MATH 21200

section 8.3

8) $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right) dx$

$$\int \sin^5\left(\frac{x}{2}\right)dx = \int \sin^4\left(\frac{x}{2}\right)\left(\sin\left(\frac{x}{2}\right)dx\right) = \int \left(\sin^2\left(\frac{x}{2}\right)\right)^2\left(\sin\left(\frac{x}{2}\right)dx\right)$$

$p = \cos\left(\frac{x}{2}\right)$

$dp = \frac{-1}{2}\sin\left(\frac{x}{2}\right)dx$

$-2dp = \sin\left(\frac{x}{2}\right)dx$

$$= \int \left(1-\cos^2\left(\frac{x}{2}\right)\right)^2\left(\sin\left(\frac{x}{2}\right)dx\right)$$

$$= \int \left(1 - 2\cos^2\left(\frac{x}{2}\right) + \cos^4\left(\frac{x}{2}\right)\right)\left(\sin\left(\frac{x}{2}\right)dx\right)$$

$$= \int (1-2p^2+p^4)(-2\,dp) = -2\left\{[p] - 2\left[\frac{p^3}{3}\right] + \left[\frac{p^5}{5}\right]\right\} + C$$

$$= -2\left\{\cos\left(\frac{x}{2}\right) - \frac{2}{3}\cos^3\left(\frac{x}{2}\right) + \frac{1}{5}\cos^5\left(\frac{x}{2}\right)\right\} + C$$

$$= -2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right) + C$$

$$\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx = \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right) + C\right]_0^{\pi}$$

$$= \left[-2\cos\left(\frac{(\pi)}{2}\right) + \frac{4}{3}\cos^3\left(\frac{(\pi)}{2}\right) - \frac{2}{5}\cos^5\left(\frac{(\pi)}{2}\right) + C\right] - \left[-2\cos\left(\frac{(0)}{2}\right) + \frac{4}{3}\cos^3\left(\frac{(0)}{2}\right) - \frac{2}{5}\cos^5\left(\frac{(0)}{2}\right) + C\right]$$

$$= \left[-2(0) + \frac{4}{3}(0)^3 - \frac{2}{5}(0)^5\right] - \left[-2(1) + \frac{4}{3}(1)^3 - \frac{2}{5}(1)^5\right] = [0] - \left[-2 + \frac{4}{3} - \frac{2}{5}\right]$$

$$= 2 - \frac{4}{3} + \frac{2}{5} = \frac{30}{15} - \frac{20}{15} + \frac{6}{15} = \frac{16}{15}$$

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10) $\int_0^{\frac{\pi}{6}} 3\cos^5(3x)\,dx$

$$\int 3\cos^5(3x)dx = \int \cos^4(3x)\left(3\cos(3x)dx\right) = \int \left(\cos^2(3x)\right)^2\left(3\cos(3x)dx\right)$$

$p = \sin(3x)$

$dp = 3\cos(3x)dx$

$$= \int \left(1-\sin^2(3x)\right)^2\left(3\cos(3x)dx\right)$$

$$= \int \left(1 - 2\sin^2(3x) + \sin^4(3x)\right)\left(3\cos(3x)dx\right)$$

10) continued

$$= \int (1-2p^2+p^4)(dp) = [p] - 2\left[\frac{p^3}{3}\right] + \left[\frac{p^5}{5}\right] + C$$

$$= \sin(3x) - \frac{2}{3}\sin^3(3x) + \frac{1}{5}\sin^5(3x) + C$$

$$\int_0^{\frac{\pi}{6}} 3\cos^5(3x)\,dx = \left[\sin(3x) - \frac{2}{3}\sin^3(3x) + \frac{1}{5}\sin^5(3x) + C\right]_0^{\frac{\pi}{6}}$$

$$= \left[\sin\left(3\left(\frac{\pi}{6}\right)\right) - \frac{2}{3}\sin^3\left(3\left(\frac{\pi}{6}\right)\right) + \frac{1}{5}\sin^5\left(3\left(\frac{\pi}{6}\right)\right) + C\right] - \left[\sin(3(0)) - \frac{2}{3}\sin^3(3(0)) + \frac{1}{5}\sin^5(3(0)) + C\right]$$

$$= \left[\sin\left(\frac{\pi}{2}\right) - \frac{2}{3}\sin^3\left(\frac{\pi}{2}\right) + \frac{1}{5}\sin^5\left(\frac{\pi}{2}\right)\right] - \left[\sin(0) - \frac{2}{3}\sin^3(0) + \frac{1}{5}\sin^5(0)\right]$$

$$= \left[(1) - \frac{2}{3}(1)^3 + \frac{1}{5}(1)^5\right] - \left[(0) - \frac{2}{3}(0) + \frac{1}{5}(0)\right] = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15}{15} - \frac{10}{15} + \frac{3}{15} = \frac{8}{15}$$

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12) method 1

$$\int \cos^3(2x)\sin^5(2x)\,dx = \int \cos^3(2x)\sin^4(2x)(\sin(2x)\,dx)$$

$p = \cos(2x)$

$dp = -2\sin(2x)\,dx$

$\frac{-1}{2}dp = \sin(2x)\,dx$

$$= \int \cos^3(2x)(\sin^2(2x))^2(\sin(2x)\,dx)$$

$$= \int \cos^3(2x)(1-\cos^2(2x))^2(\sin(2x)\,dx)$$

$$= \int \cos^3(2x)(1-2\cos^2(2x)+\cos^4(2x))(\sin(2x)\,dx)$$

$$= \int (\cos^3(2x) - 2\cos^5(2x) + \cos^7(2x))(\sin(2x)\,dx)$$

$$= \int (p^3 - 2p^5 + p^7)\left(\frac{-1}{2}dp\right)$$

$$= \frac{-1}{2}\left\{\left[\frac{p^4}{4}\right] - 2\left[\frac{p^6}{6}\right] + \left[\frac{p^8}{8}\right]\right\} + C$$

$$= \frac{-1}{2}\left\{\frac{1}{4}\cos^4(2x) - \frac{1}{3}\cos^6(2x) + \frac{1}{8}\cos^8(2x)\right. + C$$

$$= \frac{-1}{8}\cos^4(2x) + \frac{1}{6}\cos^6(2x) - \frac{1}{16}\cos^8(2x) + C$$

12) continued method 2

$$\int \cos^3(2x)\sin^5(2x)\,dx = \int \cos^2(2x)\sin^5(2x)\,(\cos(2x)\,dx)$$

$q = \sin(2x)$

$dq = 2\cos(2x)\,dx$

$\frac{1}{2}dq = \cos(2x)\,dx$

$$= \int (1-\sin^2(2x))\sin^5(2x)\,(\cos(2x)\,dx)$$

$$= \int (\sin^5(2x) - \sin^7(2x))\,(\cos(2x)\,dx)$$

$$= \int (q^5 - q^7)\left(\tfrac{1}{2}dq\right) = \frac{1}{2}\left\{\left[\frac{q^6}{6}\right] - \left[\frac{q^8}{8}\right]\right\} + C$$

$$= \frac{1}{2}\left\{\frac{1}{6}\sin^6(2x) - \frac{1}{8}\sin^8(2x)\right\} + C$$

$$= \frac{1}{12}\sin^6(2x) - \frac{1}{16}\sin^8(2x) + C$$

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14) $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos(2x))\,dx = \int \frac{1}{2}\,dx - \int \frac{1}{2}\cos(2x)\,dx$$

$p = 2x$

$dp = 2\,dx$

$\frac{1}{2}dp = dx$

$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos p\left(\tfrac{1}{2}dp\right)$$

$$= \frac{1}{2}[x] - \frac{1}{4}[\sin p] + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin(2x) + C\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right) + C\right] - \left[\frac{1}{2}(0) - \frac{1}{4}\sin(2(0)) + C\right]$$

$$= \left[\frac{\pi}{4} - \frac{1}{4}\sin(\pi)\right] - \left[0 - \frac{1}{4}\sin(0)\right]$$

$$= \left[\frac{\pi}{4} - \frac{1}{4}(0)\right] - \left[0 - \frac{1}{4}(0)\right] = \frac{\pi}{4}$$

16) $\int 7\cos^7 t\,dt = \int 7\cos^6 t\,(\cos t\,dt) = \int 7(\cos^2 t)^3(\cos t\,dt)$

$p = \sin t$

$dp = \cos t\,dt$

$= \int 7(1-\sin^2 t)^3(\cos t\,dt)$

$= \int 7(1-3\sin^2 t+3\sin^4 t-\sin^6 t)(\cos t\,dt)$

$= \int (7-21\sin^2 t+21\sin^4 t-7\sin^6 t)(\cos t\,dt)$

$= \int (7-21p^2+21p^4-7p^6)(dp) = 7[p]-21\left[\frac{p^3}{3}\right]+21\left[\frac{p^5}{5}\right]-7\left[\frac{p^7}{7}\right]+$

$= 7\sin t-7\sin^3 t+\frac{21}{5}\sin^5 t-\sin^7 t+C$

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18) $\cos^2(2\pi x) = \frac{1}{2}(1+\cos(2(2\pi x))) = \frac{1}{2}(1+\cos(4\pi x))$

$\int 8\cos^4(2\pi x)dx = \int 8(\cos^2(2\pi x))^2 dx = \int 8\left(\frac{1}{2}(1+\cos(4\pi x)\right)^2 dx$

$= \int 8\left(\frac{1}{4}(1+2\cos(4\pi x)+\cos^2(4\pi x))\right)dx$

$= \int 2(1+2\cos(4\pi x)+\cos^2(4\pi x))dx$

$= \int (2+4\cos(4\pi x)+2\cos^2(4\pi x))dx$

$= \int 2dx+\int 4\cos(4\pi x)dx+\int 2\cos^2(4\pi x)dx$

$= \int 2dx+\int 4\cos(4\pi x)dx+\int 2\left(\frac{1}{2}(1+\cos(8\pi x))\right)dx$

$= \int 2dx+\int 4\cos(4\pi x)dx+\int (1+\cos(8\pi x))dx$

$= \int 2dx+\int 4\cos(4\pi x)dx+\int 1dx+\int \cos(8\pi x)dx$

$= \int 3dx+\int 4\cos(4\pi x)dx+\int \cos(8\pi x)dx$

$= 3[x]+\left[\frac{1}{\pi}\sin(4\pi x)\right]+\left[\frac{1}{8\pi}\sin(8\pi x)\right]+C$

$= 3x+\frac{1}{\pi}\sin(4\pi x)+\frac{1}{8\pi}\sin(8\pi x)+C$

$\int 4\cos(4\pi x)dx$

$p = 4\pi x$ | $= \int \cos(4\pi x)(4dx)$

$dp = 4\pi dx$ | $= \int \cos p\left(\frac{1}{\pi}dp\right)$

$\frac{1}{\pi}dp = 4dx$ | $= \frac{1}{\pi}\sin p+D$

$= \frac{1}{\pi}\sin(4\pi x)+D$

$\int \cos(8\pi x\,dx) = \int \cos q\left(\frac{1}{8\pi}dq\right)$

$q = 8\pi x$ $\quad = \frac{1}{8\pi}\sin q+E$

$dq = 8\pi dx$ $\quad = \frac{1}{8\pi}\sin(8\pi x)+E$

$\frac{1}{8\pi}dq = dx$

20) $\int 8\sin^4 y\cos^2 y\,dy = \int 8\sin^2 y\sin^2 y\cos^2 y\,dy$

$= \int 8\sin^2 y(\sin y\cos y)^2\,dy = \int 8\left(\frac{1}{2}(1-\cos(2y))\right)\left(\frac{1}{2}\sin(2y)\right)^2 dy$

$= \int 8\left(\frac{1}{2}(1-\cos(2y))\right)\left(\frac{1}{4}\sin^2(2y)\right)dy = \int (1-\cos(2y))(\sin^2(2y))\,dy$

$= \int (\sin^2(2y) - \sin^2(2y)\cos(2y))\,dy = \int \sin^2(2y)\,dy - \int \sin^2(2y)\cos(2y)\,dy$

$= \int \frac{1}{2}(1-\cos(4y))\,dy - \int \sin^2(2y)\cos(2y)\,dy$

$= \int \frac{1}{2}\,dy - \int \frac{1}{2}\cos(4y)\,dy - \int \sin^2(2y)(\cos(2y)\,dy)$

$= \frac{1}{2}[y] - \frac{1}{2}\left[\frac{\sin(4y)}{4}\right] - \left[\frac{\sin^3(2y)}{6}\right] + C = \frac{1}{2}y - \frac{1}{8}\sin(4y) - \frac{1}{6}\sin^3(2y) + C$

$\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = \left[\frac{1}{2}y - \frac{1}{8}\sin(4y) - \frac{1}{6}\sin^3(2y) + C\right]_0^{\pi}$

$= \left[\frac{1}{2}(\pi) - \frac{1}{8}\sin(4(\pi)) - \frac{1}{6}\sin^3(2(\pi)) + C\right] - \left[\frac{1}{2}(0) - \frac{1}{8}\sin(4(0)) - \frac{1}{6}\sin^3(2(0)) + C\right]$

$= \left[\frac{\pi}{2} - \frac{1}{8}(0) - \frac{1}{6}(0)^3\right] - \left[0 - \frac{1}{8}(0) - \frac{1}{6}(0)^3\right] = \frac{\pi}{2}$

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36) $\int \sec^3 x\tan^3 x\,dx = \int \sec^2 x\tan^2 x\,(\sec x\tan x\,dx)$

$p = \sec x$

$dp = \sec x\tan x\,dx$

$= \int \sec^2 x(\sec^2 x - 1)(\sec x\tan x\,dx)$

$= \int (\sec^4 x - \sec^2 x)(\sec x\tan x\,dx)$

$= \int (p^4 - p^2)(dp) = \left[\frac{p^5}{5}\right] - \left[\frac{p^3}{3}\right] + C$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

38) $\int \sec^4 x \tan^2 x \, dx = \int \sec^2 x \tan^2 x \, (\sec^2 x \, dx)$

$p = \tan x$

$dp = \sec^2 x \, dx$

$= \int (1+\tan^2 x) \tan^2 x \, (\sec^2 x \, dx)$

$= \int (\tan^2 x + \tan^4 x)(\sec^2 x \, dx)$

$= \int (p^2 + p^4)(dp) = \left[\frac{p^3}{3}\right] + \left[\frac{p^5}{5}\right] + C$

$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

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42) $\int 3 \sec^4(3x) \, dx = \int \sec^4(3x)(3\,dx) = \int \sec^4\theta \, (d\theta)$

$\theta = 3x$

$d\theta = 3\,dx$

$= \int (\sec^2\theta)^2 d\theta = \int \sec^2\theta \sec^2\theta \, d\theta$

$= \int (1+\tan^2\theta)\sec^2\theta \, d\theta$

$= \int (\sec^2\theta + \tan^2\theta \sec^2\theta) d\theta$

$= \int \sec^2\theta \, d\theta + \int \tan^2\theta (\sec^2\theta \, d\theta)$

$= [\tan\theta] + \left[\frac{\tan^3\theta}{3}\right] + C$

$= \tan(3x) + \frac{1}{3}\tan^3(3x) + C$

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46) $\int 6\tan^4 x \, dx = \int 6 \tan^2 x \tan^2 x \, dx = \int 6(\sec^2 x - 1)\tan^2 x \, dx$

$= \int 6(\tan^2 x \sec^2 x - \tan^2 x) dx$

$= \int 6 \tan^2 x \sec^2 x \, dx - \int 6 \tan^2 x \, dx$

$= \int 6 \tan^2 \sec^2 x \, dx - \int 6(\sec^2 x - 1) dx$

46) continued

$$= \int 6\tan^2 x\,(\sec^2 x\,dx) - \int 6\sec^2 x\,dx + \int 6\,dx$$

$$= 6\left[\frac{\tan^3 x}{3}\right] - 6\left[\tan x\right] + 6\left[x\right] + C$$

$$= 2\tan^3 x - 6\tan x + 6x + C$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 6\tan^4 x\,dx = \left[2\tan^3 x - 6\tan x + 6x + C\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left[2\tan^3\left(\tfrac{\pi}{4}\right) - 6\tan\left(\tfrac{\pi}{4}\right) + 6\left(\tfrac{\pi}{4}\right) + C\right] - \left[2\tan^3\left(\tfrac{-\pi}{4}\right) - 6\tan\left(\tfrac{-\pi}{4}\right) + 6\left(\tfrac{-\pi}{4}\right) + C\right]$$

$$= \left[2(1)^3 - 6(1) + \frac{3\pi}{2}\right] - \left[2(-1)^3 - 6(-1) - \frac{3\pi}{2}\right] = \left[2 - 6 + \frac{3\pi}{2}\right] - \left[-2 + 6 - \frac{3\pi}{2}\right]$$

$$= \left[\frac{3\pi}{2} - 4\right] - \left[4 - \frac{3\pi}{2}\right] = 3\pi - 8$$

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52)

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$- \{\sin(A-B) = \sin A\cos B - \cos A\sin B\}$$

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$$

$$\frac{1}{2}\{\sin(A+B) - \sin(A-B)\} = \cos A\sin B$$

let $A = 3x$
$B = 2x$

$$\int \sin(2x)\cos(3x)\,dx = \int \cos(3x)\sin(2x)\,dx$$

$$= \int \frac{1}{2}\{\sin((3x)+(2x)) - \sin((3x)-(2x))\}\,dx$$

$$= \int \frac{1}{2}\{\sin(5x) - \sin x\}\,dx$$

$$= \frac{1}{2}\int \sin(5x)\,dx - \frac{1}{2}\int \sin x\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos(5x)}{5}\right] - \frac{1}{2}\left[-\cos x\right] + C$$

$$= \frac{1}{2}\cos x - \frac{1}{10}\cos(5x) + C$$

54) $\int \sin x \cos x \, dx = \int \frac{1}{2} \sin(2x)\, dx = \frac{1}{2}\left[\frac{-\cos(2x)}{2}\right] + C$

$= \frac{-1}{4}\cos(2x) + C$

$\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx = \left[\frac{-1}{4}\cos(2x) + C\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{-1}{4}\cos\left(2\left(\frac{\pi}{2}\right)\right) + C\right] - \left[\frac{-1}{4}\cos(2(0)) + C\right]$

$= \left[\frac{-1}{4}\cos(\pi)\right] - \left[\frac{-1}{4}\cos(0)\right] = \left[\frac{-1}{4}(-1)\right] - \left[\frac{-1}{4}(1)\right] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

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56)

$$\begin{array}{l} + \; \cos(A+B) = \cos A \cos B - \sin A \sin B \\ \{\cos(A-B) = \cos A \cos B + \sin A \sin B\} \\ \hline \cos(A+B) + \cos(A-B) = 2\cos A \cos B \\ \frac{1}{2}\{\cos(A+B) + \cos(A-B)\} = \cos A \cos B \end{array}$$

let $A = 7x$
$B = x$

$\int \cos x \cos(7x)\, dx = \int \cos(7x) \cos x \, dx = \int \frac{1}{2}\{\cos((7x)+(x)) + \cos((7x)-(x))\}\, dx$

$= \int \frac{1}{2}\{\cos(8x) + \cos(6x)\}\, dx = \frac{1}{2}\int \cos(8x)\, dx + \frac{1}{2}\int \cos(6x)\, dx$

$= \frac{1}{2}\left[\frac{\sin(8x)}{8}\right] + \frac{1}{2}\left[\frac{\sin(6x)}{6}\right] + C = \frac{1}{16}\sin(8x) + \frac{1}{12}\sin(6x) + C$

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos x \cos(7x)\, dx = \left[\frac{1}{16}\sin(8x) + \frac{1}{12}\sin(6x) + C\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

$= \left[\frac{1}{16}\sin\left(8\left(\frac{\pi}{2}\right)\right) + \frac{1}{12}\sin\left(6\left(\frac{\pi}{2}\right)\right) + C\right] - \left[\frac{1}{16}\sin\left(8\left(\frac{-\pi}{2}\right)\right) + \frac{1}{12}\sin\left(6\left(\frac{-\pi}{2}\right)\right) + C\right]$

$= \left[\frac{1}{16}\sin(4\pi) + \frac{1}{12}\sin(3\pi)\right] - \left[\frac{1}{16}\sin(-4\pi) + \frac{1}{12}\sin(-3\pi)\right]$

$= \left[\frac{1}{16}(0) + \frac{1}{12}(0)\right] - \left[\frac{1}{16}(0) + \frac{1}{12}(0)\right] = 0$