Theorem 7 –Substitution in Definite Integrals

If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x)=p$, then

$$\int_a^b \left(f(g(x))\right)\left(g'(x)\,dx\right) = \int_{g(a)}^{g(b)} f(p)\,dp.$$

Another option instead of using this theorem above is to first find the indefinite integral with substitution method and then apply the Fundamental Theorem of Calculus part 2.

Theorem 8:

Let f be continuous on the symmetric interval $[-a,a]$.

1. If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

2. If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Definition

If f and g are continuous with $f(x) \ge g(x)$ Throughout $[a,b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from *a* to *b*** is the integral of $(f-g)$ from a to b:

$$A = \int_a^b \left[f(x) - g(x)\right] dx.$$

2) $\int x\sqrt{1-x^2} = \int \sqrt{1-x^2}\,(x\,dx) = \int \sqrt{p}\left(\frac{-1}{2}dp\right) = \frac{-1}{2}\int p^{\frac{1}{2}}dp$

$p = 1-x^2$

$dp = -2x\,dx$

$-\frac{1}{2}dp = x\,dx$

$= \frac{-1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{-1}{3}(\sqrt{p})^3 + C = \frac{-1}{3}\left(\sqrt{1-x^2}\right)^3 + C$

a) $\int_0^1 x\sqrt{1-x^2}\,dx = \left[\frac{-1}{3}\left(\sqrt{1-x^2}\right)^3 + C\right]_0^1 = \left[\frac{-1}{3}\left(\sqrt{1-(1)^2}\right)^3 + C\right] - \left[\frac{-1}{3}\left(\sqrt{1-(0)^2}\right)^3 + C\right]$

$= \left[\frac{-1}{3}(0)^3\right] - \left[\frac{-1}{3}(1)^3\right] = [0] - \left[\frac{-1}{3}\right] = \frac{1}{3}$

b) $\int_{-1}^1 x\sqrt{1-x^2}\,dx = \left[\frac{-1}{3}\left(\sqrt{1-x^2}\right)^3 + C\right]_{-1}^1 = \left[\frac{-1}{3}\left(\sqrt{1-(1)^2}\right)^3 + C\right] - \left[\frac{-1}{3}\left(\sqrt{1-(-1)^2}\right)^3 + C\right]$

$= \left[\frac{-1}{3}(0)^3\right] - \left[\frac{-1}{3}(0)^3\right] = [0] - [0] = 0$

$p = 1-x^2$ $\quad x = -1 \Rightarrow p = 1-(-1)^2 = 1-1 = 0$

$dp = -2x\,dx$ $\quad x = 0 \Rightarrow p = 1-(0)^2 = 1-0 = 1$

$-\frac{1}{2}dp = x\,dx$ $\quad x = 1 \Rightarrow p = 1-(1)^2 = 1-1 = 0$

a) $\int_0^1 x\sqrt{1-x^2}\,dx = \int_1^0 \sqrt{p}\left(\frac{-1}{2}dp\right) = \int_1^0 \frac{-1}{2}p^{\frac{1}{2}}dp = \left[\frac{-1}{2}\left(\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right) + C\right]_1^0$

$= \left[\frac{-1}{3}(\sqrt{p})^3 + C\right]_1^0 = \left[\frac{-1}{3}\left(\sqrt{(0)}\right)^3 + C\right] - \left[\frac{-1}{3}\left(\sqrt{(1)}\right)^3 + C\right]$

$= \left[\frac{-1}{3}(0)\right] - \left[\frac{-1}{3}(1)\right] = \frac{1}{3}$

b) $\int_{-1}^1 x\sqrt{1-x^2}\,dx = \int_0^0 \sqrt{p}\left(\frac{-1}{2}dp\right) = 0$

4) $\int 3\cos^2 x \sin x \, dx = \int 3p^2(-1\,dp) = -p^3 + C$
$= -\cos^3 x + C$

$p = \cos x$
$dp = -\sin x \, dx$
$-1\,dp = \sin x \, dx$

a) $\int_0^{\pi} 3\cos^2 x \sin x \, dx = \left[-\cos^3 x + C\right]_0^{\pi} = \left[-\cos^3(\pi) + C\right] - \left[-\cos^3(0) + C\right]$
$= \left[-(-1)^3\right] - \left[-(1)^3\right] = [1] - [-1] = 2$

b) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx = \left[-\cos^3 x + C\right]_{2\pi}^{3\pi} = \left[-\cos^3(3\pi) + C\right] - \left[-\cos^3(2\pi) + C\right]$
$= \left[-(-1)^3\right] - \left[-(1)^3\right] = [1] - [-1] = 2$

$p = \cos x$ $\quad x = 0 \Rightarrow p = \cos(0) = 1$
$dp = -\sin x \, dx$ $\quad x = \pi \Rightarrow p = \cos(\pi) = -1$
$\quad x = 2\pi \Rightarrow p = \cos(2\pi) = 1$
$-1\,dp = \sin x \, dx$ $\quad x = 3\pi \Rightarrow p = \cos(3\pi) = -1$

a) $\int_0^{\pi} 3\cos^2 x \sin x \, dx = \int_1^{-1} 3p^2(-1\,dp) = \left[-p^3 + C\right]_1^{-1}$
$= \left[-(-1)^3 + C\right] - \left[-(1)^3 + C\right] = [1] - [-1] = 2$

b) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx = \int_1^{-1} 3p^2(-1\,dp) = \left[-p^3 + C\right]_1^{-1}$
$= \left[-(-1)^3 + C\right] - \left[-(1)^3 + C\right] = [1] - [-1] = 2$

6) $\int x(x^2+1)^{\frac{1}{3}}dx = \int (x^2+1)^{\frac{1}{3}}(x\,dx) = \int p^{\frac{1}{3}}\left(\frac{1}{2}dp\right)$

$p = x^2+1$

$dp = 2x\,dx$

$\frac{1}{2}dp = x\,dx$

$$= \frac{1}{2}\left[\frac{p^{\frac{4}{3}}}{\frac{4}{3}}\right] + C = \frac{3}{8}\left(\sqrt[3]{p}\right)^4 + C$$

$$= \frac{3}{8}\left(\sqrt[3]{x^2+1}\right)^4 + C$$

a) $\int_0^{\sqrt{7}} x(x^2+1)^{\frac{1}{3}}dx = \left[\frac{3}{8}\left(\sqrt[3]{x^2+1}\right)^4 + C\right]_0^{\sqrt{7}} = \left[\frac{3}{8}\left(\sqrt[3]{(\sqrt{7})^2+1}\right)^4 + C\right] - \left[\frac{3}{8}\left(\sqrt[3]{(0)^2+1}\right)^4 + C\right]$

$$= \left[\frac{3}{8}\left(\sqrt[3]{7+1}\right)^4\right] - \left[\frac{3}{8}\left(\sqrt[3]{1}\right)^4\right] = \frac{3}{8}(2)^4 - \frac{3}{8}(1) = \frac{48}{8} - \frac{3}{8} = \frac{45}{8}$$

b) $\int_{-\sqrt{7}}^{0} x(x^2+1)^{\frac{1}{3}}dx = \left[\frac{3}{8}\left(\sqrt[3]{x^2+1}\right)^4 + C\right]_{-\sqrt{7}}^{0} = \left[\frac{3}{8}\left(\sqrt[3]{(0)^2+1}\right)^4 + C\right] - \left[\frac{3}{8}\left(\sqrt[3]{(-\sqrt{7})^2+1}\right)^4 + C\right]$

$$= \left[\frac{3}{8}(1)^4\right] - \left[\frac{3}{8}\left(\sqrt[3]{8}\right)^4\right] = \frac{3}{8} - \frac{48}{8} = \frac{-45}{8}$$

$p = x^2+1 \qquad x = \sqrt{7} \Rightarrow p = (\sqrt{7})^2+1 = 7+1 = 8$

$dp = 2x\,dx \qquad x = 0 \Rightarrow p = (0)^2+1 = 1$

$\frac{1}{2}dp = x\,dx \qquad x = -\sqrt{7} \Rightarrow p = (-\sqrt{7})^2+1 = 7+1 = 8$

a) $\int_0^{\sqrt{7}} x(x^2+1)^{\frac{1}{3}}dx = \int_1^8 p^{\frac{1}{3}}\left(\frac{1}{2}dp\right) = \left[\frac{1}{2}\left(\frac{p^{\frac{4}{3}}}{\frac{4}{3}}\right) + C\right]_1^8 = \left[\frac{3}{8}\left(\sqrt[3]{p}\right)^4 + C\right]_1^8$

$$= \left[\frac{3}{8}\left(\sqrt[3]{(8)}\right)^4 + C\right] - \left[\frac{3}{8}\left(\sqrt[3]{(1)}\right)^4 + C\right] = \frac{48}{8} - \frac{3}{8} = \frac{45}{8}$$

b) $\int_{-\sqrt{7}}^{0} x(x^2+1)^{\frac{1}{3}}dx = \int_8^1 p^{\frac{1}{3}}\left(\frac{1}{2}dp\right) = \left[\frac{1}{2}\left(\frac{p^{\frac{4}{3}}}{\frac{4}{3}}\right) + C\right]_8^1 = \left[\frac{3}{8}\left(\sqrt[3]{p}\right)^4 + C\right]_8^1$

$$= \left[\frac{3}{8}\left(\sqrt[3]{(1)}\right)^4 + C\right] - \left[\frac{3}{8}\left(\sqrt[3]{(8)}\right)^4 + C\right]$$

$$= \frac{3}{8} - \frac{48}{8} = \frac{-45}{8}$$

8) $\int \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv = \int \frac{10}{(1+v^{3/2})^2}(\sqrt{v}\,dv) = \int \frac{10}{p^2}\left(\frac{2}{3}\,dp\right)$

$p = 1+v^{\frac{3}{2}}$ $\qquad = \frac{20}{3}\int p^{-2}\,dp = \frac{20}{3}\left[\frac{p^{-1}}{-1}\right]+C$

$dp = \frac{3}{2}v^{\frac{1}{2}}\,dv$

$dp = \frac{3}{2}\sqrt{v}\,dv$ $\qquad = \frac{-20}{3p}+C = \frac{-20}{3(1+v^{3/2})}+C = \frac{-20}{3(1+(\sqrt{v})^3)}+C$

$\frac{2}{3}\,dp = \sqrt{v}\,dv$

a) $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv = \left[\frac{-20}{3(1+(\sqrt{v})^3)}+C\right]_0^1 = \left[\frac{-20}{3(1+(\sqrt{(1)})^3)}+C\right]-\left[\frac{-20}{3(1+(\sqrt{(0)})^3)}+C\right]$

$= \left[\frac{-20}{3(2)}\right]-\left[\frac{-20}{3(1)}\right] = \frac{-20}{6}+\frac{20}{3} = \frac{20}{3}-\frac{10}{3} = \frac{10}{3}$

b) $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv = \left[\frac{-20}{3(1+(\sqrt{v})^3)}+C\right]_1^4 = \left[\frac{-20}{3(1+(\sqrt{(4)})^3)}+C\right]-\left[\frac{-20}{3(1+(\sqrt{(1)})^3)}+C\right]$

$= \left[\frac{-20}{3(1+8)}\right]-\left[\frac{-20}{3(2)}\right] = \frac{-20}{27}+\frac{20}{6} = \frac{20}{6}-\frac{20}{27} = \frac{10}{3}-\frac{20}{27} = \frac{90}{27}-\frac{20}{27} = \frac{70}{27}$

$p = 1+v^{\frac{3}{2}} = 1+(\sqrt{v})^3$ $\qquad v=0 \Rightarrow p = 1+(\sqrt{(0)})^3 = 1+0 = 1$

$dp = \frac{3}{2}v^{\frac{1}{2}}\,dv$ $\qquad v=1 \Rightarrow p = 1+(\sqrt{(1)})^3 = 1+1 = 2$

$dp = \frac{3}{2}\sqrt{v}\,dv$ $\qquad v=4 \Rightarrow p = 1+(\sqrt{(4)})^3 = 1+(2)^3 = 1+8 = 9$

$\frac{2}{3}\,dp = \sqrt{v}\,dv$

a) $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv = \int_1^2 \frac{10}{p^2}\left(\frac{2}{3}\,dp\right) = \frac{20}{3}\int_1^2 p^{-2}\,dp = \left[\frac{20}{3}\left(\frac{p^{-1}}{-1}\right)+C\right]_1^2$

$= \left[\frac{-20}{3p}+C\right]_1^2 = \left[\frac{-20}{3(2)}+C\right]-\left[\frac{-20}{3(1)}+C\right] = \frac{-20}{6}+\frac{20}{3} = \frac{20}{3}-\frac{10}{3} = \frac{10}{3}$

b) $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv = \int_2^9 \frac{10}{p^2}\left(\frac{2}{3}\,dp\right) = \frac{20}{3}\int_2^9 p^{-2}\,dp = \left[\frac{20}{3}\left(\frac{p^{-1}}{-1}\right)+C\right]_2^9$

$= \left[\frac{-20}{3p}+C\right]_2^9 = \left[\frac{-20}{3(9)}+C\right]-\left[\frac{-20}{3(2)}+C\right] = \left[\frac{-20}{27}\right]-\left[\frac{-10}{3}\right] = \frac{10}{3}-\frac{20}{27}$

$= \frac{90}{27}-\frac{20}{27} = \frac{70}{27}$

10) $\int \frac{x^3}{\sqrt{x^4+9}}\,dx = \int \frac{1}{\sqrt{x^4+9}}\,(x^3\,dx) = \int \frac{1}{\sqrt{p}}\left(\frac{1}{4}\,dp\right) = \frac{1}{4}\int p^{-\frac{1}{2}}\,dp$

$p = x^4+9$ $\qquad = \frac{1}{4}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = \frac{1}{2}\sqrt{p} + C = \frac{1}{2}\sqrt{x^4+9} + C$

$dp = 4x^3\,dx$

$\frac{1}{4}\,dp = x^3\,dx$

a) $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \left[\frac{1}{2}\sqrt{x^4+9} + C\right]_0^1 = \left[\frac{1}{2}\sqrt{(1)^4+9} + C\right] - \left[\frac{1}{2}\sqrt{(0)^4+9} + C\right]$

$= \left[\frac{1}{2}\sqrt{10}\right] - \left[\frac{1}{2}\sqrt{9}\right] = \frac{\sqrt{10}}{2} - \frac{3}{2} = \frac{\sqrt{10}-3}{2}$

b) $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx = \left[\frac{1}{2}\sqrt{x^4+9} + C\right]_{-1}^0 = \left[\frac{1}{2}\sqrt{(0)^4+9} + C\right] - \left[\frac{1}{2}\sqrt{(-1)^4+9} + C\right]$

$= \left[\frac{1}{2}\sqrt{9}\right] - \left[\frac{1}{2}\sqrt{10}\right] = \frac{3}{2} - \frac{\sqrt{10}}{2} = \frac{3-\sqrt{10}}{2}$

$p = x^4+9$ $\qquad x = 1 \Rightarrow p = (1)^4+9 = 1+9 = 10$

$dp = 4x^3\,dx$ $\qquad x = 0 \Rightarrow p = (0)^4+9 = 0+9 = 9$

$\frac{1}{4}\,dp = x^3\,dx$ $\qquad x = -1 \Rightarrow p = (-1)^4+9 = 1+9 = 10$

a) $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_9^{10} \frac{1}{\sqrt{p}}\left(\frac{1}{4}\,dp\right) = \frac{1}{4}\int_9^{10} p^{-\frac{1}{2}}\,dp = \left[\frac{1}{4}\left(\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right) + C\right]_9^{10}$

$= \left[\frac{1}{2}\sqrt{p} + C\right]_9^{10} = \left[\frac{1}{2}\sqrt{(10)} + C\right] - \left[\frac{1}{2}\sqrt{(9)} + C\right] = \frac{\sqrt{10}}{2} - \frac{3}{2} = \frac{\sqrt{10}-3}{2}$

b) $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_{10}^{9} \frac{1}{\sqrt{p}}\left(\frac{1}{4}\,dp\right) = \frac{1}{4}\int_{10}^{9} p^{-\frac{1}{2}}\,dp = \left[\frac{1}{4}\left(\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right) + C\right]_{10}^{9}$

$= \left[\frac{1}{2}\sqrt{p} + C\right]_{10}^{9} = \left[\frac{1}{2}\sqrt{(9)} + C\right] - \left[\frac{1}{2}\sqrt{(10)} + C\right] = \frac{3}{2} - \frac{\sqrt{10}}{2} = \frac{3-\sqrt{10}}{2}$

12) $\int (1-\cos 3t)\sin 3t\,dt = \int p\left(\frac{1}{3}dp\right) = \frac{1}{3}\left[\frac{p^2}{2}\right]+C$

$= \frac{1}{6}p^2 + C = \frac{1}{6}(1-\cos 3t)^2 + C$

$p = 1-\cos 3t$

$dp = -[-\sin 3t(3)]dt$

$dp = 3\sin 3t\,dt$

$\frac{1}{3}dp = \sin 3t\,dt$

a) $\int_0^{\frac{\pi}{6}}(1-\cos 3t)\sin 3t\,dt = \left[\frac{1}{6}(1-\cos 3t)^2+C\right]_0^{\frac{\pi}{6}}$

$= \left[\frac{1}{6}\left(1-\cos\left(3\left(\frac{\pi}{6}\right)\right)\right)^2+C\right] - \left[\frac{1}{6}(1-\cos(3(0)))^2+C\right]$

$= \left[\frac{1}{6}\left(1-\cos\left(\frac{\pi}{2}\right)\right)^2\right] - \left[\frac{1}{6}(1-\cos(0))^2\right] = \left[\frac{1}{6}(1-0)^2\right] - \left[\frac{1}{6}(1-1)^2\right] = \frac{1}{6}$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1-\cos 3t)\sin 3t\,dt = \left[\frac{1}{6}(1-\cos 3t)^2+C\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \left[\frac{1}{6}\left(1-\cos\left(3\left(\frac{\pi}{3}\right)\right)\right)^2+C\right] - \left[\frac{1}{6}\left(1-\cos\left(3\left(\frac{\pi}{6}\right)\right)\right)^2+C\right]$

$= \left[\frac{1}{6}(1-\cos(\pi))^2\right] - \left[\frac{1}{6}\left(1-\cos\left(\frac{\pi}{2}\right)\right)^2\right] = \left[\frac{1}{6}(1-(-1))^2\right] - \left[\frac{1}{6}(1-0)^2\right] = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

$p = 1-\cos 3t$ $\qquad t=0 \Rightarrow p = 1-\cos(3(0)) = 1-\cos(0) = 1-1 = 0$

$dp = -[-\sin 3t(3)]dt$ $\qquad t=\frac{\pi}{6} \Rightarrow p = 1-\cos\left(3\left(\frac{\pi}{6}\right)\right) = 1-\cos\left(\frac{\pi}{2}\right) = 1-0 = 1$

$\frac{1}{3}dp = \sin 3t\,dt$ $\qquad t=\frac{\pi}{3} \Rightarrow p = 1-\cos\left(3\left(\frac{\pi}{3}\right)\right) = 1-\cos(\pi) = 1-(-1) = 2$

a) $\int_0^{\frac{\pi}{6}}(1-\cos 3t)\sin 3t\,dt = \int_0^1 p\left(\frac{1}{3}dp\right) = \left[\frac{1}{3}\left(\frac{p^2}{2}\right)+C\right]_0^1 = \left[\frac{1}{6}p^2+C\right]_0^1$

$= \left[\frac{1}{6}(1)^2+C\right] - \left[\frac{1}{6}(0)^2+C\right] = \frac{1}{6} - 0 = \frac{1}{6}$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1-\cos 3t)\sin 3t\,dt = \int_1^2 p\left(\frac{1}{3}dp\right) = \left[\frac{1}{3}\left(\frac{p^2}{2}\right)+C\right]_1^2 = \left[\frac{1}{6}p^2+C\right]_1^2$

$= \left[\frac{1}{6}(2)^2+C\right] - \left[\frac{1}{6}(1)^2+C\right] = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

14) $\int (2+\tan\frac{t}{2})\sec^2\frac{t}{2}\,dt = \int p(2\,dp) = 2\left[\frac{p^2}{2}\right]+C$

$p = 2+\tan\frac{t}{2}$

$dp = \sec^2\frac{t}{2}\left(\frac{1}{2}\right)dt$ $\quad = p^2+C = (2+\tan(\frac{t}{2}))^2+C$

$2\,dp = \sec^2\frac{t}{2}\,dt$

a) $\int_{-\frac{\pi}{2}}^{0}(2+\tan\frac{t}{2})\sec^2\frac{t}{2}\,dt = \left[(2+\tan(\frac{t}{2}))^2+C\right]_{-\frac{\pi}{2}}^{0}$

$= \left[(2+\tan(\frac{(0)}{2}))^2+C\right] - \left[(2+\tan(\frac{(-\frac{\pi}{2})}{2}))^2+C\right]$

$= \left[(2+\tan(0))^2\right] - \left[(2+\tan(\frac{-\pi}{4}))^2\right] = \left[(2+(0))^2\right] - \left[(2+(-1))^2\right]$

$= 4-1 = 3$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2+\tan\frac{t}{2})\sec^2\frac{t}{2}\,dt = \left[(2+\tan(\frac{t}{2}))^2+C\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \left[(2+\tan(\frac{(\frac{\pi}{2})}{2}))^2+C\right] - \left[(2+\tan(\frac{(-\frac{\pi}{2})}{2})+C\right]$

$= \left[(2+\tan(\frac{\pi}{4}))^2\right] - \left[(2+\tan(\frac{-\pi}{4}))^2\right] = \left[(2+(1))^2\right] - \left[(2+(-1))^2\right] = 9-1 = 8$

$p = 2+\tan\frac{t}{2}$ $\qquad t=0 \Rightarrow p = 2+\tan\left(\frac{(0)}{2}\right) = 2+\tan(0) = 2+0 = 2$

$dp = \sec^2\frac{t}{2}\left(\frac{1}{2}\right)dt$ $\qquad t=\frac{-\pi}{2} \Rightarrow p = 2+\tan\left(\frac{(-\frac{\pi}{2})}{2}\right) = 2+\tan(\frac{-\pi}{4}) = 2+(-1) = 1$

$2\,dp = \sec^2\frac{t}{2}\,dt$ $\qquad t=\frac{\pi}{2} \Rightarrow p = 2+\tan\left(\frac{(\frac{\pi}{2})}{2}\right) = 2+\tan(\frac{\pi}{4}) = 2+(1) = 3$

a) $\int_{-\frac{\pi}{2}}^{0}(2+\tan\frac{t}{2})\sec^2\frac{t}{2}\,dt = \int_1^2 p(2\,dp) = \int_1^2 2p\,dp = \left[p^2+C\right]_1^2$

$= \left[(2)^2+C\right] - \left[(1)^2+C\right] = [4]-[1] = 3$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(2+\tan\frac{t}{2})\sec^2\frac{t}{2}\,dt = \int_1^3 p(2\,dp) = \int_1^3 2p\,dp = \left[p^2+C\right]_1^3$

$= \left[(3)^2+C\right] - \left[(1)^2+C\right] = [9]-[1] = 8$

16) $$\int \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int \frac{1}{(1+\sqrt{y})^2}\left(\frac{1}{2\sqrt{y}}dy\right) = \int \frac{1}{p^2}\,dp$$

$p = 1+\sqrt{y} = 1+y^{\frac{1}{2}}$

$dp = \frac{1}{2}y^{-\frac{1}{2}}dy$

$dp = \frac{1}{2\sqrt{y}}dy$

$$= \int p^{-2}\,dp = \left[\frac{p^{-1}}{-1}\right] + C = \frac{-1}{p} + C$$

$$= \frac{-1}{(1+\sqrt{y})} + C$$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \left[\frac{-1}{(1+\sqrt{y})} + C\right]_1^4 = \left[\frac{-1}{(1+\sqrt{(4)})} + C\right] - \left[\frac{-1}{(1+\sqrt{(1)})} + C\right]$$

$$= \left[\frac{-1}{1+2}\right] - \left[\frac{-1}{1+1}\right] = \frac{-1}{3} + \frac{1}{2} = \frac{-2}{6} + \frac{3}{6} = \frac{1}{6}$$

$p = 1+\sqrt{y} = 1+y^{\frac{1}{2}}$

$dp = \frac{1}{2}y^{-\frac{1}{2}}dy$

$dp = \frac{1}{2\sqrt{y}}dy$

$y=1 \Rightarrow p = 1+\sqrt{(1)} = 1+1 = 2$

$y=4 \Rightarrow p = 1+\sqrt{(4)} = 1+2 = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int_1^4 \frac{1}{(1+\sqrt{y})^2}\left(\frac{1}{2\sqrt{y}}dy\right) = \int_2^3 \frac{1}{p^2}\,dp$$

$$= \int_2^3 p^{-2}\,dp = \left[\frac{p^{-1}}{-1} + C\right]_2^3 = \left[\frac{-1}{p} + C\right]_2^3$$

$$= \left[\frac{-1}{(3)} + C\right] - \left[\frac{-1}{(2)} + C\right] = \frac{-1}{3} + \frac{1}{2} = \frac{-2}{6} + \frac{3}{6} = \frac{1}{6}$$

18) $\int \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta = \int \frac{1}{\tan^5\left(\frac{\theta}{6}\right)}\left(\sec^2\left(\frac{\theta}{6}\right)d\theta\right)$

$p = \tan\left(\frac{\theta}{6}\right)$

$dp = \sec^2\left(\frac{\theta}{6}\right)\left(\frac{1}{6}\right)d\theta$

$6\,dp = \sec^2\left(\frac{\theta}{6}\right)d\theta$

$= \int \frac{1}{p^5}(6\,dp) = 6\int p^{-5}\,dp$

$= 6\left[\frac{p^{-4}}{-4}\right] + C = \frac{-3}{2p^4} + C$

$= \frac{-3}{2\tan^4\left(\frac{\theta}{6}\right)} + C$

$$\int_{\pi}^{\frac{3\pi}{2}} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta = \left[\frac{-3}{2\tan^4\left(\frac{\theta}{6}\right)} + C\right]_{\pi}^{\frac{3\pi}{2}}$$

$$= \left[\frac{-3}{2\tan^4\left(\frac{\left(\frac{3\pi}{2}\right)}{6}\right)} + C\right] - \left[\frac{-3}{2\tan^4\left(\frac{(\pi)}{6}\right)} + C\right] = \left[\frac{-3}{2\tan^4\left(\frac{\pi}{4}\right)}\right] - \left[\frac{-3}{2\tan^4\left(\frac{\pi}{6}\right)}\right]$$

$$= \left[\frac{-3}{2(1)^4}\right] - \left[\frac{-3}{2\left(\frac{1}{\sqrt{3}}\right)^4}\right] = \left[\frac{-3}{2}\right] - \left[\frac{-3}{2\left(\frac{1}{9}\right)}\right] = \frac{-3}{2} + \frac{27}{2} = \frac{24}{2} = 12$$

$p = \tan\left(\frac{\theta}{6}\right)$ $\qquad \theta = \pi \Rightarrow p = \tan\left(\frac{(\pi)}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$dp = \sec^2\left(\frac{\theta}{6}\right)\left(\frac{1}{6}\right)d\theta$ $\qquad \theta = \frac{3\pi}{2} \Rightarrow p = \tan\left(\frac{\left(\frac{3\pi}{2}\right)}{6}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$6\,dp = \sec^2\left(\frac{\theta}{6}\right)d\theta$

$$\int_{\pi}^{\frac{3\pi}{2}} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta = \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{\tan^5\left(\frac{\theta}{6}\right)}\left(\sec^2\left(\frac{\theta}{6}\right)d\theta\right) = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{p^5}(6\,dp)$$

$$= 6\int_{\frac{1}{\sqrt{3}}}^{1} p^{-5}\,dp = \left[6\left(\frac{p^{-4}}{-4}\right) + C\right]_{\frac{1}{\sqrt{3}}}^{1} = \left[\frac{-3}{2p^4} + C\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \left[\frac{-3}{2(1)^4} + C\right] - \left[\frac{-3}{2\left(\frac{1}{\sqrt{3}}\right)^4} + C\right] = \left[\frac{-3}{2}\right] - \left[\frac{-3}{2\left(\frac{1}{9}\right)}\right]$$

$$= \frac{-3}{2} + \frac{27}{2} = \frac{24}{2} = 12$$

20) $\int (1-\sin 2t)^{\frac{3}{2}} \cos 2t \, dt = \int p^{\frac{3}{2}} \left(\frac{-1}{2}\, dp\right)$

$p = 1 - \sin 2t$

$dp = -[\cos 2t\,(2)]\,dt$

$-\frac{1}{2}\, dp = \cos 2t \, dt$

$= \frac{-1}{2}\left[\frac{p^{\frac{5}{2}}}{\frac{5}{2}}\right] + C = \frac{-1}{5}\left(\sqrt{p}\right)^5 + C$

$= \frac{-1}{5}\left(\sqrt{1-\sin 2t}\right)^5 + C$

$\int_0^{\frac{\pi}{4}} (1-\sin 2t)^{\frac{3}{2}} \cos 2t \, dt = \left[\frac{-1}{5}\left(\sqrt{1-\sin 2t}\right)^5 + C\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{-1}{5}\left(\sqrt{1-\sin(2(\frac{\pi}{4}))}\right)^5 + C\right] - \left[\frac{-1}{5}\left(\sqrt{1-\sin(2(0))}\right)^5 + C\right]$

$= \left[\frac{-1}{5}\left(\sqrt{1-\sin(\frac{\pi}{2})}\right)^5\right] - \left[\frac{-1}{5}\left(\sqrt{1-\sin(0)}\right)^5\right] = \left[\frac{-1}{5}\left(\sqrt{1-(1)}\right)^5\right] - \left[\frac{-1}{5}\left(\sqrt{1-(0)}\right)^5\right]$

$= \left[\frac{-1}{5}(0)^5\right] - \left[\frac{-1}{5}(1)^5\right] = \frac{1}{5}$

$p = 1 - \sin 2t$ $\qquad t = 0 \Rightarrow p = 1 - \sin(2(0)) = 1 - (0) = 1$

$dp = -2\cos 2t \, dt$ $\qquad t = \frac{\pi}{4} \Rightarrow p = 1 - \sin\left(2\left(\frac{\pi}{4}\right)\right) = 1 - \sin\left(\frac{\pi}{2}\right) = 1 - (1) = 0$

$-\frac{1}{2}\, dp = \cos 2t \, dt$

$\int_0^{\frac{\pi}{4}} (1-\sin 2t)^{\frac{3}{2}} \cos 2t \, dt = \int_1^0 p^{\frac{3}{2}} \left(\frac{-1}{2}\, dp\right) = \left[\frac{-1}{2}\left(\frac{p^{\frac{5}{2}}}{\frac{5}{2}}\right) + C\right]_1^0$

$= \left[\frac{-1}{5}\left(\sqrt{p}\right)^5 + C\right]_1^0 = \left[\frac{-1}{5}\left(\sqrt{(0)}\right)^5 + C\right] - \left[\frac{-1}{5}\left(\sqrt{(1)}\right)^5 + C\right]$

$= \frac{1}{5}$

22) $\int (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

$p = y^3+6y^2-12y+9$

$dp = 3y^2+12y-12\,dy$

$dp = 3(y^2+4y-4)\,dy$

$\frac{1}{3}\,dp = (y^2+4y-4)\,dy$

$= \int p^{-\frac{1}{2}}\left(\frac{1}{3}dp\right) = \frac{1}{3}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$= \frac{2}{3}\sqrt{p} + C$

$= \frac{2}{3}\sqrt{y^3+6y^2-12y+9} + C$

$$\int_0^1 (y^3+6y^2-12y+9)^{-\frac{1}{2}}(y^2+4y-4)\,dy = \left[\frac{2}{3}\sqrt{y^3+6y^2-12y+9} + C\right]_0^1$$

$$= \left[\frac{2}{3}\sqrt{(1)^3+6(1)^2-12(1)+9} + C\right] - \left[\frac{2}{3}\sqrt{(0)^3+6(0)^2-12(0)+9} + C\right]$$

$$= \left[\frac{2}{3}\sqrt{4}\right] - \left[\frac{2}{3}\sqrt{9}\right] = \frac{4}{3} - \frac{6}{3} = \frac{-2}{3}$$

$p = y^3+6y^2-12y+9$ $\qquad y=0 \Rightarrow p = (0)^3+6(0)^2-12(0)+9 = 9$

$dp = 3y^2+12y-12\,dy$ $\qquad y=1 \Rightarrow p = (1)^3+6(1)^2-12(1)+9 = 4$

$dp = 3(y^2+4y-4)\,dy$

$\frac{1}{3}\,dp = (y^2+4y-4)\,dy$

$$\int_0^1 (y^3+6y^2-12y+9)^{-\frac{1}{2}}(y^2+4y-4)\,dy = \int_9^4 p^{-\frac{1}{2}}\left(\frac{1}{3}dp\right)$$

$$= \left[\frac{1}{3}\left(\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right) + C\right]_9^4 = \left[\frac{2}{3}\sqrt{p} + C\right]_9^4$$

$$= \left[\frac{2}{3}\sqrt{(4)} + C\right] - \left[\frac{2}{3}\sqrt{(9)} + C\right] = \frac{4}{3} - \frac{6}{3} = \frac{-2}{3}$$

24) $\int t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt = \int \sin^2\left(1+\frac{1}{t}\right)\left(t^{-2}dt\right)$

$p = 1+\frac{1}{t} = 1+t^{-1}$

$dp = -1t^{-2}dt$

$-1dp = t^{-2}dt$

$= \int \sin^2 p\,(-1dp) = -1\int\left(\frac{1-\cos(2p)}{2}\right)dp$

$= \int \frac{-1}{2}dp + \int \frac{1}{2}\cos(2p)\,dp$

$q = 2p \quad dq = 2dp$

$\frac{1}{2}dq = dp$

$= \int -\frac{1}{2}dp + \int \frac{1}{2}\cos q\left(\frac{1}{2}dq\right)$

$= \frac{-1}{2}p + \frac{1}{4}\sin q + C = \frac{-1}{2}p + \frac{1}{4}\sin(2p) + C$

$= \frac{-1}{2}\left(1+\frac{1}{t}\right) + \frac{1}{4}\sin\left(2\left(1+\frac{1}{t}\right)\right) + C$

$\int_{-1}^{-\frac{1}{2}} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt = \left[\frac{-1}{2}\left(1+\frac{1}{t}\right)+\frac{1}{4}\sin\left(2\left(1+\frac{1}{t}\right)\right)+C\right]_{-1}^{-\frac{1}{2}}$

$= \left[\frac{-1}{2}\left(1+\frac{1}{\left(-\frac{1}{2}\right)}\right)+\frac{1}{4}\sin\left(2\left(1+\frac{1}{\left(-\frac{1}{2}\right)}\right)\right)+C\right] - \left[\frac{-1}{2}\left(1+\frac{1}{(-1)}\right)+\frac{1}{4}\sin\left(2\left(1+\frac{1}{(-1)}\right)\right)+C\right]$

$= \left[\frac{-1}{2}(1-2)+\frac{1}{4}\sin(2(1-2))\right] - \left[\frac{-1}{2}(0)+\frac{1}{4}\sin(2(0))\right]$

$= \left[\frac{-1}{2}(-1)+\frac{1}{4}\sin(-2)\right] - [0+0] = \frac{1}{2}+\frac{1}{4}\sin(-2)$

$p = 1+\frac{1}{t} = 1+t^{-1}$ $\qquad t=-1 \Rightarrow p = 1+\frac{1}{(-1)} = 0$

$dp = -1t^{-2}dt$ $\qquad t = \frac{-1}{2} \Rightarrow p = 1+\frac{1}{\left(\frac{-1}{2}\right)} = 1-2 = -1$

$-1dp = t^{-2}dt$

$\int_{-1}^{-\frac{1}{2}} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt = \int_{-1}^{-\frac{1}{2}}\sin^2\left(1+\frac{1}{t}\right)\left(t^{-2}dt\right) = \int_0^{-1}\sin^2 p\,(-1dp)$

$= \int_0^{-1} -\left(\frac{1-\cos(2p)}{2}\right)dp = \int_0^{-1}\frac{-1}{2}dp + \int_0^{-1}\frac{1}{2}\cos(2p)\,dp$

$q = 2p \qquad p=0 \Rightarrow q = 2(0) = 0$

$dq = 2dp \qquad p=-1 \Rightarrow q = 2(-1) = -2$

$\frac{1}{2}dq = dp$

$= \int_0^{-1}\frac{-1}{2}dp + \int_0^{-2}\frac{1}{2}\cos q\left(\frac{1}{2}dq\right)$

$= \left[\frac{-1}{2}p + C\right]_0^{-1} + \left[\frac{1}{4}\sin q + D\right]_0^{-2}$

$= \left\{\left[\frac{-1}{2}(-1)+C\right] - \left[\frac{-1}{2}(0)+C\right]\right\} + \left\{\left[\frac{1}{4}\sin(-2)+D\right] - \left[\frac{1}{4}\sin(0)+D\right]\right\}$

$= \frac{1}{2} + \frac{1}{4}\sin(-2)$

26) $\int (1+e^{\cot\theta})\csc^2\theta\, d\theta = \int (1+e^{p})(-1\,dp)$

$p = \cot\theta$ $\quad = -\{[p]+[e^{p}]\} + C$

$dp = -\csc^2\theta\, d\theta$ $\quad = -\cot\theta - e^{\cot\theta} + C$

$-1\,dp = \csc^2\theta\, d\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+e^{\cot\theta})\csc^2\theta\, d\theta = \left[-\cot\theta - e^{\cot\theta} + C\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[-\cot\left(\tfrac{\pi}{2}\right) - e^{\cot\left(\frac{\pi}{2}\right)} + C\right] - \left[-\cot\left(\tfrac{\pi}{4}\right) - e^{\cot\left(\frac{\pi}{4}\right)} + C\right]$$

$$= \left[-(0) - e^{(0)}\right] - \left[-(1) - e^{(1)}\right] = [-1] - [-1-e] = e$$

$p = \cot\theta$

$\theta = \frac{\pi}{4} \Rightarrow p = \cot\left(\frac{\pi}{4}\right) = 1$

$dp = -\csc^2\theta\, d\theta$

$\theta = \frac{\pi}{2} \Rightarrow p = \cot\left(\frac{\pi}{2}\right) = 0$

$-1\,dp = \csc^2\theta\, d\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+e^{\cot\theta})\csc^2\theta\, d\theta = \int_1^0 (1+e^{p})(-1\,dp)$$

$$= \int_1^0 (-1-e^{p})\, dp = \left[-p - e^{p} + C\right]_1^0$$

$$= \left[-(0) - e^{(0)} + C\right] - \left[-(1) - e^{(1)} + C\right]$$

$$= [-1] - [-1-e] = e$$

$$28)\ \int \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = \int \frac{1}{1-4\cos\theta}(4\sin\theta\,d\theta) = \int \frac{1}{p}\,dp$$

$p = 1-4\cos\theta$ $\qquad = \ln|p| + C = \ln|1-4\cos\theta| + C$

$dp = -4[-\sin\theta]\,d\theta$

$dp = 4\sin\theta\,d\theta$

$$\int_0^{\frac{\pi}{3}} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = \Big[\ln|1-4\cos\theta| + C\Big]_0^{\frac{\pi}{3}}$$

$$= \Big[\ln|1-4\cos(\tfrac{\pi}{3})| + C\Big] - \Big[\ln|1-4\cos(0)| + C\Big]$$

$$= \Big[\ln|1-4(\tfrac{1}{2})|\Big] - \Big[\ln|1-4(1)|\Big] = \Big[\ln|1-2|\Big] - \Big[\ln|1-4|\Big]$$

$$= \ln|-1| - \ln|-3| = \ln 1 - \ln 3 = 0 - \ln 3 = -\ln 3$$

$p = 1-4\cos\theta$ $\qquad \theta = 0 \Rightarrow p = 1-4\cos(0) = 1-4(1) = -3$

$dp = -4[-\sin\theta]\,d\theta$ $\qquad \theta = \frac{\pi}{3} \Rightarrow p = 1-4\cos(\frac{\pi}{3}) = 1-4(\frac{1}{2}) = 1-2 = -1$

$dp = 4\sin\theta\,d\theta$

$$\int_0^{\frac{\pi}{3}} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{1-4\cos\theta}(4\sin\theta\,d\theta) = \int_{-3}^{-1} \frac{1}{p}(dp)$$

$$= \Big[\ln|p| + C\Big]_{-3}^{-1} = \Big[\ln|(-1)| + C\Big] - \Big[\ln|(-3)| + C\Big]$$

$$= \ln|-1| - \ln|-3| = \ln 1 - \ln 3 = 0 - \ln 3 = -\ln 3$$

30) $\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\left(\frac{1}{x}dx\right) = \int \frac{1}{p}\,dp$

$p = \ln x \qquad = \ln|p| + C = \ln|\ln x| + C$

$dp = \frac{1}{x}dx$

$\int_2^4 \frac{dx}{x\ln x} = \Big[\ln|\ln x| + C\Big]_2^4 = \Big[\ln|\ln(4)| + C\Big] - \Big[\ln|\ln(2)| + C\Big]$

$= \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln(2^2)}{\ln 2}\right)$

$= \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$

$p = \ln x \qquad x = 2 \Rightarrow p = \ln(2) = \ln 2$

$dp = \frac{1}{x}dx \qquad x = 4 \Rightarrow p = \ln(4) = \ln 4$

$\int_2^4 \frac{dx}{x\ln x} = \int_2^4 \frac{1}{\ln x}\left(\frac{1}{x}dx\right) = \int_{\ln 2}^{\ln 4} \frac{1}{p}\,dp = \Big[\ln|p| + C\Big]_{\ln 2}^{\ln 4}$

$= \Big[\ln|(\ln 4)| + C\Big] - \Big[\ln|(\ln 2)| + C\Big]$

$= \ln(\ln 4) - \ln(\ln 2) = \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{\ln(2^2)}{\ln 2}\right)$

$= \ln\left(\frac{2\ln 2}{\ln 2}\right) = \ln 2$

32) $\int \frac{dx}{2x\sqrt{\ln x}} = \int \frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}dx\right) = \int \frac{1}{2\sqrt{p}}\,dp$

$p = \ln x \qquad = \int \frac{1}{2}p^{-\frac{1}{2}}dp = \frac{1}{2}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = \sqrt{p} + C$

$dp = \frac{1}{x}dx \qquad = \sqrt{\ln x} + C$

$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \left[\sqrt{\ln x} + C\right]_2^{16} = \left[\sqrt{\ln(16)} + C\right] - \left[\sqrt{\ln(2)} + C\right]$

$= \left[\sqrt{\ln 16}\right] - \left[\sqrt{\ln 2}\right] = \sqrt{\ln(2^4)} - \sqrt{\ln 2} = \sqrt{4\ln 2} - \sqrt{\ln 2}$

$= 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$

$p = \ln x \qquad x = 2 \Rightarrow p = \ln(2) = \ln 2$

$dp = \frac{1}{x}dx \qquad x = 16 \Rightarrow p = \ln(16) = \ln(2^4) = 4\ln 2$

$\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} = \int_2^{16} \frac{1}{2\sqrt{\ln x}}\left(\frac{1}{x}dx\right) = \int_{\ln 2}^{4\ln 2} \frac{1}{2\sqrt{p}}\,dp$

$= \int_{\ln 2}^{4\ln 2} \frac{1}{2}p^{-\frac{1}{2}}dp = \left[\frac{1}{2}\left(\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right) + C\right]_{\ln 2}^{4\ln 2} = \left[\sqrt{p} + C\right]_{\ln 2}^{4\ln 2}$

$= \left[\sqrt{(4\ln 2)} + C\right] - \left[\sqrt{(\ln 2)} + C\right]$

$= \sqrt{4\ln 2} - \sqrt{\ln 2} = 2\sqrt{\ln 2} - \sqrt{\ln 2} = \sqrt{\ln 2}$

34) $\int \cot t\,dt = \int \frac{\cos t}{\sin t}\,dt = \int \frac{1}{\sin t}(\cos t\,dt)$

$p = \sin t \qquad = \int \frac{1}{p}\,dp = \ln|p| + C$

$dp = \cos t\,dt \qquad = \ln|\sin t| + C$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\,dt = \Big[\ln|\sin t| + C\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \Big[\ln|\sin(\tfrac{\pi}{2})| + C\Big] - \Big[\ln|\sin(\tfrac{\pi}{4})| + C\Big]$$

$$= \Big[\ln|1|\Big] - \Big[\ln|\tfrac{1}{\sqrt{2}}|\Big] = \ln 1 - \ln(\tfrac{1}{\sqrt{2}}) = -\ln(\tfrac{1}{\sqrt{2}})$$

$$= -\{\ln 1 - \ln\sqrt{2}\} = -\{0 - \ln\sqrt{2}\} = \ln\sqrt{2} = \tfrac{1}{2}\ln 2$$

$p = \sin t \qquad t = \frac{\pi}{4} \Rightarrow p = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$dp = \cos t\,dt \qquad t = \frac{\pi}{2} \Rightarrow p = \sin(\frac{\pi}{2}) = 1$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos t}{\sin t}\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin t}(\cos t\,dt) = \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{p}\,dp$$

$$= \Big[\ln|p| + C\Big]_{\frac{1}{\sqrt{2}}}^{1} = \Big[\ln|(1)| + C\Big] - \Big[\ln|(\tfrac{1}{\sqrt{2}})| + C\Big]$$

$$= \Big[0\Big] - \Big[\ln(\tfrac{1}{\sqrt{2}})\Big] = -\ln(\tfrac{1}{\sqrt{2}}) = -\{\ln 1 - \ln\sqrt{2}\}$$

$$= -\{0 - \ln\sqrt{2}\} = \ln\sqrt{2} = \tfrac{1}{2}\ln 2$$

36) $\int 6\tan 3x\,dx = \int 6\frac{\sin 3x}{\cos 3x}\,dx = \int \frac{2}{\cos 3x}(3\sin 3x\,dx)$

$p = \cos 3x$ $\qquad = \int \frac{2}{p}(-1\,dp) = -2\ln|p| + C$

$dp = [-\sin 3x(3)]\,dx$ $\qquad = -2\ln|\cos 3x| + C$

$-1\,dp = 3\sin 3x\,dx$

$$\int_0^{\frac{\pi}{12}} 6\tan 3x\,dx = \left[-2\ln|\cos 3x| + C\right]_0^{\frac{\pi}{12}}$$

$$= \left[-2\ln\left|\cos\left(3\left(\tfrac{\pi}{12}\right)\right)\right| + C\right] - \left[-2\ln|\cos(3(0))| + C\right]$$

$$= \left[-2\ln\left|\cos\left(\tfrac{\pi}{4}\right)\right|\right] - \left[-2\ln|\cos(0)|\right] = \left[-2\ln\left|\left(\tfrac{1}{\sqrt{2}}\right)\right|\right] - \left[-2\ln|(1)|\right]$$

$$= \left[-2\ln\left(\tfrac{1}{\sqrt{2}}\right)\right] - \left[-2(0)\right] = -2\ln\left(\tfrac{1}{\sqrt{2}}\right) = -2\left\{\ln(1) - \ln(\sqrt{2})\right\}$$

$$= -2\left\{0 - \ln\sqrt{2}\right\} = 2\ln\sqrt{2} = 2\left(\tfrac{1}{2}\ln 2\right) = \ln 2$$

$p = \cos 3x$ $\qquad x = 0 \Rightarrow p = \cos(3(0)) = \cos(0) = 1$

$dp = -3\sin 3x\,dx$ $\qquad x = \frac{\pi}{12} \Rightarrow p = \cos\left(3\left(\frac{\pi}{12}\right)\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$-1\,dp = 3\sin 3x\,dx$

$$\int_0^{\frac{\pi}{12}} 6\tan 3x\,dx = \int_0^{\frac{\pi}{12}} 6\frac{\sin 3x}{\cos 3x}\,dx = \int_0^{\frac{\pi}{12}} \frac{2}{\cos 3x}(3\sin 3x\,dx) = \int_1^{\frac{1}{\sqrt{2}}} \frac{2}{p}(-1\,dp)$$

$$= \left[-2\ln|p| + C\right]_1^{\frac{1}{\sqrt{2}}} = \left[-2\ln\left|\left(\tfrac{1}{\sqrt{2}}\right)\right| + C\right] - \left[-2\ln|(1)| + C\right]$$

$$= \left[-2\ln\left(\tfrac{1}{\sqrt{2}}\right)\right] - \left[-2(0)\right] = -2\ln\left(\tfrac{1}{\sqrt{2}}\right) = -2\left\{\ln 1 - \ln\sqrt{2}\right\}$$

$$= -2\left\{0 - \ln\sqrt{2}\right\} = 2\ln\sqrt{2} = 2\left(\tfrac{1}{2}\ln 2\right) = \ln 2$$

38) $\int \frac{\csc^2 x\,dx}{1+(\cot x)^2} = \int \frac{1}{1+(\cot x)^2}(\csc^2 x\,dx) = \int \frac{1}{1+p^2}(-1\,dp)$

$p = \cot x$

$dp = -\csc^2 x\,dx$

$-1\,dp = \csc^2 x\,dx$

using the formula

$= -\left[\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right] + C$

$= -\tan^{-1}(\cot x) + C$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\csc^2 x\,dx}{1+(\cot x)^2} = \left[-\tan^{-1}(\cot x) + C\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \left[-\tan^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right) + C\right] - \left[-\tan^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right) + C\right]$

$= \left[-\tan^{-1}\left(\frac{1}{1}\right)\right] - \left[-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)\right] = \left[-\left(\frac{\pi}{4}\right)\right] - \left[-\left(\frac{\pi}{3}\right)\right] = \frac{-\pi}{4} + \frac{\pi}{3}$

$= \frac{-3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{12}$

$p = \cot x$ $\quad x = \frac{\pi}{6} \Rightarrow p = \cot\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$

$dp = -\csc^2 x\,dx$ $\quad x = \frac{\pi}{4} \Rightarrow p = \cot\left(\frac{\pi}{4}\right) = \frac{1}{1} = 1$

$-1\,dp = \csc^2 x\,dx$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\csc^2 x\,dx}{1+(\cot x)^2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{1+(\cot x)^2}(\csc^2 x\,dx) = \int_{\sqrt{3}}^{1} \frac{1}{1+p^2}(-1\,dp)$

$= \left[-\left(\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right) + C\right]_{\sqrt{3}}^{1} = \left[-\tan^{-1} p + C\right]_{\sqrt{3}}^{1}$

$= \left[-\tan^{-1}(1) + C\right] - \left[-\tan^{-1}(\sqrt{3}) + C\right]$

$= \left[-\left(\frac{\pi}{4}\right)\right] - \left[-\left(\frac{\pi}{3}\right)\right] = \frac{-\pi}{4} + \frac{\pi}{3} = \frac{-3\pi}{12} + \frac{4\pi}{12} = \frac{\pi}{12}$

40) $\int \frac{4\,dt}{t(1+\ln^2 t)} = \int 4\left(\frac{1}{1+(\ln t)^2}\right)\left(\frac{1}{t}\,dt\right) = \int 4\left(\frac{1}{1+p^2}\right)dp$

$p = \ln t$

$dp = \frac{1}{t}\,dt$

using the formula $= 4\left[\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right] + C = 4\tan^{-1} p + C$

$= 4\tan^{-1}(\ln t) + C$

$\int_1^{e^{\frac{\pi}{4}}} \frac{4\,dt}{t(1+\ln^2 t)} = \left[4\tan^{-1}(\ln t) + C\right]_1^{e^{\frac{\pi}{4}}}$

$= \left[4\tan^{-1}(\ln(e^{\frac{\pi}{4}})) + C\right] - \left[4\tan^{-1}(\ln(1)) + C\right]$

$= \left[4\tan^{-1}\left(\frac{\pi}{4}\right)\right] - \left[4\tan^{-1}(0)\right] = 4\tan^{-1}\left(\frac{\pi}{4}\right) - 4(0) = 4\tan^{-1}\left(\frac{\pi}{4}\right)$

$p = \ln t \qquad t = 1 \Rightarrow p = \ln(1) = 0$

$dp = \frac{1}{t}\,dt \qquad t = e^{\frac{\pi}{4}} \Rightarrow p = \ln(e^{\frac{\pi}{4}}) = \frac{\pi}{4}$

$\int_1^{e^{\frac{\pi}{4}}} \frac{4\,dt}{t(1+\ln^2 t)} = \int_1^{e^{\frac{\pi}{4}}} 4\left(\frac{1}{1+(\ln t)^2}\right)\left(\frac{1}{t}\,dt\right) = \int_0^{\frac{\pi}{4}} 4\left(\frac{1}{1+p^2}\right)dp$

$= \left[4\left(\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right) + C\right]_0^{\frac{\pi}{4}} = \left[4\tan^{-1} p + C\right]_0^{\frac{\pi}{4}}$

$= \left[4\tan^{-1}\left(\frac{\pi}{4}\right) + C\right] - \left[4\tan^{-1}(0) + C\right]$

$= 4\tan^{-1}\left(\frac{\pi}{4}\right) - 4(0)$

$= 4\tan^{-1}\left(\frac{\pi}{4}\right)$

42) $\int \frac{ds}{\sqrt{9-4s^2}} = \int \frac{1}{\sqrt{(3)^2-(2s)^2}}\, ds = \int \frac{1}{\sqrt{(3)^2-p^2}}\left(\frac{1}{2}dp\right)$

$p = 2s$
$dp = 2\,ds$
$\frac{1}{2}dp = ds$

using the formula $= \frac{1}{2}\left[\sin^{-1}\left(\frac{p}{3}\right)\right] + C = \frac{1}{2}\sin^{-1}\left(\frac{2s}{3}\right) + C$

$$\int_0^{\frac{3\sqrt{2}}{4}} \frac{ds}{\sqrt{9-4s^2}} = \left[\frac{1}{2}\sin^{-1}\left(\frac{2s}{3}\right) + C\right]_0^{\frac{3\sqrt{2}}{4}} = \left[\frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{3\sqrt{2}}{4}\right)}{3}\right) + C\right] - \left[\frac{1}{2}\sin^{-1}\left(\frac{2(0)}{3}\right) + C\right]$$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{3\sqrt{2}}{6}\right)\right] - \left[\frac{1}{2}\sin^{-1}(0)\right] = \frac{1}{2}\sin^{-1}\left(\frac{3\sqrt{2}}{6}\right)$$

$p = 2s$ $\qquad s = 0 \Longrightarrow p = 2(0) = 0$
$dp = 2\,ds$
$\frac{1}{2}dp = ds$ $\qquad s = \frac{3\sqrt{2}}{4} \Longrightarrow p = 2\left(\frac{3\sqrt{2}}{4}\right) = \frac{3\sqrt{2}}{2}$

$$\int_0^{\frac{3\sqrt{2}}{4}} \frac{ds}{\sqrt{9-4s^2}} = \int_0^{\frac{3\sqrt{2}}{4}} \frac{1}{\sqrt{(3)^2-(2s)^2}}\, ds = \int_0^{\frac{3\sqrt{2}}{2}} \frac{1}{\sqrt{(3)^2-p^2}}\left(\frac{1}{2}dp\right)$$

$$= \left[\frac{1}{2}\left(\sin^{-1}\left(\frac{p}{3}\right)\right) + C\right]_0^{\frac{3\sqrt{2}}{2}} = \left[\frac{1}{2}\sin^{-1}\left(\frac{p}{3}\right) + C\right]_0^{\frac{3\sqrt{2}}{2}}$$

$$= \left[\frac{1}{2}\sin^{-1}\left(\frac{\left(\frac{3\sqrt{2}}{2}\right)}{3}\right) + C\right] - \left[\frac{1}{2}\sin^{-1}\left(\frac{(0)}{3}\right) + C\right]$$

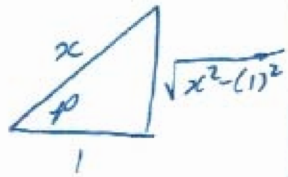
$$= \frac{1}{2}\sin^{-1}\left(\frac{3\sqrt{2}}{6}\right) - \frac{1}{2}(0) = \frac{1}{2}\sin^{-1}\left(\frac{3\sqrt{2}}{6}\right)$$

44) $$\int \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}} = \int \cos(\sec^{-1}x)\left(\frac{1}{x\sqrt{x^2-1}}\,dx\right)$$

$\varphi = \sec^{-1}x$

$\Downarrow$

$\sec\varphi = x = \frac{x}{1}$



$\sec\varphi\tan\varphi\,\frac{d\varphi}{dx} = 1$

$$\frac{d\varphi}{dx} = \frac{1}{\sec\varphi\tan\varphi} = \frac{1}{\left(\frac{x}{1}\right)\left(\frac{\sqrt{x^2-(1)^2}}{1}\right)}$$

$$d\varphi = \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= \int \cos\varphi\,d\varphi = \sin\varphi + C$$

$$= \sin(\sec^{-1}x) + C$$

$$\int_{\frac{2}{\sqrt{3}}}^{2} \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}} = \Big[\sin(\sec^{-1}x) + C\Big]_{\frac{2}{\sqrt{3}}}^{2}$$

$$= \Big[\sin(\sec^{-1}(2)) + C\Big] - \Big[\sin\left(\sec^{-1}\left(\tfrac{2}{\sqrt{3}}\right)\right) + C\Big]$$

$$= \Big[\sin\left(\tfrac{\pi}{3}\right)\Big] - \Big[\sin\left(\tfrac{\pi}{6}\right)\Big] = \Big[\tfrac{\sqrt{3}}{2}\Big] - \Big[\tfrac{1}{2}\Big] = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

$\varphi = \sec^{-1}x$

$d\varphi = \frac{1}{x\sqrt{x^2-1}}\,dx$ {see above}

$x = \frac{2}{\sqrt{3}} \Rightarrow \varphi = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

$x = 2 \Rightarrow \varphi = \sec^{-1}(2) = \frac{\pi}{3}$

$$\int_{\frac{2}{\sqrt{3}}}^{2} \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}} = \int_{\frac{2}{\sqrt{3}}}^{2} \cos(\sec^{-1}x)\left(\frac{1}{x\sqrt{x^2-1}}\,dx\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos\varphi\,d\varphi$$

$$= \Big[\sin\varphi + C\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \Big[\sin\left(\tfrac{\pi}{3}\right) + C\Big] - \Big[\sin\left(\tfrac{\pi}{6}\right) + C\Big]$$

$$= \Big[\tfrac{\sqrt{3}}{2}\Big] - \Big[\tfrac{1}{2}\Big] = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

46) $\int \frac{y\,dy}{\sqrt{5y+1}} = \int \frac{y}{\sqrt{5y+1}}\,dy = \int \frac{\left(\frac{p-1}{5}\right)}{\sqrt{p}}\left(\frac{1}{5}\,dp\right)$

$p = 5y+1 \Rightarrow 5y = p-1$

$dp = 5\,dy \qquad y = \frac{p-1}{5}$

$\frac{1}{5}\,dp = dy$

$= \frac{1}{5^2}\int\left(\frac{p}{\sqrt{p}} - \frac{1}{\sqrt{p}}\right)dp = \frac{1}{25}\int\left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)dp$

$= \frac{1}{25}\left\{\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] - \left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right]\right\} + C$

$= \frac{1}{25}\left\{\frac{2}{3}(\sqrt{p})^3 - 2\sqrt{p}\right\} + C$

$= \frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{5y+1}\right)^3 - 2\sqrt{5y+1}\right\} + C$

$\int_0^3 \frac{y\,dy}{\sqrt{5y+1}} = \left[\frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{5y+1}\right)^3 - 2\sqrt{5y+1}\right\} + C\right]_0^3$

$= \left[\frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{5(3)+1}\right)^3 - 2\sqrt{5(3)+1}\right\} + C\right] - \left[\frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{5(0)+1}\right)^3 - 2\sqrt{5(0)+1}\right\} + C\right]$

$= \left[\frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{16}\right)^3 - 2\sqrt{16}\right\}\right] - \left[\frac{1}{25}\left\{\frac{2}{3}\left(\sqrt{1}\right)^3 - 2\sqrt{1}\right\}\right]$

$= \frac{1}{25}\left\{\left[\frac{2}{3}(4)^3 - 2(4)\right] - \left[\frac{2}{3} - 2\right]\right\} = \frac{1}{25}\left\{\left[\frac{128}{3} - 8\right] - \left[\frac{2}{3} - 2\right]\right\} = \frac{1}{25}\left\{\frac{126}{3} - 6\right\}$

$= \frac{1}{25}\{42 - 6\} = \frac{1}{25}\{36\} = \frac{36}{25}$

$p = 5y+1 \quad y = \frac{p-1}{5} \qquad y = 0 \Rightarrow p = 5(0)+1 = 1$

$\frac{1}{5}\,dp = dy$ {see above} $\qquad y = 3 \Rightarrow p = 5(3)+1 = 15+1 = 16$

$\int_0^3 \frac{y\,dy}{\sqrt{5y+1}} = \int_0^3 \frac{y}{\sqrt{5y+1}}\,dy = \int_1^{16} \frac{\left(\frac{p-1}{5}\right)}{\sqrt{p}}\left(\frac{1}{5}\,dp\right) = \frac{1}{25}\int_1^{16}\left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)dp$

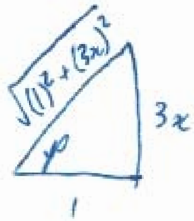
$= \frac{1}{25}\left[\left(\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right) - \left(\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right) + C\right]_1^{16} = \frac{1}{25}\left[\frac{2}{3}(\sqrt{p})^3 - 2\sqrt{p} + C\right]_1^{16}$

$= \frac{1}{25}\left\{\left[\frac{2}{3}\left(\sqrt{(16)}\right)^3 - 2\sqrt{(16)} + C\right] - \left[\frac{2}{3}\left(\sqrt{(1)}\right)^3 - 2\sqrt{(1)} + C\right]\right\}$

$= \frac{1}{25}\left\{\left[\frac{128}{3} - 8\right] - \left[\frac{2}{3} - 2\right]\right\} = \frac{1}{25}\left\{\frac{126}{3} - 6\right\} = \frac{1}{25}\{42 - 6\} = \frac{1}{25}\{36\} = \frac{36}{25}$

$$48)\ \int \frac{\cos(\tan^{-1}3x)}{1+9x^2}\,dx = \int \cos(\tan^{-1}3x)\left(\frac{1}{1+9x^2}\,dx\right)$$

$$p = \tan^{-1}3x$$
$$\Downarrow$$
$$\tan p = 3x = \frac{(3x)}{1}$$



$$= \int \cos p\left(\tfrac{1}{3}\,dp\right) = \tfrac{1}{3}\sin p + C$$
$$= \tfrac{1}{3}\sin(\tan^{-1}3x) + C$$

$$\sec^2 p\,\frac{dp}{dx} = 3$$

$$\frac{dp}{dx} = \frac{3}{\sec^2 p} = \frac{3}{\left(\frac{\sqrt{(1)^2+(3x)^2}}{1}\right)^2} = \frac{3}{1+9x^2} \Rightarrow dp = \frac{3}{1+9x^2}\,dx \Rightarrow \frac{1}{3}\,dp = \frac{1}{1+9x^2}\,dx$$

$$\int_{-\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\cos(\tan^{-1}3x)}{1+9x^2}\,dx = \left[\tfrac{1}{3}\sin(\tan^{-1}(3x)) + C\right]_{-\sqrt{3}}^{\frac{1}{\sqrt{3}}}$$

$$= \left[\tfrac{1}{3}\sin\left(\tan^{-1}3\left(\tfrac{1}{\sqrt{3}}\right)\right) + C\right] - \left[\tfrac{1}{3}\sin\left(\tan^{-1}3(-\sqrt{3})\right) + C\right]$$

$$= \left[\tfrac{1}{3}\sin(\tan^{-1}(\sqrt{3})\right] - \left[\tfrac{1}{3}\sin(\tan^{-1}(-3\sqrt{3})\right] = \left[\tfrac{1}{3}\sin\left(\tfrac{\pi}{3}\right)\right] - \left[\tfrac{1}{3}\sin(\tan^{-1}(-3\sqrt{3})\right]$$

$$= \left[\tfrac{1}{3}\left(\tfrac{\sqrt{3}}{2}\right)\right] - \left[\tfrac{1}{3}\sin(\tan^{-1}(-3\sqrt{3})\right] = \frac{\sqrt{3}}{6} - \frac{1}{3}\sin(\tan^{-1}(-3\sqrt{3}))$$

$$p = \tan^{-1}(3x)$$
$$dp = \frac{3}{1+9x^2}\,dx$$
$$\frac{1}{3}\,dp = \frac{1}{1+9x^2}\,dx \quad \{\text{see above}\}$$

$$x = \tfrac{1}{\sqrt{3}} \Rightarrow p = \tan^{-1}\left(3\left(\tfrac{1}{\sqrt{3}}\right)\right) = \tan^{-1}\left(\tfrac{3}{\sqrt{3}}\right)$$
$$= \tan^{-1}(\sqrt{3}) = \tfrac{\pi}{3}$$
$$x = -\sqrt{3} \Rightarrow p = \tan^{-1}(3(-\sqrt{3})) = \tan^{-1}(-3\sqrt{3})$$

$$\int_{-\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\cos(\tan^{-1}(3x))}{1+9x^2}\,dx = \int_{\tan^{-1}(-3\sqrt{3})}^{\frac{\pi}{3}} \cos p\left(\tfrac{1}{3}\,dp\right) = \left[\tfrac{1}{3}\sin p + C\right]_{\tan^{-1}(-3\sqrt{3})}^{\frac{\pi}{3}}$$

$$= \left[\tfrac{1}{3}\sin\left(\tfrac{\pi}{3}\right) + C\right] - \left[\tfrac{1}{3}\sin(\tan^{-1}(-3\sqrt{3})) + C\right]$$

$$= \left[\tfrac{1}{3}\left(\tfrac{\sqrt{3}}{2}\right)\right] - \left[\tfrac{1}{3}\sin(\tan^{-1}(-3\sqrt{3}))\right] = \frac{\sqrt{3}}{6} - \frac{1}{3}\sin(\tan^{-1}(-3\sqrt{3}))$$

50) Upper Curve: $y=(1-\cos x)\sin x$

lower curve: $y=0$

$$A=\int_0^\pi \{((1-\cos x)\sin x)-(0)\}\,dx = \int_0^\pi (1-\cos x)\sin x\,dx$$

$p = 1-\cos x$

$dp = -[-\sin x]\,dx$

$dp = \sin x\,dx$

$$\int (1-\cos x)\sin x\,dx = \int p\,(dp)$$

$$= \frac{p^2}{2}+C = \frac{1}{2}p^2+C = \frac{1}{2}(1-\cos x)^2+C$$

$$A=\int_0^\pi (1-\cos x)\sin x\,dx = \left[\frac{1}{2}(1-\cos x)^2+C\right]_0^\pi$$

$$=\left[\frac{1}{2}(1-\cos(\pi))^2+C\right]-\left[\frac{1}{2}(1-\cos(0))^2+C\right]$$

$$=\left[\frac{1}{2}(1-(-1))^2\right]-\left[\frac{1}{2}(1-(1))^2\right]$$

$$=\left[\frac{1}{2}(2)^2\right]-\left[\frac{1}{2}(0)^2\right]=2$$

---

52) on $\left(-\pi, \frac{-\pi}{2}\right)$: Upper Curve: $y=0$

lower curve: $y=\frac{\pi}{2}(\cos x)(\sin(\pi+\pi\sin x))$

region$_1$ $=(0)-\left(\frac{\pi}{2}(\cos x)(\sin(\pi+\pi\sin x))\right) = -\frac{\pi}{2}(\cos x)\sin(\pi+\pi\sin x)$

on $\left(\frac{-\pi}{2}, 0\right)$: Upper Curve: $y=\frac{\pi}{2}(\cos x)(\sin(\pi+\pi\sin x))$

lower curve: $y=0$

region$_2$ $=\left(\frac{\pi}{2}(\cos x)(\sin(\pi+\pi\sin x))\right)-(0) = \frac{\pi}{2}(\cos x)\sin(\pi+\pi\sin x)$

52) continued

$$\int \frac{\pi}{2}(\cos x)\sin(\pi+\pi\sin x)\,dx = \int \frac{\pi}{2}\sin(\pi+\pi\sin x)(\cos x\,dx)$$

$p = \pi + \pi \sin x$ | $= \int \frac{\pi}{2}\sin p\left(\frac{1}{\pi}dp\right) = \int \frac{1}{2}\sin p\,dp$

$dp = \pi[\cos x]\,dx$ | $= \frac{1}{2}[-\cos p] + C = \frac{-1}{2}\cos(\pi+\pi\sin x) + C$

$\frac{1}{\pi}dp = \cos x\,dx$

$$A_1 = \int_{-\pi}^{-\frac{\pi}{2}} -\frac{\pi}{2}(\cos x)\sin(\pi+\pi\sin x)\,dx = \left[-\left(\frac{-1}{2}\cos(\pi+\pi\sin x)\right)+C\right]_{-\pi}^{-\frac{\pi}{2}}$$

$$= \left[\frac{1}{2}\cos\left(\pi+\pi\sin\left(\frac{-\pi}{2}\right)\right)+C\right] - \left[\frac{1}{2}\cos(\pi+\pi\sin(-\pi))+C\right]$$

$$= \left[\frac{1}{2}\cos(\pi+\pi(-1))\right] - \left[\frac{1}{2}\cos(\pi+\pi(0))\right] = \left[\frac{1}{2}\cos(0)\right] - \left[\frac{1}{2}\cos(\pi)\right]$$

$$= \left[\frac{1}{2}(1)\right] - \left[\frac{1}{2}(-1)\right] = \left[\frac{1}{2}\right] - \left[\frac{-1}{2}\right] = \frac{1}{2} + \frac{1}{2} = 1$$

$$A_2 = \int_{-\frac{\pi}{2}}^{0} \frac{\pi}{2}(\cos x)\sin(\pi+\pi\sin x)\,dx = \left[\frac{-1}{2}\cos(\pi+\pi\sin x)+C\right]_{-\frac{\pi}{2}}^{0}$$

$$= \left[\frac{-1}{2}\cos(\pi+\pi\sin(0))+C\right] - \left[\frac{-1}{2}\cos\left(\pi+\pi\sin\left(\frac{-\pi}{2}\right)\right)+C\right]$$

$$= \left[\frac{-1}{2}\cos(\pi+\pi(0))\right] - \left[\frac{-1}{2}\cos(\pi+\pi(-1))\right] = \left[\frac{-1}{2}\cos(\pi)\right] - \left[\frac{-1}{2}\cos(0)\right]$$

$$= \left[\frac{-1}{2}(-1)\right] - \left[\frac{-1}{2}(1)\right] = \left[\frac{1}{2}\right] - \left[\frac{-1}{2}\right] = \frac{1}{2} + \frac{1}{2} = 1$$

$$A = A_1 + A_2 = (1) + (1) = 2$$

54) Upper Curve: $y = \frac{1}{2}\sec^2 t$

lower curve: $y = -4\sin^2 t = -4\left(\frac{1-\cos(2t)}{2}\right) = -2(1-\cos(2t))$

$= -2 + 2\cos(2t)$

region $= \left(\frac{1}{2}\sec^2 t\right) - (-2 + 2\cos(2t)) = \frac{1}{2}\sec^2 t + 2 - 2\cos(2t)$

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2}\sec^2 t + 2 - 2\cos(2t)\right) dt = \left[\frac{1}{2}\tan t + 2t - \sin(2t) + C\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \left[\frac{1}{2}\tan\left(\frac{\pi}{3}\right) + 2\left(\frac{\pi}{3}\right) - \sin\left(2\left(\frac{\pi}{3}\right)\right) + C\right] - \left[\frac{1}{2}\tan\left(\frac{-\pi}{3}\right) + 2\left(\frac{-\pi}{3}\right) - \sin\left(2\left(\frac{-\pi}{3}\right)\right) + C\right]$$

$$= \left[\frac{1}{2}\left(\frac{\sqrt{3}}{1}\right) + \frac{2\pi}{3} - \left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{1}{2}\left(\frac{-\sqrt{3}}{1}\right) - \frac{2\pi}{3} - \left(\frac{-\sqrt{3}}{2}\right)\right] = \left[\frac{2\pi}{3}\right] - \left[\frac{-2\pi}{3}\right] = \frac{4\pi}{3}$$

---

56) Upper Curve: $x = y^2$

lower curve: $x = y^3$

region $= (y^2) - (y^3) = y^2 - y^3$

$$A = \int_0^1 (y^2 - y^3)\,dy = \left[\frac{y^3}{3} - \frac{y^4}{4} + C\right]_0^1 = \left[\frac{(1)^3}{3} - \frac{(1)^4}{4} + C\right] - \left[\frac{(0)^3}{3} - \frac{(0)^4}{4} + C\right]$$

$$= \left[\frac{1}{3} - \frac{1}{4}\right] - [0] = \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{1}{12}$$

---

58) Upper Curve: $y = x^2$

lower curve: $y = -2x^4$

region $= (x^2) - (-2x^4) = x^2 + 2x^4$

$$A = \int_{-1}^{1} (x^2 + 2x^4)\,dx = \left[\frac{1}{3}x^3 + \frac{2}{5}x^5 + C\right]_{-1}^{1} = \left[\frac{1}{3}(1)^3 + \frac{2}{5}(1)^5 + C\right] - \left[\frac{1}{3}(-1)^3 + \frac{2}{5}(-1)^5 + C\right]$$

$$= \left[\frac{1}{3} + \frac{2}{5}\right] - \left[\frac{-1}{3} - \frac{2}{5}\right] = \frac{2}{3} + \frac{4}{5} = \frac{10}{15} + \frac{12}{15} = \frac{22}{15}$$

60) best when we set up x as a function of y

Upper Curve: $x+y=2 \Rightarrow x=2-y$ $\{$interval: $0 \le y \le 1\}$

lower curve: $y=x^2 \Rightarrow x=\pm\sqrt{y} \Rightarrow x=+\sqrt{y}=y^{\frac{1}{2}}$

region $=(2-y)-(\sqrt{y})=2-y-\sqrt{y}=2-y-y^{\frac{1}{2}}$

$$A=\int_0^1 (2-y-y^{\frac{1}{2}})\,dy=\left[2y-\frac{y^2}{2}-\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)+C\right]_0^1=\left[2y-\frac{1}{2}y^2-\frac{2}{3}(\sqrt{y})^3+C\right]_0^1$$

$$=\left[2(1)-\frac{1}{2}(1)^2-\frac{2}{3}(\sqrt{(1)})^3+C\right]-\left[2(0)-\frac{1}{2}(0)^2-\frac{2}{3}(\sqrt{(0)})^3+C\right]$$

$$=\left[2-\frac{1}{2}-\frac{2}{3}\right]-\left[0\right]=\frac{12}{6}-\frac{3}{6}-\frac{4}{6}=\frac{5}{6}$$

---

62) on $(-2,0)$ Upper Curve: $y=2x^3-x^2-5x$

lower curve: $y=-x^2+3x$

region$_1=(2x^3-x^2-5x)-(-x^2+3x)=2x^3-x^2-5x+x^2-3x=2x^3-8x$

on $(0,2)$ Upper Curve: $y=-x^2+3x$

lower curve: $y=2x^3-x^2-5x$

region$_2=(-x^2+3x)-(2x^3-x^2-5x)=-x^2+3x-2x^3+x^2+5x=8x-2x^3=-(2x^3-8x)$

$$\int(2x^3-8x)\,dx=2\left[\frac{x^4}{4}\right]-8\left[\frac{x^2}{2}\right]+C=\frac{1}{2}x^4-4x^2+C$$

$$A_1=\int_{-2}^0(2x^3-8x)\,dx=\left[\frac{1}{2}x^4-4x^2+C\right]_{-2}^0=\left[\frac{1}{2}(0)^4-4(0)^2+C\right]-\left[\frac{1}{2}(-2)^4-4(-2)^2+C\right]$$

$$=[0]-[8-16]=0-(-8)=8$$

$$A_2=\int_0^2-(2x^3-8x)\,dx=\left[-\left(\frac{1}{2}x^4-4x^2\right)+C\right]_0^2=\left[-\left(\frac{1}{2}(2)^4-4(2)^2\right)+C\right]-\left[-\left(\frac{1}{2}(0)^4-4(0)^2\right)+C\right]$$

$$=[-(8-16)]-[-(0)]=[-(-8)]=8 \qquad A=A_1+A_2=(8)+(8)=16$$

64) intersection points: $\frac{x^3}{3} - x = \frac{x}{3}$ $\qquad \frac{1}{3}x(x+2)(x-2)=0$

$$\frac{x^3}{3} - \frac{4x}{3} = 0 \Rightarrow \frac{1}{3}x=0 \mid x+2=0 \mid x-2=0$$

$$x=0 \mid x=-2 \mid x=2$$

$$\frac{1}{3}x(x^2-4)=0$$

on $(-2,0)$: Upper Curve: $y = \frac{x^3}{3} - x$ $\qquad$ region$_1 = \left(\frac{x^3}{3}-x\right)-\left(\frac{x}{3}\right)$

lower Curve: $y = \frac{x}{3}$ $\qquad = \left(\frac{x^3}{3} - \frac{4x}{3}\right)$

on $(0,2)$ Upper Curve: $y = \frac{x}{3}$ $\qquad$ region$_2 = \left(\frac{x}{3}\right)-\left(\frac{x^3}{3}-x\right) = \frac{4x}{3} - \frac{x^3}{3}$

lower curve: $y = \frac{x^3}{3} - x$ $\qquad = -\left(\frac{x^3}{3} - \frac{4x}{3}\right)$

on $(2,3)$ Upper Curve: $y = \frac{x^3}{3} - x$

lower curve: $y = \frac{x}{3}$ $\qquad$ region$_3 = \left(\frac{x^3}{3}-x\right)-\left(\frac{x}{3}\right) = \left(\frac{x^3}{3} - \frac{4x}{3}\right)$

$$\int \left(\frac{x^3}{3} - \frac{4x}{3}\right)dx = \frac{1}{3}\left[\frac{x^4}{4}\right] - \frac{4}{3}\left[\frac{x^2}{2}\right] + C = \frac{1}{12}x^4 - \frac{2}{3}x^2 + C$$

$$A_1 = \int_{-2}^{0}\left(\frac{x^3}{3} - \frac{4x}{3}\right)dx = \left[\frac{1}{12}x^4 - \frac{2}{3}x^2 + C\right]_{-2}^{0} = \left[\frac{1}{12}(0)^4 - \frac{2}{3}(0)^2 + C\right] - \left[\frac{1}{12}(-2)^4 - \frac{2}{3}(-2)^2 + C\right]$$

$$= [0] - \left[\frac{4}{3} - \frac{8}{3}\right] = [0] - \left[\frac{-4}{3}\right] = \frac{4}{3}$$

$$A_2 = \int_0^2 -\left(\frac{x^3}{3} - \frac{4x}{3}\right)dx = \left[-\left(\frac{1}{12}x^4 - \frac{2}{3}x^2\right) + C\right]_0^2 = \left[-\left(\frac{1}{12}(2)^4 - \frac{2}{3}(2)^2\right) + C\right] - \left[-\left(\frac{1}{12}(0)^4 - \frac{2}{3}(0)^2\right) + C\right]$$

$$= \left[-\left(\frac{4}{3} - \frac{8}{3}\right)\right] - [-(0)] = \left[-\left(\frac{-4}{3}\right)\right] = \frac{4}{3}$$

$$A_3 = \int_2^3 \left(\frac{x^3}{3} - \frac{4x}{3}\right)dx = \left[\frac{1}{12}x^4 - \frac{2}{3}x^2 + C\right]_2^3 = \left[\frac{1}{12}(3)^4 - \frac{2}{3}(3)^2 + C\right] - \left[\frac{1}{12}(2)^4 - \frac{2}{3}(2)^2 + C\right]$$

$$= \left[\frac{27}{4} - 6\right] - \left[\frac{4}{3} - \frac{8}{3}\right] = \left[\frac{27}{4} - \frac{24}{4}\right] - \left[\frac{-4}{3}\right] = \frac{3}{4} + \frac{4}{3}$$

$$A = A_1 + A_2 + A_3 = \left(\frac{4}{3}\right) + \left(\frac{4}{3}\right) + \left(\frac{3}{4} + \frac{4}{3}\right) = \frac{3}{4} + \frac{12}{3} = \frac{3}{4} + 4 = \frac{3}{4} + \frac{16}{4} = \frac{19}{4}$$

66) $y = 2x - x^2$ $\quad y = -3$

intersection:

$2x - x^2 = -3$

$0 = x^2 - 2x - 3$

$0 = (x+1)(x-3)$

$x+1=0 \mid x-3=0$

$x=-1 \mid x=3$

testing at $x=0$

Upper curve: $y = 2x - x^2$

lower curve: $y = -3$

region $= (2x - x^2) - (-3) = (3 + 2x - x^2)$

$$A = \int_{-1}^{3} (3 + 2x - x^2)\,dx = \left[3x + x^2 - \frac{x^3}{3} + C\right]_{-1}^{3}$$

$$= \left[3(3) + (3)^2 - \frac{(3)^3}{3} + C\right] - \left[3(-1) + (-1)^2 - \frac{(-1)^3}{3} + C\right]$$

$$= [9 + 9 - 9] - \left[-3 + 1 + \frac{1}{3}\right] = [9] - \left[-2 + \frac{1}{3}\right] = 11 - \frac{1}{3} = \frac{33}{3} - \frac{1}{3} = \frac{32}{3}$$

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68) $y = x^2 - 2x$ $\quad y = x$

intersection:

$x^2 - 2x = x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0 \mid x-3=0$

$x=3$

testing at $x=1$

Upper curve: $y = x$

lower curve: $y = x^2 - 2x$

region $= (x) - (x^2 - 2x) = (3x - x^2)$

$$A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 + C\right]_0^3$$

$$= \left[\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 + C\right] - \left[\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3 + C\right]$$

$$= \left[\frac{27}{2} - 9\right] - [0] = \frac{27}{2} - \frac{18}{2} = \frac{9}{2}$$

70) $y = 7 - 2x^2$ $\quad y = x^2 + 4$

intersection:

$7 - 2x^2 = x^2 + 4$

$0 = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$0 = 3(x+1)(x-1)$

$x + 1 = 0 \mid x - 1 = 0$

$x = -1 \mid x = 1$

testing at $x = 0$

Upper Curve: $y = 7 - 2x^2$

lower curve: $y = x^2 + 4$

region $= (7 - 2x^2) - (x^2 + 4) = (3 - 3x^2)$

$$A = \int_{-1}^{1} (3 - 3x^2)\,dx = \left[3x - x^3 + C\right]_{-1}^{1} = \left[3(1) - (1)^3 + C\right] - \left[3(-1) - (-1)^3 + C\right]$$

$$= [3 - 1] - [-3 + 1] = [2] - [-2] = 4$$

---

72) $y = x\sqrt{a^2 - x^2}$, $a > 0$ $\quad y = 0$

intersection:

$x\sqrt{a^2 - x^2} = 0$

$x = 0 \mid \sqrt{a^2 - x^2} = 0$

$a^2 - x^2 = 0$

$(a + x)(a - x) = 0$

$a + x = 0 \mid a - x = 0$

$x = -a \mid x = a$

on $(-a, 0)$: Upper Curve: $y = 0$

lower curve: $y = x\sqrt{a^2 - x^2}$

region$_1$ $= (0) - (x\sqrt{a^2 - x^2}) = -(x\sqrt{a^2 - x^2})$

on $(0, a)$: Upper Curve: $y = x\sqrt{a^2 - x^2}$

lower curve: $y = 0$

region$_2$ $= (x\sqrt{a^2 - x^2}) - (0) = (x\sqrt{a^2 - x^2})$

$$\int x\sqrt{a^2 - x^2}\,dx = \int (a^2 - x^2)^{\frac{1}{2}}(x\,dx) = \int p^{\frac{1}{2}}\left(\tfrac{-1}{2}\,dp\right) = \frac{-1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{-1}{3}\left(\sqrt{p}\right)^3 + C$$

$p = a^2 - x^2$

$dp = -2x\,dx \Rightarrow \frac{-1}{2}\,dp = x\,dx$

$$= \frac{-1}{3}\left(\sqrt{a^2 - x^2}\right)^3 + C$$

$$A_1 = \int_{-a}^{0} -\left(x\sqrt{a^2 - x^2}\right)dx = \left[-\left(\frac{-1}{3}\left(\sqrt{a^2 - x^2}\right)^3\right) + C\right]_{-a}^{0} = \left[\frac{1}{3}\left(\sqrt{a^2 - x^2}\right)^3 + C\right]_{-a}^{0}$$

$$= \left[\frac{1}{3}\left(\sqrt{a^2 - (0)^2}\right)^3 + C\right] - \left[\frac{1}{3}\left(\sqrt{a^2 - (-a)^2}\right)^3 + C\right] = \left[\frac{1}{3}\left(\sqrt{a^2}\right)^3\right] - \left[\frac{1}{3}\left(\sqrt{0}\right)^3\right] = \frac{a^3}{3}$$

$$A_2 = \int_{0}^{a} \left(x\sqrt{a^2 - x^2}\right)dx = \left[\frac{-1}{3}\left(\sqrt{a^2 - x^2}\right)^3 + C\right]_{0}^{a} = \left[\frac{-1}{3}\left(\sqrt{a^2 - (a)^2}\right)^3 + C\right] - \left[\frac{-1}{3}\left(\sqrt{a^2 - (0)^2}\right)^3 + C\right]$$

$$= \left[\frac{-1}{3}\left(\sqrt{0}\right)^3\right] - \left[\frac{-1}{3}\left(\sqrt{a^2}\right)^3\right] = \frac{a^3}{3}$$

$$A = A_1 + A_2 = \left(\frac{a^3}{3}\right) + \left(\frac{a^3}{3}\right) = \frac{2a^3}{3}$$

74) $y = |x^2-4| = \begin{cases} +(x^2-4) = A & x \le -2 \text{ or } 2 \le x \\ -(x^2-4) = C & -2 < x < 2 \end{cases}$ $\underset{B}{y = \frac{x^2}{2} + 4}$

break at $x = -2$ and $x = 2$

intersections:

$+(x^2-4) = \frac{x^2}{2} + 4$

$\frac{1}{2}x^2 - 8 = 0$

$\frac{1}{2}(x^2-16) = 0$

$\frac{1}{2}(x+4)(x-4) = 0$

$x+4=0 \mid x-4=0$

$x=-4 \mid x=4$

$-(x^2-4) = \frac{x^2}{2} + 4$

$-x^2+4 = \frac{x^2}{2} + 4$

$0 = \frac{3}{2}x^2$

$0 = x^2$

$x = 0$

region 1: on $(-4,-2)$: Upper: B, lower: A

region 2: on $(-2,0)$: Upper: B, lower: C; on $(0,2)$: Upper: B, lower: C — Combine intervals

region 3: on $(2,4)$ Upper: B, lower: A

on $(-2,2)$:

$\text{region}_2 = \left(\frac{x^2}{2}+4\right) - \left(-(x^2-4)\right) = \frac{x^2}{2} + 4 + x^2 - 4 = \frac{3}{2}x^2$

on $(-4,-2)$ and $(2,4)$

$\text{region}_1 = \text{region}_3 = \left(\frac{x^2}{2}+4\right) - \left(+(x^2-4)\right) = \frac{x^2}{2} + 4 - x^2 + 4 = \left(8 - \frac{1}{2}x^2\right)$

$\int\left(8 - \frac{1}{2}x^2\right)dx = 8[x] - \frac{1}{2}\left[\frac{x^3}{3}\right] + C = 8x - \frac{1}{6}x^3 + C$

$A_1 = \int_{-4}^{-2}\left(8 - \frac{1}{2}x^2\right)dx = \left[8x - \frac{1}{6}x^3 + C\right]_{-4}^{-2} = \left[8(-2) - \frac{1}{6}(-2)^3 + C\right] - \left[8(-4) - \frac{1}{6}(-4)^3 + C\right]$

$= \left[-16 + \frac{4}{3}\right] - \left[-32 + \frac{32}{3}\right] = -16 + \frac{4}{3} + 32 - \frac{32}{3} = 16 - \frac{28}{3}$

$A_3 = \int_{2}^{4}\left(8 - \frac{1}{2}x^2\right)dx = \left[8x - \frac{1}{6}x^3 + C\right]_{2}^{4} = \left[8(4) - \frac{1}{6}(4)^3 + C\right] - \left[8(2) - \frac{1}{6}(2)^3 + C\right]$

$= \left[32 - \frac{32}{3}\right] - \left[16 - \frac{4}{3}\right] = 32 - \frac{32}{3} - 16 + \frac{4}{3} = 16 - \frac{28}{3}$

$A_2 = \int_{-2}^{2}\left(\frac{3}{2}x^2\right)dx = \left[\frac{1}{2}x^3 + C\right]_{-2}^{2} = \left[\frac{1}{2}(2)^3 + C\right] - \left[\frac{1}{2}(-2)^3 + C\right] = [4] - [-4] = 8$

$A = A_1 + A_2 + A_3 = \left(16 - \frac{28}{3}\right) + (8) + \left(16 - \frac{28}{3}\right) = 40 - \frac{56}{3} = \frac{120}{3} - \frac{56}{3} = \frac{64}{3}$

76) $x = y^2$ $\quad$ $x = y + 2$ $\quad$ best when x as function of y

intersection:

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y + 1 = 0 \mid y - 2 = 0$

$y = -1 \mid y = 2$

Upper Curve: $x = y + 2$

lower curve: $x = y^2$

region: $(y+2) - (y^2) = (2 + y - y^2)$

$$A = \int_{-1}^{2} (2 + y - y^2)\, dy = \left[2y + \frac{y^2}{2} - \frac{y^3}{3} + C\right] = \left[2(2) + \frac{(2)^2}{2} - \frac{(2)^3}{3} + C\right] - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} + C\right]$$

$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = \left[6 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

$$= 8 - \frac{9}{3} - \frac{1}{2} = 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{10}{2} - \frac{1}{2} = \frac{9}{2}$$

---

78) $x - y^2 = 0$ $\quad$ $x + 2y^2 = 3$ $\quad$ best when x as function of y

$x = y^2$ $\quad$ $x = 3 - 2y^2$

intersection:

$y^2 = 3 - 2y^2$

$3y^2 - 3 = 0$

$3(y^2 - 1) = 0$

$3(y+1)(y-1) = 0$

$y + 1 = 0 \mid y - 1 = 0$

$y = -1 \mid y = 1$

Upper Curve: $x = 3 - 2y^2$

lower curve: $x = y^2$

region $= (3 - 2y^2) - (y^2) = (3 - 3y^2)$

$$A = \int_{-1}^{1} (3 - 3y^2)\, dy = \left[2y - y^3 + C\right]_{-1}^{1}$$

$$= \left[2(1) - (1)^3 + C\right] - \left[2(-1) - (-1)^3 + C\right]$$

$$= [2 - 1] - [-2 + 1]$$

$$= [1] - [-1]$$

$$= 2$$

80) $x - y^{\frac{2}{3}} = 0$ $\quad x + y^4 = 2$ best when x as function of y

$x = y^{\frac{2}{3}}$ $\quad x = 2 - y^4$

intersection

$y^{\frac{2}{3}} = 2 - y^4$

$y^4 + y^{\frac{2}{3}} - 2 = 0$

$y^{\frac{12}{3}} + y^{\frac{2}{3}} - 2 = 0$

$(y^{\frac{2}{3}})^6 + (y^{\frac{2}{3}}) - 2 = 0$

this statement will equal 0 when $y = 1$ and $y = -1$

Upper Curve: $x = 2 - y^4$

lower curve: $x = y^{\frac{2}{3}}$

region $= (2 - y^4) - (y^{\frac{2}{3}})$

$= (2 - y^4 - y^{\frac{2}{3}})$

$$A = \int_{-1}^{1} (2 - y^4 - y^{\frac{2}{3}})\,dy = \left[2y - \frac{y^5}{5} - \left(\frac{y^{\frac{5}{3}}}{\frac{5}{3}}\right) + C\right]_{-1}^{1}$$

$$= \left[2y - \frac{y^5}{5} - \frac{3}{5}(\sqrt[3]{y})^5 + C\right]_{-1}^{1} = \left[2(1) - \frac{(1)^5}{5} - \frac{3}{5}(\sqrt[3]{(1)})^5 + C\right] - \left[2(-1) - \frac{(-1)^5}{5} - \frac{3}{5}(\sqrt[3]{(-1)})^5 + C\right]$$

$$= \left[2 - \frac{1}{5} - \frac{3}{5}\right] - \left[-2 + \frac{1}{5} + \frac{3}{5}\right] = \left[2 - \frac{4}{5}\right] - \left[-2 + \frac{4}{5}\right] = 4 - \frac{8}{5} = \frac{20}{5} - \frac{8}{5} = \frac{12}{5}$$

---

82) $x = y^3 - y^2$ $\quad x = 2y$

intersection

$y^3 - y^2 = 2y$

$y^3 - y^2 - 2y = 0$

$y(y^2 - y - 2) = 0$

$y(y+1)(y-2) = 0$

$y = 0 \mid y + 1 = 0,\ y = -1 \mid y - 2 = 0,\ y = 2$

on $(-1, 0)$ Upper: $x = y^3 - y^2$

lower: $x = 2y$

region$_1 = (y^3 - y^2) - (2y) = (y^3 - y^2 - 2y)$

on $(0, 2)$ Upper: $x = 2y$

lower: $x = y^3 - y^2$

region$_2 = (2y) - (y^3 - y^2) = -y^3 + y^2 + 2y = -(y^3 - y^2 - 2y)$

$\int (y^3 - y^2 - 2y)\,dy = \frac{y^4}{4} - \frac{y^3}{3} - y^2 + C$

82) continued

$$A_1 = \int_{-1}^{0} (y^3 - y^2 - 2y)\,dy = \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2 + C\right]_{-1}^{0}$$

$$= \left[\frac{(0)^4}{4} - \frac{(0)^3}{3} - (0)^2 + C\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2 + C\right]$$

$$= [0] - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = -\left[\frac{3}{12} + \frac{4}{12} - \frac{12}{12}\right] = -\left[\frac{-5}{12}\right] = \frac{5}{12}$$

$$A_2 = \int_{0}^{2} -(y^3 - y^2 - 2y)\,dy = \left[-\left(\frac{y^4}{4} - \frac{y^3}{3} - y^2\right) + C\right]_0^2 = \left[\frac{-y^4}{4} + \frac{y^3}{3} + y^2 + C\right]_0^2$$

$$= \left[\frac{-(2)^4}{4} + \frac{(2)^3}{3} + (2)^2 + C\right] - \left[\frac{(0)^4}{4} + \frac{(0)^3}{3} + (0)^2 + C\right] = \left[-4 + \frac{8}{3} + 4\right] - [0] = \frac{8}{3}$$

$$A = A_1 + A_2 = \left(\frac{5}{12}\right) + \left(\frac{8}{3}\right) = \frac{5}{12} + \frac{32}{12} = \frac{37}{12}$$

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84) $x^3 - y = 0$ $\quad$ $3x^2 - y = 4$

$y = x^3$ $\quad$ $y = 3x^2 - 4$

Upper Curve: $y = x^3$

lower curve: $y = 3x^2 - 4$

region $= (x^3) - (3x^2 - 4) = (x^3 - 3x^2 + 4)$

$$A = \int_{-1}^{2} (x^3 - 3x^2 + 4)\,dx = \left[\frac{x^4}{4} - x^3 + 4x + C\right]_{-1}^{2}$$

$$= \left[\frac{(2)^4}{4} - (2)^3 + 4(2) + C\right] - \left[\frac{(-1)^4}{4} - (-1)^3 + 4(-1) + C\right]$$

$$= [4 - 8 + 8] - \left[\frac{1}{4} + 1 - 4\right]$$

$$= [4] - \left[\frac{1}{4} - 3\right] = 7 - \frac{1}{4}$$

$$= \frac{28}{4} - \frac{1}{4} = \frac{27}{4}$$

intersection

$x^3 = 3x^2 - 4$

$x^3 - 3x^2 + 4 = 0$

by trial and error we can find $x = 2$ is a solution and its factor is $(x-2) = 0$

$$\begin{array}{r} x^2 - x - 2 \\ x - 2 \enclose{longdiv}{x^3 - 3x^2 + 0x + 4} \\ -(x^3 - 2x^2) \\ \hline -x^2 + 0x \\ -(-x^2 + 2x) \\ \hline -2x + 4 \\ (-2x + 4) \\ \hline 0 \end{array}$$

$x^3 - 3x^2 + 4 = 0$

$(x-2)(x^2 - x - 2) = 0$

$(x-2)(x+1)(x-2) = 0$

$(x+1)(x-2)^2 = 0$

$x + 1 = 0$	$(x-2)^2 = 0$
$x = -1$	$x - 2 = 0$
	$x = 2$

86) $x + y^2 = 3$ $\quad$ $4x + y^2 = 0$ $\quad$ intersection

$x = 3 - y^2$ $\quad$ $4x = -y^2$ $\quad$ $3 - y^2 = \frac{-y^2}{4}$

$x = \frac{-y^2}{4}$ $\quad$ $0 = \frac{3}{4}y^2 - 3\left(\frac{4}{4}\right)$

Upper Curve: $x = 3 - y^2$ $\quad$ $0 = \frac{3}{4}(y^2 - 4)$

lower curve: $x = \frac{-y^2}{4}$ $\quad$ $0 = \frac{3}{4}(y+2)(y-2)$

$y + 2 = 0$ | $y - 2 = 0$

region $= (3 - y^2) - \left(\frac{-y^2}{4}\right) = \left(3 - \frac{3}{4}y^2\right)$ $\quad$ $y = -2$ | $y = 2$

$$A = \int_{-2}^{2}\left(3 - \frac{3}{4}y^2\right)dy = \left[3y - \frac{1}{4}y^3 + C\right]_{-2}^{2} = \left[3(2) - \frac{1}{4}(2)^3 + C\right] - \left[3(-2) - \frac{1}{4}(-2)^3 + C\right]$$

$$= [6 - 2] - [-6 + 2] = [4] - [-4] = 8$$

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88) $y = 8\cos x$ $\quad$ $y = \sec^2 x$ $\quad$ $\frac{-\pi}{3} \le x \le \frac{\pi}{3}$

Upper Curve: $y = 8\cos x$

region $= (8\cos x) - (\sec^2 x) = (8\cos x - \sec^2 x)$

lower curve: $y = \sec^2 x$

$$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(8\cos x - \sec^2 x)\,dx = \left[8\sin x - \tan x + C\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \left[8\sin\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{3}\right) + C\right] - \left[8\sin\left(\frac{-\pi}{3}\right) - \tan\left(\frac{-\pi}{3}\right) + C\right]$$

$$= \left[8\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{1}\right)\right] - \left[8\left(\frac{-\sqrt{3}}{2}\right) - \left(\frac{-\sqrt{3}}{1}\right)\right]$$

$$= [4\sqrt{3} - \sqrt{3}] - [-4\sqrt{3} + \sqrt{3}]$$

$$= [3\sqrt{3}] - [-3\sqrt{3}] = 6\sqrt{3}$$

90) $y = \sin\left(\frac{\pi}{2}x\right)$ $\quad y = x$

intersection

$\sin\left(\frac{\pi}{2}x\right) = x$

by trial and error the statement is true for $x = 0$, $x = 1$, and $x = -1$

on $(-1, 0)$, Upper: $y = x$

lower: $y = \sin\left(\frac{\pi}{2}x\right)$

$$\text{region}_1 = (x) - \left(\sin\left(\tfrac{\pi}{2}x\right)\right) = \left(x - \sin\left(\tfrac{\pi}{2}x\right)\right)$$

on $(0, 1)$ Upper: $y = \sin\left(\frac{\pi}{2}x\right)$

lower: $y = x$

$$\text{region}_2 = \left(\sin\left(\tfrac{\pi}{2}x\right)\right) - (x) = \left(\sin\left(\tfrac{\pi}{2}x\right) - x\right) = -\left(x - \sin\left(\tfrac{\pi}{2}x\right)\right)$$

$$\int \left(x - \sin\left(\tfrac{\pi}{2}x\right)\right)dx = \int x\,dx - \int \sin\left(\tfrac{\pi}{2}x\right)dx = \int x\,dx - \int \sin p\left(\tfrac{2}{\pi}\,dp\right)$$

$p = \frac{\pi}{2}x$

$$= \left[\frac{x^2}{2}\right] - \left[\frac{2}{\pi}(-\cos p)\right] + C = \frac{x^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + C$$

$dp = \frac{\pi}{2}dx$

$\frac{2}{\pi}dp = dx$

$$A_1 = \int_{-1}^{0}\left(x - \sin\left(\tfrac{\pi}{2}x\right)\right)dx = \left[\frac{x^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + C\right]_{-1}^{0}$$

$$= \left[\frac{(0)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}(0)\right) + C\right] - \left[\frac{(-1)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}(-1)\right) + C\right]$$

$$= \left[0 + \frac{2}{\pi}(1)\right] - \left[\frac{1}{2} + \frac{2}{\pi}(0)\right] = \frac{2}{\pi} - \frac{1}{2}$$

$$A_2 = \int_{0}^{1} -\left(x - \sin\left(\tfrac{\pi}{2}x\right)\right)dx = \left[-\left(\frac{x^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\right) + C\right]_{0}^{1}$$

$$= \left[-\left(\frac{(1)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}(1)\right)\right) + C\right] - \left[-\left(\frac{(0)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}(0)\right)\right) + C\right]$$

$$= \left[-\left(\frac{1}{2} + \frac{2}{\pi}(0)\right)\right] - \left[-\left(0 + \frac{2}{\pi}(1)\right)\right] = \left[-\frac{1}{2}\right] - \left[-\frac{2}{\pi}\right] = \frac{2}{\pi} - \frac{1}{2}$$

$$A = A_1 + A_2 = \left(\frac{2}{\pi} - \frac{1}{2}\right) + \left(\frac{2}{\pi} - \frac{1}{2}\right) = \frac{4}{\pi} - 1 = \frac{4 - \pi}{\pi}$$

92) $x = \tan^2 y \qquad x = -\tan^2 y \qquad \frac{-\pi}{4} \le y \le \frac{\pi}{4}$

by observation, when $y=0$ these functions are both 0.
Now testing at $x = \frac{\pi}{6}$ and $x = \frac{-\pi}{6}$ we can find that

Upper Curve: $x = \tan^2 y \qquad$ region $= (\tan^2 y) - (-\tan^2 y)$

lower curve: $x = -\tan^2 y \qquad = 2\tan^2 y$

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\tan^2 y \, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2(\sec^2 y - 1)\, dy = \left[2(\tan y - y) + C\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left[2\left(\tan\left(\tfrac{\pi}{4}\right) - \left(\tfrac{\pi}{4}\right)\right) + C\right] - \left[2\left(\tan\left(\tfrac{-\pi}{4}\right) - \left(\tfrac{-\pi}{4}\right)\right) + C\right]$$

$$= \left[2\left((1) - \tfrac{\pi}{4}\right)\right] - \left[2\left((-1) + \tfrac{\pi}{4}\right)\right] = \left[2 - \tfrac{\pi}{2}\right] - \left[-2 + \tfrac{\pi}{2}\right] = 4 - \pi$$

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94) $y = \sec^2\left(\frac{\pi}{3}x\right) \qquad y = x^{\frac{1}{3}} \qquad -1 \le x \le 1$

Upper Curve: $y = \sec^2\left(\frac{\pi}{3}x\right) \qquad$ lower curve: $y = x^{\frac{1}{3}}$

region $= \left(\sec^2\left(\frac{\pi}{3}x\right)\right) - \left(x^{\frac{1}{3}}\right) = \left(\sec^2\left(\frac{\pi}{3}x\right) - x^{\frac{1}{3}}\right)$

$p = \frac{\pi}{3}x$

$dp = \frac{\pi}{3}dx \Rightarrow \frac{3}{\pi}dp = dx$

$$\int \left(\sec^2\left(\tfrac{\pi}{3}x\right) - x^{\frac{1}{3}}\right)dx = \int \sec^2\left(\tfrac{\pi}{3}x\right)dx - \int x^{\frac{1}{3}}\,dx$$

$$= \int \sec^2 p\left(\tfrac{3}{\pi}dp\right) - \int x^{\frac{1}{3}}dx = \left[\tfrac{3}{\pi}\tan p\right] - \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right] + C = \tfrac{3}{\pi}\tan\left(\tfrac{\pi}{3}x\right) - \tfrac{3}{4}\left(\sqrt[3]{x}\right)^4 + C$$

$$A = \int_{-1}^{1}\left(\sec^2\left(\tfrac{\pi}{3}x\right) - x^{\frac{1}{3}}\right)dx = \left[\tfrac{3}{\pi}\tan\left(\tfrac{\pi}{3}x\right) - \tfrac{3}{4}\left(\sqrt[3]{x}\right)^4 + C\right]_{-1}^{1}$$

$$= \left[\tfrac{3}{\pi}\tan\left(\tfrac{\pi}{3}(1)\right) - \tfrac{3}{4}\left(\sqrt[3]{(1)}\right)^4 + C\right] - \left[\tfrac{3}{\pi}\tan\left(\tfrac{\pi}{3}(-1)\right) - \tfrac{3}{4}\left(\sqrt[3]{(-1)}\right)^4 + C\right]$$

$$= \left[\tfrac{3}{\pi}\left(\tfrac{\sqrt{3}}{1}\right) - \tfrac{3}{4}(1)^4\right] - \left[\tfrac{3}{\pi}\left(\tfrac{-\sqrt{3}}{1}\right) - \tfrac{3}{4}(-1)^4\right] = \frac{3\sqrt{3}}{\pi} - \frac{3}{4} + \frac{3\sqrt{3}}{\pi} + \frac{3}{4} = \frac{6\sqrt{3}}{\pi}$$

96) $x - y^{\frac{1}{3}} = 0$ $\quad$ $x - y^{\frac{1}{5}} = 0$

$x = y^{\frac{1}{3}}$ $\quad$ $x = y^{\frac{1}{5}}$

intersection

$y^{\frac{1}{3}} = y^{\frac{1}{5}}$

by observation the intersecting points occurs when $y = 0$, $y = 1$, and $y = -1$

on $(-1, 0)$: Upper: $x = y^{\frac{1}{3}}$

lower: $x = y^{\frac{1}{5}}$

$$\text{region}_1 = (y^{\frac{1}{3}}) - (y^{\frac{1}{5}}) = (y^{\frac{1}{3}} - y^{\frac{1}{5}}) = -(y^{\frac{1}{5}} - y^{\frac{1}{3}})$$

on $(0,1)$: Upper: $x = y^{\frac{1}{5}}$

lower: $x = y^{\frac{1}{3}}$

$$\text{region}_2 = (y^{\frac{1}{5}}) - (y^{\frac{1}{3}}) = (y^{\frac{1}{5}} - y^{\frac{1}{3}})$$

$$\int (y^{\frac{1}{5}} - y^{\frac{1}{3}}) \, dy = \left[\frac{y^{\frac{6}{5}}}{\frac{6}{5}}\right] - \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right] + C = \frac{5}{6}(\sqrt[5]{y})^6 - \frac{3}{4}(\sqrt[3]{y})^4 + C$$

$$A_1 = \int_{-1}^{0} -(y^{\frac{1}{5}} - y^{\frac{1}{3}}) \, dy = \left[-\left(\frac{5}{6}(\sqrt[5]{y})^6 - \frac{3}{4}(\sqrt[3]{y})\right) + C\right]_{-1}^{0}$$

$$= \left[-\left(\frac{5}{6}(\sqrt[5]{(0)})^6 - \frac{3}{4}(\sqrt[3]{(0)})^4\right) + C\right] - \left[-\left(\frac{5}{6}(\sqrt[5]{(-1)})^6 - \frac{3}{4}(\sqrt[3]{(-1)})^4\right) + C\right]$$

$$= [0] - \left[-\left(\frac{5}{6}(1) - \frac{3}{4}(1)\right)\right] = -\left[\frac{-5}{6} + \frac{3}{4}\right] = -\left[\frac{-10}{12} + \frac{9}{12}\right] = -\left[\frac{-1}{12}\right] = \frac{1}{12}$$

$$A_2 = \int_0^1 (y^{\frac{1}{5}} - y^{\frac{1}{3}}) \, dy = \left[\frac{5}{6}(\sqrt[5]{y})^6 - \frac{3}{4}(\sqrt[3]{y})^4 + C\right]_0^1$$

$$= \left[\frac{5}{6}(\sqrt[5]{(1)})^6 - \frac{3}{4}(\sqrt[3]{(1)})^4 + C\right] - \left[\frac{5}{6}(\sqrt[5]{(0)})^6 - \frac{3}{4}(\sqrt[3]{(0)})^4 + C\right]$$

$$= \left[\frac{5}{6}(1) - \frac{3}{4}(1)\right] - [0] = \frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}$$

$$A = A_1 + A_2 = \left(\frac{1}{12}\right) + \left(\frac{1}{12}\right) = \frac{2}{12} = \frac{1}{6}$$

98) $y = \sin x \quad y = \cos x$ in QI

intersection

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$ } in QI $x = \frac{\pi}{4}$

$\tan x = 1$

on left y-axis: $x = 0$

Upper Curve: $y = \cos x$

lower curve: $y = \sin x$

region $= (\cos x) - (\sin x)$
$= (\cos x - \sin x)$

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx = \left[\sin x + \cos x + C\right]_0^{\frac{\pi}{4}}$$
$$= \left[\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right) + C\right] - \left[\sin(0) + \cos(0) + C\right] = \left[\left(\tfrac{1}{\sqrt{2}}\right) + \left(\tfrac{1}{\sqrt{2}}\right)\right] - \left[(0) + (1)\right]$$
$$= \left[\tfrac{2}{\sqrt{2}}\right] - \left[1\right] = \sqrt{2} - 1$$

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100) $y = \tan x \quad$ x-axis: $y = 0 \qquad -\frac{\pi}{4} \le x \le \frac{\pi}{3}$

on $\left(-\frac{\pi}{4}, 0\right)$: Upper: $y = 0$ lower: $y = \tan x$ region$_1 = (0) - (\tan x) = -\tan x$

on $\left(0, \frac{\pi}{3}\right)$: Upper: $y = \tan x$ lower: $y = 0$ region$_2 = (\tan x) - (0) = \tan x$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{\cos x}(\sin x\,dx) = \int \frac{1}{p}(-1\,dp) = -\ln|p| + C$$
$p = \cos x \quad dp = -\sin x\,dx \Rightarrow -1\,dp = \sin x\,dx \qquad = -\ln|\cos x| + C$

$$A_1 = \int_{-\frac{\pi}{4}}^{0} -\tan x\,dx = \left[-(-\ln|\cos x|) + C\right]_{-\frac{\pi}{4}}^{0} = \left[-(-\ln|\cos(0)|) + C\right] - \left[-(-\ln|\cos(\tfrac{-\pi}{4})|) + C\right]$$
$$= \left[-(-\ln|(1)|)\right] - \left[-(-\ln|(\tfrac{1}{\sqrt{2}})|)\right] = \left[-(0)\right] - \left[(\ln(\tfrac{1}{\sqrt{2}}))\right] = -\left[\ln 1 - \ln\sqrt{2}\right] = \ln\sqrt{2}$$

$$A_2 = \int_0^{\frac{\pi}{3}} \tan x\,dx = \left[-\ln|\cos x| + C\right]_0^{\frac{\pi}{3}} = \left[-\ln|\cos(\tfrac{\pi}{3})| + C\right] - \left[-\ln|\cos(0)| + C\right]$$
$$= \left[-\ln|(\tfrac{1}{2})|\right] - \left[-\ln|(1)|\right] = \left[-(\ln 1 - \ln 2)\right] - \left[0\right] = \ln 2$$

$$A = A_1 + A_2 = (\ln\sqrt{2}) + (\ln 2) = \ln 2^{\frac{1}{2}} + \ln 2 = \tfrac{1}{2}\ln 2 + \ln 2 = \tfrac{3}{2}\ln 2$$

102) Upper Curve: $y = e^{\frac{x}{2}}$ lower curve: $y = e^{\frac{-x}{2}}$

intersects at $x = 0$ and on the right $x = 2\ln 2$

region $= (e^{\frac{x}{2}}) - (e^{\frac{-x}{2}}) = (e^{\frac{x}{2}} - e^{\frac{-x}{2}})$

$$\int (e^{\frac{x}{2}} - e^{\frac{-x}{2}})dx = \int e^{\frac{x}{2}}dx - \int e^{\frac{-x}{2}}dx = \int e^{p}(2dp) - \int e^{q}(-2dq)$$

$p = \frac{x}{2}$ $\quad q = \frac{-x}{2}dx$ $\quad = (2e^{p}) - (-2e^{q}) + C$

$dp = \frac{1}{2}dx$ $\quad dq = \frac{-1}{2}dx$

$2dp = dx$ $\quad -2dq = dx$ $\quad = 2e^{\frac{x}{2}} + 2e^{\frac{-x}{2}} + C = 2e^{\frac{x}{2}} + \frac{2}{e^{\frac{x}{2}}} + C$

$$A = \int_0^{2\ln 2} (e^{\frac{x}{2}} - e^{\frac{-x}{2}})dx = \left[2e^{\frac{x}{2}} + \frac{2}{e^{\frac{x}{2}}} + C\right]_0^{2\ln 2}$$

$$= \left[2e^{\frac{(2\ln 2)}{2}} + \frac{2}{e^{\frac{(2\ln 2)}{2}}} + C\right] - \left[2e^{\frac{(0)}{2}} + \frac{2}{e^{\frac{(0)}{2}}} + C\right]$$

$$= \left[2e^{\ln 2} + \frac{2}{e^{\ln 2}}\right] - \left[2e^{0} + \frac{2}{e^{0}}\right] = \left[2(2) + \frac{2}{(2)}\right] - \left[2(1) + \frac{2}{(1)}\right]$$

$$= [4 + 1] - [2 + 2] = 5 - 4 = 1$$

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104) Upper Curve: $y = 2^{1-x}$ lower curve: $y = 0$ $\quad -1 \le x \le 1$

region $= (2^{1-x}) - (0) = 2^{1-x}$ $\quad \int 2^{1-x}dx = \int 2^{p}(-1dp) = -1\left[\frac{2^{p}}{\ln 2}\right] + C$

$p = 1 - x \Rightarrow dp = -1dx \Rightarrow -1dp = dx$ $\quad = \frac{-1}{\ln 2}2^{1-x} + C$

$$A = \int_{-1}^{1} 2^{1-x}dx = \left[\frac{-1}{\ln 2}2^{1-x} + C\right]_{-1}^{1} = \left[\frac{-1}{\ln 2}2^{1-(1)} + C\right] - \left[\frac{-1}{\ln 2}2^{1-(-1)} + C\right]$$

$$= \left[\frac{-1}{\ln 2}2^{0}\right] - \left[\frac{-1}{\ln 2}2^{2}\right] = \left[\frac{-1}{\ln 2}\right] - \left[\frac{-4}{\ln 2}\right] = \frac{-1}{\ln 2} + \frac{4}{\ln 2} = \frac{3}{\ln 2}$$