \frac{\sqrt{(x^2+2x+1)+1}}{x^2+2x+1}\,dx = \int \frac{\sqrt{(x+1)^2+(1)^2}}{(x+1)^2}\,dx$$



$$= \int \frac{(\sec\theta)}{(\tan\theta)^2}(\sec^2\theta\,d\theta) = \int \frac{\sec^3\theta}{\tan^2\theta}\,d\theta$$

$\frac{x+1}{1} = \tan\theta$

$x+1 = \tan\theta$

$x = -1 + \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\frac{\sqrt{(x+1)^2+(1)^2}}{1} = \sec\theta$

$\sqrt{(x+1)^2+(1)^2} = \sec\theta$

$$= \int \frac{\left(\frac{1}{\cos^3\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta\cos^3\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta\cos^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta\cos^2\theta}(\cos\theta\,d\theta)$$

$$= \int \frac{1}{\sin^2\theta(1-\sin^2\theta)}(\cos\theta\,d\theta) = \int \frac{1}{p^2(1-p^2)}(dp)$$

$p = \sin\theta$

$dp = \cos\theta\,d\theta$

$$= \int \frac{1}{(p)^2(1+p)^1(1-p)^1}\,dp$$

this exercise requires to use
partial fraction decomposition which is section 8.5.

52) continued

$$\frac{1}{(p)^2(1+p)^1(1-p)^1} = \frac{A}{(p)^1} + \frac{B}{(p)^2} + \frac{C}{(1+p)^1} + \frac{D}{(1-p)^1}$$

$$1 = A\left(p(1+p)(1-p)\right) + B\left((1+p)(1-p)\right) + C\left(p^2(1-p)\right) + D\left(p^2(1+p)\right)$$

$$1 = A(p-p^3) + B(1-p^2) + C(p^2-p^3) + D(p^2+p^3)$$

constant term: $1 = B$

p-term: $0 = A$

p^2-term: $0 = -B + C + D$, $0 = -(1) + C + D$, $1 = C + D$

p^3-term: $0 = -C + D$, $C = D$

$1 = C + D$
$1 = C + (C)$
$1 = 2C$
$\frac{1}{2} = C$
$D = \frac{1}{2}$

$$= \int \frac{1}{(p)^2(1+p)^1(1-p)^1}\,dp = \int \left(\frac{(0)}{(p)^1} + \frac{(1)}{(p)^2} + \frac{(\frac{1}{2})}{(1+p)^1} + \frac{(\frac{1}{2})}{(1-p)^1}\right) dp$$

$$= [0] + \left[\frac{p^{-1}}{-1}\right] + \frac{1}{2}\left[\ln|1+p|\right] + \frac{1}{2}\left[-\ln|1-p|\right] + C$$

$$= \frac{-1}{p} + \frac{1}{2}\ln|1+p| - \frac{1}{2}\ln|1-p| + C$$

$$= \frac{-1}{(\sin\theta)} + \frac{1}{2}\ln|1+(\sin\theta)| - \frac{1}{2}\ln|1-(\sin\theta)| + C$$

$$= -\csc\theta + \frac{1}{2}\ln|1+\sin\theta| - \frac{1}{2}\ln|1-\sin\theta| + C$$

$$= -\left(\frac{\sqrt{(x+1)^2+(1)^2}}{x+1}\right) + \frac{1}{2}\ln\left|1+\left(\frac{x+1}{\sqrt{(x+1)^2+(1)^2}}\right)\right| - \frac{1}{2}\ln\left|1-\left(\frac{x+1}{\sqrt{(x+1)^2+(1)^2}}\right)\right| + C$$

$$= \frac{-\sqrt{x^2+2x+2}}{x+1} + \frac{1}{2}\ln\left|\frac{x+1+\sqrt{x^2+2x+2}}{\sqrt{x^2+2x+2}}\right| - \frac{1}{2}\ln\left|\frac{\sqrt{x^2+2x+2}-x-1}{\sqrt{x^2+2x+2}}\right| + C$$

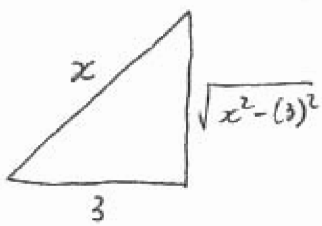
$$= \frac{-\sqrt{x^2+2x+2}}{x+1} + \frac{1}{2}\ln\left|\frac{\frac{x+1+\sqrt{x^2+2x+2}}{\sqrt{x^2+2x+2}}}{\frac{\sqrt{x^2+2x+2}-x-1}{\sqrt{x^2+2x+2}}}\right| + C = \frac{1}{2}\ln\left|\frac{x+1+\sqrt{x^2+2x+2}}{\sqrt{x^2+2x+2}-x-1}\right| - \frac{\sqrt{x^2+2x+2}}{x+1} + C$$

54)

$$\sqrt{x^2-9}\,\frac{dy}{dx} = 1 \qquad x > 3 \qquad y(5) = \ln 3$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-(3)^2}}$$

$$y = \int \frac{1}{\sqrt{x^2-(3)^2}}\,dx = \int \frac{1}{(3\tan\theta)}(3\sec\theta\tan\theta\, d\theta)$$



$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\left(\frac{x}{3}\right) + \left(\frac{\sqrt{x^2-(3)^2}}{3}\right)\right| + C$$

$$\frac{x}{3} = \sec\theta \qquad \frac{\sqrt{x^2-(3)^2}}{3} = \tan\theta$$

$$= \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C$$

$$x = 3\sec\theta \qquad \sqrt{x^2-(3)^2} = 3\tan\theta$$

$$x = 3\sec\theta\tan\theta\, d\theta$$

$$y(x) = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C$$

$$(\ln 3) = y(5) = \ln\left|\frac{(5)+\sqrt{(5)^2-9}}{3}\right| + C$$

$$\ln 3 = \ln\left|\frac{5+\sqrt{25-9}}{3}\right| + C$$

$$\ln 3 = \ln\left|\frac{5+\sqrt{16}}{3}\right| + C$$

$$\ln 3 = \ln\left|\frac{5+4}{3}\right| + C$$

$$\ln 3 = \ln\left|\frac{9}{3}\right| + C$$

$$\ln 3 = \ln|3| + C$$

$$\ln 3 = \ln 3 + C$$

$$0 = C$$

$$y(x) = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + (0)$$

$$\boxed{y(x) = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right|}$$