Theorem 6 – The Substitution Rule

If $p = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int \left(f(g(x)) \right) \left(g'(x)\, dx \right) = \int \left(f(g(x)) \right) \left(\frac{dg}{dx}\, dx \right) = \int f(p)\, dp\,.$$

The Substitution Method to evaluate $\int \left(f(g(x)) \right) \left(g'(x)\, dx \right)$

1. Substitute $p = g(x)$ and $dp = \left(\frac{dp}{dx} \right) dx = g'(x)\, dx$ to obtain $\int f(p)\, dp$.

2. Integrate with respect to p.

3. Replace p by $g(x)$.

A more detailed integration table is given below, and all integration here is with respect to p:

1. $\int k\, dp = kp + C$ (any number k)	10. $\int \sec p \tan p\, dp = \sec p + C$
2. $\int p^n\, dt = \frac{p^{n+1}}{n+1} + C \quad (n \neq 1)$	11. $\int \csc p \cot p\, dt = -\csc p + C$
3. $\int \frac{1}{p}\, dp = \ln\lvert p \rvert + C$	12. $\int \tan p\, dp = \ln\lvert \sec p \rvert + C = -\ln\lvert \cos p \rvert + C$
4. $\int e^p\, dt = e^p + C$	13. $\int \cot p\, dp = \ln\lvert \sin p \rvert + C$
5. $\int a^p\, dt = \frac{a^p}{\ln a} + C \quad (a > 0,\, a \neq 1)$	14. $\int \csc p\, dp = -\ln\lvert \csc p + \cot p \rvert + C$ $= \ln\lvert \csc p - \cot p \rvert + C$
6. $\int \sin p\, dt = -\cos p + C$	15. $\int \sec p\, dp = \ln\lvert \sec p + \tan p \rvert + C$
7. $\int \cos p\, dt = \sin p + C$	16. $\int \frac{1}{a^2 + p^2}\, dp = \frac{1}{a} \tan^{-1}\left(\frac{p}{a} \right) + C$
8. $\int \sec^2 p\, dt = \tan p + C$	17. $\int \frac{1}{\sqrt{a^2 - p^2}}\, dp = \sin^{-1}\left(\frac{p}{a} \right) + C$
9. $\int \csc^2 p\, dt = -\cot p + C$	

2) $\int 7\sqrt{7x-1}\,dx = \int \sqrt{7x-1}\,(7\,dx) = \int \sqrt{p}\,dp = \int p^{\frac{1}{2}}\,dp$

$p = 7x-1$

$dp = 7\,dx$

$= \left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{2}{3}p^{\frac{3}{2}} + C = \frac{2}{3}(\sqrt{p})^3 + C$

$= \frac{2}{3}(\sqrt{7x-1})^3 + C$

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4) $\int \frac{4x^3}{(x^4+1)^2}\,dx = \int \frac{1}{(x^4+1)^2}(4x^3\,dx) = \int \frac{1}{p^2}\,dp = \int p^{-2}\,dp$

$p = x^4+1$

$dp = 4x^3\,dx$

$= \left[\frac{p^{-1}}{-1}\right] + C = \frac{-1}{p} + C = \frac{-1}{x^4+1} + C$

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6) $\int \frac{(1+\sqrt{x})^{\frac{1}{3}}}{\sqrt{x}}\,dx = \int (1+\sqrt{x})^3\left(\frac{1}{\sqrt{x}}\,dx\right) = \int p^3(2\,dp)$

$p = 1+\sqrt{x} = 1 + x^{\frac{1}{2}}$

$dp = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$dp = \frac{1}{2\sqrt{x}}\,dx$

$2\,dp = \frac{1}{\sqrt{x}}\,dx$

$= 2\left[\frac{p^4}{4}\right] + C = \frac{1}{2}p^4 + C$

$= \frac{1}{2}(1+\sqrt{x})^4 + C$

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8) $\int x\sin(2x^2)\,dx = \int \sin(2x^2)\,(x\,dx)$

$p = 2x^2$

$dp = 4x\,dx$

$\frac{1}{4}\,dp = x\,dx$

$= \int \sin p\left(\frac{1}{4}\,dp\right)$

$= \frac{1}{4}\left[-\cos p\right] + C$

$= \frac{-1}{4}\cos(2x^2) + C$

10) $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt = \int p^2(2\,dp)$

$p = 1-\cos\frac{t}{2}$ $\quad = 2\left[\frac{p^3}{3}\right]+C$

$dp = [0]-\left[-\sin\frac{t}{2}\left(\frac{1}{2}\right)\right]$ $\quad = \frac{2}{3}p^3+C$

$dp = \frac{1}{2}\sin\frac{t}{2}\,dt$

$2\,dp = \sin\frac{t}{2}\,dt$ $\quad = \frac{2}{3}\left(1-\cos\frac{t}{2}\right)^3+C$

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12) $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy = \int 3(y^4+4y^2+1)^2(4(y^3+2y)\,dy)$

$p = y^4+4y^2+1$ $\quad = \int 3p^2\,dp$

$dp = 4y^3+8y\ dy$ $\quad = 3\left[\frac{p^3}{3}\right]+C = p^3+C$

$dp = 4(y^3+2y)\,dy$ $\quad = (y^4+4y^2+1)^3+C$

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14) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = \int \cos^2\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\,dx\right) = \int \cos^2 p\,(-1\,dp)$

$p = \frac{1}{x} = x^{-1}$ $\quad = -\int \cos^2 p\,dp = -\int\left(\frac{1+\cos(2p)}{2}\right)dp$

$dp = -1x^{-2}\,dx$ $\quad = -\int\left(\frac{1}{2}+\frac{1}{2}\cos(2p)\right)dp = \int \frac{-1}{2}\,dp - \int \frac{1}{2}\cos(2p)\,dp$

$dp = \frac{-1}{x^2}\,dx$ $\quad = \frac{-1}{2}p - \left[\frac{1}{4}\sin(2p)\right]+C$

$-1\,dp = \frac{1}{x^2}\,dx$ $\quad = \frac{-1}{2}\left(\frac{1}{x}\right) - \frac{1}{4}\sin\left(2\left(\frac{1}{x}\right)\right)+C$

$\int \frac{1}{2}\cos(2p)\,dp = \int \frac{1}{2}\cos q\left(\frac{1}{2}\,dq\right)$ $\quad = \frac{-1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right)+C$

$q = 2p$ $\quad = \int \frac{1}{4}\cos q\,dq$

$dq = 2\,dp$ $\quad = \frac{1}{4}\sin q + C$

$\frac{1}{2}\,dq = dp$ $\quad = \frac{1}{4}\sin(2p)+C$

16-a) $\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{\sqrt{5x+8}}\,dx = \int \frac{1}{\sqrt{p}}\left(\frac{1}{5}dp\right) = \frac{1}{5}\int p^{-\frac{1}{2}}dp$

$p = 5x+8$ $\qquad = \frac{1}{5}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = \frac{2}{5}\sqrt{p} + C$

$dp = 5dx$ $\qquad = \frac{2}{5}\sqrt{5x+8} + C$

$\frac{1}{5}dp = dx$

16-b) $\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{\sqrt{5x+8}}\,dx = \int \left(\frac{2}{5}dq\right) = \frac{2}{5}q + C$

$q = \sqrt{5x+8} = (5x+8)^{\frac{1}{2}}$

$= \frac{2}{5}\sqrt{5x+8} + C$

$dq = \frac{1}{2}(5x+8)^{-\frac{1}{2}}(5)\,dx$

$dq = \frac{5}{2\sqrt{5x+8}}\,dx$

$\frac{2}{5}dq = \frac{1}{\sqrt{5x+8}}\,dx$

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18) $\int \frac{1}{\sqrt{5s+4}}\,ds = \int \frac{1}{\sqrt{p}}\left(\frac{1}{5}dp\right) = \frac{1}{5}\int p^{-\frac{1}{2}}dp$

$p = 5s+4$ $\qquad = \frac{1}{5}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = \frac{2}{5}\sqrt{p} + C$

$dp = 5\,ds$ $\qquad = \frac{2}{5}\sqrt{5s+4} + C$

$\frac{1}{5}dp = ds$

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20) $\int 3y\sqrt{7-3y^2}\,dy = \int 3\sqrt{7-3y^2}\,(y\,dy) = \int 3\sqrt{p}\left(\frac{-1}{6}dp\right)$

$p = 7-3y^2$ $\qquad = \frac{-1}{2}\int p^{\frac{1}{2}}dp = \frac{-1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$dp = -6y\,dy$ $\qquad = \frac{-1}{3}\left(\sqrt{p}\right)^3 + C = \frac{-1}{3}\left(\sqrt{7-3y^2}\right)^3 + C$

$\frac{-1}{6}dp = y\,dy$

22) $\int \sqrt{\sin x}\,\cos^3 x\,dx = \int \sqrt{\sin x}\,\cos^2 x\,(\cos x\,dx)$

$p = \sin x$

$dp = \cos x\,dx$

$= \int \sqrt{\sin x}\,(1-\sin^2 x)(\cos x\,dx)$

$= \int (\sin x)^{\frac{1}{2}}(1-\sin^2 x)(\cos x\,dx)$

$= \int \left((\sin x)^{\frac{1}{2}} - (\sin x)^{\frac{5}{2}}\right)(\cos x\,dx)$

$= \int \left(p^{\frac{1}{2}} - p^{\frac{5}{2}}\right)dp = \left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] - \left[\frac{p^{\frac{7}{2}}}{\frac{7}{2}}\right] + C$

$= \frac{2}{3}(\sqrt{p})^3 - \frac{2}{7}(\sqrt{p})^7 + C = \frac{2}{3}(\sqrt{\sin x})^3 - \frac{2}{7}(\sqrt{\sin x})^7 + C$

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24) $\int \tan^2 x\,\sec^2 x\,dx = \int p^2\,dp = \left[\frac{p^3}{3}\right] + C = \frac{1}{3}p^3 + C$

$p = \tan x$

$= \frac{1}{3}\tan^3 x + C$

$dp = \sec^2 x\,dx$

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26) $\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx = \int p^7(2\,dp) = 2\left[\frac{p^8}{8}\right] + C$

$p = \tan\left(\frac{x}{2}\right)$

$= \frac{1}{4}p^8 + C$

$dp = \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx$

$= \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$

$2\,dp = \sec^2\left(\frac{x}{2}\right)dx$

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28) $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr = \int \left(7-\frac{r^5}{10}\right)^3 (r^4\,dr) = \int p^3(-2\,dp)$

$p = 7-\frac{r^5}{10}$

$= -2\left[\frac{p^4}{4}\right] + C = \frac{-1}{2}p^4 + C$

$dp = -\frac{1}{10}(5r^4)\,dr$

$= \frac{-1}{2}\left(7-\frac{r^5}{10}\right)^4 + C$

$dp = \frac{-1}{2}r^4\,dr$

$-2\,dp = r^4\,dr$

30) $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = \int \csc p \cot p\ (2\,dp)$

$p = \frac{v-\pi}{2} = \frac{v}{2} - \frac{\pi}{2}$

$dp = \frac{1}{2}dv$

$2\,dp = dv$

$= 2[-\csc p] + C$

$= -2\csc\left(\frac{v-\pi}{2}\right) + C$

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32) $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz = \int \frac{1}{\sqrt{\sec z}}(\sec z \tan z\,dz)$

$p = \sec z$

$dp = \sec z \tan z\,dz$

$= \int \frac{1}{\sqrt{p}}\,dp = \int p^{-\frac{1}{2}}\,dp = \left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$= 2\sqrt{p} + C = 2\sqrt{\sec z} + C$

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34) $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt = \int \cos(\sqrt{t}+3)\left(\frac{1}{\sqrt{t}}\,dt\right)$

$p = \sqrt{t} + 3 = t^{\frac{1}{2}} + 3$

$dp = \frac{1}{2}t^{-\frac{1}{2}}dt$

$dp = \frac{1}{2\sqrt{t}}\,dt$

$2\,dp = \frac{1}{\sqrt{t}}\,dt$

$= \int \cos p\,(2\,dp)$

$= 2\sin p + C$

$= 2\sin(\sqrt{t}+3) + C$

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36) $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\,\sin^2\sqrt{\theta}}\,d\theta = \int \frac{1}{\sin^2\sqrt{\theta}}\left(\frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta\right) = \int \frac{1}{p^2}(2\,dp)$

$p = \sin\sqrt{\theta} = \sin(\theta^{\frac{1}{2}})$

$dp = \cos(\theta^{\frac{1}{2}})\left(\frac{1}{2}\theta^{-\frac{1}{2}}\right)d\theta$

$dp = \frac{\cos\sqrt{\theta}}{2\sqrt{\theta}}\,d\theta$

$2\,dp = \frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$= 2\int p^{-2}\,dp = 2\left[\frac{p^{-1}}{-1}\right] + C = \frac{-2}{p} + C$

$= \frac{-2}{\sin\sqrt{\theta}} + C = -2\csc\sqrt{\theta} + C$

38) $\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \sqrt{\left(\frac{1}{x^4}\right)\left(\frac{x-1}{x}\right)}\,dx = \int \sqrt{\frac{1}{x^4}}\sqrt{\frac{x}{x}-\frac{1}{x}}\,dx$

$p = 1-\frac{1}{x} = 1-x^{-1}$

$dp = -[-1x^{-2}]dx$

$dp = \frac{1}{x^2}dx$

$= \int \frac{1}{x^2}\sqrt{1-\frac{1}{x}}\,dx = \int \sqrt{1-\frac{1}{x}}\left(\frac{1}{x^2}dx\right) = \int \sqrt{p}\,dp$

$= \int p^{\frac{1}{2}}dp = \left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{2}{3}\left(\sqrt{p}\right)^3 + C$

$= \frac{2}{3}\left(\sqrt{1-\frac{1}{x}}\right)^3 + C$

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40) $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx = \int \frac{1}{x^3}\sqrt{\frac{x^2}{x^2}-\frac{1}{x^2}}\,dx = \int \frac{1}{x^3}\sqrt{1-\frac{1}{x^2}}\,dx$

$p = 1-\frac{1}{x^2} = 1-x^{-2}$

$dp = -[-2x^{-3}]dx$

$dp = \frac{2}{x^3}dx$

$\frac{1}{2}dp = \frac{1}{x^3}dx$

$= \int \sqrt{1-\frac{1}{x^2}}\left(\frac{1}{x^3}dx\right) = \int \sqrt{p}\left(\frac{1}{2}dp\right)$

$= \frac{1}{2}\int p^{\frac{1}{2}}dp = \frac{1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{1}{3}\left(\sqrt{p}\right)^3 + C = \frac{1}{3}\left(\sqrt{1-\frac{1}{x^2}}\right)^3 + C$

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42) $\int \sqrt{\frac{x^4}{x^3-1}}\,dx = \int \frac{\sqrt{x^4}}{\sqrt{x^3-1}}\,dx = \int \frac{x^2}{\sqrt{x^3-1}}\,dx = \int \frac{1}{\sqrt{x^3-1}}\left(x^2\,dx\right)$

$p = x^3-1$

$dp = 3x^2\,dx$

$\frac{1}{3}dp = x^2\,dx$

$= \int \frac{1}{\sqrt{p}}\left(\frac{1}{3}dp\right) = \frac{1}{3}\int p^{-\frac{1}{2}}dp = \frac{1}{3}\left[\frac{p^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$

$= \frac{2}{3}\sqrt{p} + C = \frac{2}{3}\sqrt{x^3-1} + C$

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44) $\int x\sqrt{4-x}\,dx = \int (4-p)\sqrt{p}\,(-1\,dp) = \int \left(-4\sqrt{p} + p^{\frac{3}{2}}\right)dp$

$p = 4-x \Rightarrow x = 4-p$

$dp = -1\,dx$

$-1\,dp = dx$

$= \int \left(p^{\frac{3}{2}} - 4p^{\frac{1}{2}}\right)dp = \left[\frac{p^{\frac{5}{2}}}{\frac{5}{2}}\right] - 4\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{2}{5}\left(\sqrt{p}\right)^5 - \frac{8}{3}\left(\sqrt{p}\right)^3 + C$

$= \frac{2}{5}\left(\sqrt{4-x}\right)^5 - \frac{8}{3}\left(\sqrt{4-x}\right)^3 + C$

46) $\int (x+5)(x-5)^{\frac{1}{3}}\,dx = \int (p+10)\,p^{\frac{1}{3}}\,dp = \int \left(p^{\frac{4}{3}} + 10p^{\frac{1}{3}}\right)dp$

$p = x-5 \Rightarrow x = p+5 \Rightarrow x+5 = (p+5)+5$
$= p+10$
$dp = dx$

$= \left[\dfrac{p^{\frac{7}{3}}}{\frac{7}{3}}\right] + 10\left[\dfrac{p^{\frac{4}{3}}}{\frac{4}{3}}\right] + C$

$= \frac{3}{7}\left(\sqrt[3]{p}\right)^7 + \frac{15}{2}\left(\sqrt[3]{p}\right)^4 + C$

$= \frac{3}{7}\left(\sqrt[3]{x-5}\right)^7 + \frac{15}{2}\left(\sqrt[3]{x-5}\right)^4 + C$

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48) $\int 3x^5\sqrt{x^3+1}\,dx = \int x^3\sqrt{x^3+1}\,(3x^2\,dx)$

$p = x^3+1 \Rightarrow x^3 = p-1$
$dp = 3x^2\,dx$

$= \int (p-1)\sqrt{p}\,dp = \int \left(p^{\frac{3}{2}} - p^{\frac{1}{2}}\right)dp$

$= \left[\dfrac{p^{\frac{5}{2}}}{\frac{5}{2}}\right] - \left[\dfrac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{2}{5}\left(\sqrt{p}\right)^5 - \frac{2}{3}\left(\sqrt{p}\right)^3 + C$

$= \frac{2}{5}\left(\sqrt{x^3+1}\right)^5 - \frac{2}{3}\left(\sqrt{x^3+1}\right)^3 + C$

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50) $\int \dfrac{x}{(2x-1)^{2/3}}\,dx = \int \dfrac{\left(\frac{1}{2}(p+1)\right)}{p^{2/3}}\left(\frac{1}{2}\,dp\right) = \frac{1}{4}\int \left(\dfrac{p}{p^{2/3}} + \dfrac{1}{p^{2/3}}\right)dp$

$p = 2x-1 \Rightarrow 2x = p+1$
$dp = 2\,dx \quad x = \frac{1}{2}(p+1)$
$\frac{1}{2}\,dp = dx$

$= \frac{1}{4}\int \left(p^{\frac{1}{3}} + p^{-\frac{2}{3}}\right)dp = \frac{1}{4}\left\{\left[\dfrac{p^{\frac{4}{3}}}{\frac{4}{3}}\right] + \left[\dfrac{p^{\frac{1}{3}}}{\frac{1}{3}}\right]\right\} + C$

$= \frac{1}{4}\left\{\frac{3}{4}\left(\sqrt[3]{p}\right)^4 + 3\left(\sqrt[3]{p}\right)\right\} + C$

$= \frac{3}{16}\left(\sqrt[3]{2x-1}\right)^4 + \frac{3}{4}\left(\sqrt[3]{2x-1}\right) + C$

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52) $\int (\sin 2\theta)\,e^{\sin^2\theta}\,d\theta = \int e^{\sin^2\theta}(2\sin\theta\cos\theta\,d\theta)$

$p = \sin^2\theta$
$dp = 2\sin\theta\cos\theta\,d\theta$

$= \int e^p\,dp = e^p + C$

$= e^{\sin^2\theta} + C$

54) $\int \frac{1}{x^2} e^{\frac{1}{x}} \sec(1+e^{\frac{1}{x}}) \tan(1+e^{\frac{1}{x}})\, dx$

$p = 1+e^{\frac{1}{x}} = 1+e^{x^{-1}}$

$dp = e^{x^{-1}}(-1x^{-2})\,dx$

$dp = \frac{-e^{\frac{1}{x}}}{x^2}\,dx$

$-1\,dp = \frac{1}{x^2} e^{\frac{1}{x}}\,dx$

$= \int \sec(1+e^{\frac{1}{x}}) \tan(1+e^{\frac{1}{x}}) \left(\frac{1}{x^2} e^{\frac{1}{x}}\,dx\right)$

$= \int \sec p \tan p\,(-1\,dp)$

$= -\sec p + C$

$= -\sec(1+e^{\frac{1}{x}}) + C$

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56) $\int \frac{\ln\sqrt{t}}{t}\,dt = \int \frac{\ln(t^{\frac{1}{2}})}{t}\,dt = \int \frac{\frac{1}{2}\ln t}{t}\,dt = \frac{1}{2}\int \frac{\ln t}{t}\,dt$

$p = \ln t$

$dp = \frac{1}{t}\,dt$

$= \frac{1}{2}\int \ln t\left(\frac{1}{t}\,dt\right) = \frac{1}{2}\int p\,dp = \frac{1}{2}\left[\frac{p^2}{2}\right] + C$

$= \frac{1}{4}p^2 + C = \frac{1}{4}(\ln t)^2 + C$

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58) $\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{dx}{x\sqrt{(x^2)^2-1}}\left(\frac{x}{x}\right) = \int \frac{1}{x^2\sqrt{(x^2)-1}}(x\,dx)$

$p = x^2$

$dp = 2x\,dx$

$\frac{1}{2}dp = x\,dx$

$= \int \frac{1}{p\sqrt{p^2-1}}\left(\frac{1}{2}dp\right) = \frac{1}{2}\int \frac{1}{p\sqrt{p^2-1}}\,dp$

using the formula of integration for now. You will learn how to integrate without formula in MATH 21200 (Calculus 2)

$= \frac{1}{2}\left[\sec^{-1} p\right] + C = \frac{1}{2}\sec^{-1}(x^2) + C$

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60) $\int \frac{1}{\sqrt{e^{2\theta}-1}}\,d\theta = \int \frac{1}{\sqrt{(e^\theta)^2-1}}\left(\frac{e^\theta}{e^\theta}\right)d\theta = \int \frac{1}{e^\theta\sqrt{(e^\theta)^2-1}}(e^\theta\,d\theta) = \int \frac{1}{p\sqrt{p^2-1}}\,dp$

$p = e^\theta$

$dp = e^\theta\,d\theta$

same reason as example 58

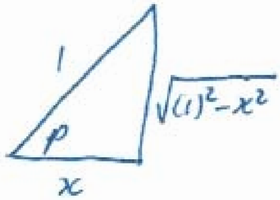
$= \sec^{-1} p + C = \sec^{-1}(e^\theta) + C$

62) $$\int \frac{e^{\cos^{-1}x}\,dx}{\sqrt{1-x^2}} = \int e^{\cos^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\,dx\right)$$

$\rho = \cos^{-1}x$

$\Downarrow$

$\cos\rho = x$



$-\sin\rho\,\frac{d\rho}{dx} = 1$

$\frac{d\rho}{dx} = \frac{1}{-\sin\rho} = \frac{-1}{\left(\frac{\sqrt{(1)^2-x^2}}{1}\right)} = \frac{-1}{\sqrt{1-x^2}}$

$d\rho = \frac{-1}{\sqrt{1-x^2}}\,dx$

$(-1\,d\rho) = \frac{1}{\sqrt{1-x^2}}\,dx$

$= \int e^{\rho}(-1\,d\rho)$

$= -1\,e^{\rho} + C$

$= -e^{\cos^{-1}x} + C$

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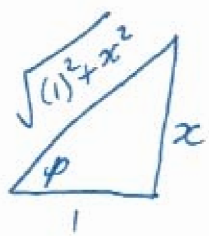
64) $$\int \frac{\sqrt{\tan^{-1}x}\,dx}{1+x^2} = \int \sqrt{\tan^{-1}x}\left(\frac{1}{1+x^2}\,dx\right)$$

$\rho = \tan^{-1}x$

$\Downarrow$

$\tan\rho = x$



$\sec^2\rho\,\frac{d\rho}{dx} = 1$

$\frac{d\rho}{dx} = \frac{1}{\sec^2\rho} = \frac{1}{\left(\frac{\sqrt{(1)^2+x^2}}{1}\right)^2}$

$d\rho = \frac{1}{(1)^2+x^2}\,dx$

$d\rho = \frac{1}{1+x^2}\,dx$

$= \int \sqrt{\rho}\,d\rho = \int \rho^{\frac{1}{2}}\,d\rho$

$= \left[\frac{\rho^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$

$= \frac{2}{3}\left(\sqrt{\rho}\right)^3 + C$

$= \frac{2}{3}\left(\sqrt{\tan^{-1}x}\right)^3 + C$

66) $\displaystyle\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}} = \int \frac{1}{\sin^{-1}y}\left(\frac{1}{\sqrt{1-y^2}}\,dy\right)$

$\varphi = \sin^{-1}y$

$\Downarrow$

$\sin\varphi = y$

$\cos\varphi \dfrac{d\varphi}{dy} = 1$

$\dfrac{d\varphi}{dy} = \dfrac{1}{\cos\varphi} = \dfrac{1}{\left(\frac{\sqrt{(1)^2-y^2}}{1}\right)}$

$d\varphi = \dfrac{1}{\sqrt{1-y^2}}\,dy$

$= \displaystyle\int \frac{1}{\varphi}\,d\varphi = \ln|\varphi| + C$

$= \ln|\sin^{-1}y| + C$

68-a) $\displaystyle\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\cos(x-1)\,dx$

$\varphi = x-1$

$d\varphi = dx$

$= \displaystyle\int \sqrt{1+\sin^2\varphi}\ \sin\varphi\cos\varphi\,d\varphi$

$= \displaystyle\int \sqrt{1+\sin^2\varphi}\ \sin\varphi\ (\cos\varphi\,d\varphi)$

$v = \sin\varphi$

$dv = \cos\varphi\,d\varphi$

$= \displaystyle\int \sqrt{1+v^2}\ v\ (dv)$

$= \displaystyle\int \sqrt{1+v^2}\ (v\,dv)$

$w = 1+v^2$

$dw = 2v\,dv$

$\frac{1}{2}dw = v\,dv$

$= \displaystyle\int \sqrt{w}\left(\frac{1}{2}dw\right) = \frac{1}{2}\int w^{\frac{1}{2}}\,dw$

$= \dfrac{1}{2}\left[\dfrac{w^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \dfrac{1}{3}\left(\sqrt{w}\right)^3 + C$

$= \dfrac{1}{3}\left(\sqrt{1+v^2}\right)^3 + C = \dfrac{1}{3}\left(\sqrt{1+\sin^2\varphi}\right)^3 + C$

$= \dfrac{1}{3}\left(\sqrt{1+\sin^2(x-1)}\right)^3 + C$

68-b) $\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\cos(x-1)\,dx$

$$= \int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\left(\cos(x-1)\,dx\right)$$

$p = \sin(x-1)$

$dp = \cos(x-1)(1)\,dx$

$dp = \cos(x-1)\,dx$

$$= \int \sqrt{1+p^2}\ p\ dp$$

$$= \int \sqrt{1+p^2}\ (p\,dp)$$

$v = 1+v^2$

$dv = 2\,v\,dv$

$\frac{1}{2}dv = v\,dv$

$$= \int \sqrt{v}\left(\tfrac{1}{2}dv\right) = \frac{1}{2}\int v^{\frac{1}{2}}dv$$

$$= \frac{1}{2}\left[\frac{v^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{3}\left(\sqrt{v}\right)^3 + C$$

$$= \frac{1}{3}\left(\sqrt{1+v^2}\right)^3 + C$$

$$= \frac{1}{3}\left(\sqrt{1+\sin^2(x-1)}\right)^3 + C$$

68-c) $\int \sqrt{1+\sin^2(x-1)}\ \sin(x-1)\cos(x-1)\,dx$

$p = 1+\sin^2(x-1)$

$dp = 2\sin(x-1)\cos(x-1)(1)\,dx$

$\frac{1}{2}dp = \sin(x-1)\cos(x-1)\,dx$

$$= \int \sqrt{1+\sin^2(x-1)}\ \left(\sin(x-1)\cos(x-1)\,dx\right)$$

$$= \int \sqrt{p}\ \left(\tfrac{1}{2}dp\right)$$

$$= \frac{1}{2}\int p^{\frac{1}{2}}\,dp = \frac{1}{2}\left[\frac{p^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{3}\left(\sqrt{p}\right)^3 + C$$

$$= \frac{1}{3}\left(\sqrt{1+\sin^2(x-1)}\right)^3 + C$$

70) $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta = \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\,\sqrt{\cos^3\sqrt{\theta}}}\,d\theta$

$p = \cos\sqrt{\theta} = \cos(\theta^{\frac{1}{2}})$

$dp = -\sin(\theta^{\frac{1}{2}})(\frac{1}{2}\theta^{-\frac{1}{2}})d\theta$

$dp = \dfrac{-\sin\sqrt{\theta}}{2\sqrt{\theta}}\,d\theta$

$-2dp = \dfrac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$$= \int \frac{1}{\sqrt{\cos^3\sqrt{\theta}}}\left(\frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta\right)$$

$$= \int \frac{1}{\sqrt{p^3}}(-2\,dp) = \int -2p^{-\frac{3}{2}}dp$$

$$= -2\left[\frac{p^{-\frac{1}{2}}}{-\frac{1}{2}}\right] + C = \frac{4}{\sqrt{p}} + C$$

$$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$$

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72) $\displaystyle\int \csc x\,dx = \int \csc x\,(1)\,dx = \int \csc x\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx$

$p = \csc x + \cot x$

$dp = -\csc x\cot x - \csc^2 x\,dx$

$dp = -1(\csc x\cot x + \csc^2 x)\,dx$

$-1\,dp = (\csc x\cot x + \csc^2 x)\,dx$

$$= \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}\,dx$$

$$= \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx$$

$$= \int \frac{1}{\csc x + \cot x}\left((\csc x\cot x + \csc^2 x)\,dx\right)$$

$$= \int \frac{1}{p}(-1\,dp) = -\ln|p| + C$$

$$= -\ln|\csc x + \cot x| + C$$

74) $\frac{dy}{dx} = 4x(x^2+8)^{\frac{-1}{3}}$, $y(0)=0$

$y = \int 4x(x^2+8)^{\frac{-1}{3}}dx = \int 2(x^2+8)^{\frac{-1}{3}}(2x\,dx) = \int 2p^{\frac{-1}{3}}dp$

$p = x^2+8$

$dp = 2x\,dx$

$= 2\left[\frac{p^{\frac{2}{3}}}{\frac{2}{3}}\right] + C = 3\left(\sqrt[3]{p}\right)^2 + C$

$= 3\left(\sqrt[3]{x^2+8}\right)^2 + C$

$(0) = 3\left(\sqrt[3]{(0)^2+8}\right)^2 + C$

$0 = 3(2)^2 + C$

$0 = 12 + C$

$-12 = C$

$y = 3\left(\sqrt[3]{x^2+8}\right)^2 + (-12)$

$\underline{\underline{y = 3\left(\sqrt[3]{x^2+8}\right)^2 - 12}}$

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76) $\frac{dr}{d\theta} = 3\cos^2\left(\frac{\pi}{4}-\theta\right)$, $r(0) = \frac{\pi}{8}$

$r = \int 3\cos^2\left(\frac{\pi}{4}-\theta\right)d\theta = \int 3\cos^2 p\,(-1dp) = \int -3\cos^2 p\,dp$

$p = \frac{\pi}{4} - \theta$ $\quad q = 2p$

$dp = -1d\theta$ $\quad dq = 2\,dp$

$-1dp = d\theta$ $\quad \frac{1}{2}dq = dp$

$= \int -3\left(\frac{1+\cos(2p)}{2}\right)dp = \int \left(\frac{-3}{2} - \frac{-3}{2}\cos(2p)\right)dp$

$= \int \frac{-3}{2}dp - \frac{3}{2}\int \cos(2p)\,dp = \int \frac{-3}{2}dp - \frac{3}{2}\int \cos q\left(\frac{1}{2}dq\right)$

$= \frac{-3}{2}p - \frac{3}{4}\sin q + C = \frac{-3}{2}p - \frac{3}{4}\sin(2p) + C$

$= \frac{-3}{2}\left(\frac{\pi}{4}-\theta\right) - \frac{3}{4}\sin\left(2\left(\frac{\pi}{4}-\theta\right)\right) + C$

$\frac{\pi}{8} = \frac{-3}{2}\left(\frac{\pi}{4}-(0)\right) - \frac{3}{4}\sin\left(2\left(\frac{\pi}{4}-(0)\right)\right) + C$

$\frac{\pi}{8} = \frac{-3}{2}\left(\frac{\pi}{4}\right) - \frac{3}{4}\sin\left(\frac{\pi}{2}\right) + C$

$\frac{\pi}{8} = \frac{-3\pi}{8} - \frac{3}{4}(1) + C$

$C = \frac{\pi}{8} + \frac{3\pi}{8} + \frac{3}{4} = \frac{4\pi}{8} + \frac{3}{4}$

$r = \frac{-3}{2}\left(\frac{\pi}{4}-\theta\right) - \frac{3}{4}\sin\left(2\left(\frac{\pi}{4}-\theta\right)\right) + \left(\frac{4\pi}{8} + \frac{3}{4}\right)$

$= \frac{-3\pi}{8} + \frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{4\pi}{8} + \frac{3}{4}$

$r = \frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{8} + \frac{3}{4}$

$= \frac{3}{2}\theta - \frac{3}{4}\cos(2\theta) + \frac{\pi}{8} + \frac{3}{4}$

78) $\frac{d^2y}{dx^2} = 4\sec^2(2x)\tan(2x)$ $\quad y'(0) = \frac{dy}{dx}\Big|_{x=0} = 4$, $y(0) = -1$

$$\frac{dy}{dx} = \int 4\sec^2(2x)\tan(2x)\,dx = \int 2\sec(2x)\left(2\sec(2x)\tan(2x)\,dx\right)$$

$p = \sec(2x)$

$dp = \sec(2x)\tan(2x)(2)dx$

$dp = 2\sec(2x)\tan(2x)\,dx$

$= \int 2p\,(dp) = p^2 + C$

$= (\sec(2x))^2 + C = \sec^2(2x) + C$

$4 = \frac{dy}{dx}\Big|_{x=0} = \sec^2(2(0)) + C$

$4 = \sec^2(0) + C$

$4 = (1)^2 + C$

$4 = C$

$\frac{dy}{dx} = \sec^2(2x) + (4)$

$\frac{dy}{dx} = \sec^2(2x) + 4$

$$y = \int(\sec^2(2x) + 4)\,dx = \int \sec^2(2x)\,dx + \int 4\,dx$$

$q = 2x$

$dq = 2\,dx$

$\frac{1}{2}dq = dx$

$= \int \sec^2 q\left(\frac{1}{2}dq\right) + \int 4\,dx$

$= \frac{1}{2}[\tan q] + 4x + D$

$= \frac{1}{2}\tan(2x) + 4x + D$

$-1 = y(0) = \frac{1}{2}\tan(2(0)) + 4(0) + D$

$-1 = \frac{1}{2}(0) + 0 + D$

$-1 = D$

$y = \frac{1}{2}\tan(2x) + 4x + (-1)$

$$\underline{y = \frac{1}{2}\tan(2x) + 4x - 1}$$

80) $a = \frac{d^2s}{dt^2} = \pi^2 \cos \pi t$ m/sec^2 $\qquad s(0) = 0$ m, $v(0) = 8$ m/sec

$s(1) = ?$

$v(t) = \frac{ds}{dt} = \int \pi^2 \cos \pi t \, dt = \int \pi \cos(\pi t)(\pi \, dt)$

$p = \pi t$ $\qquad = \int \pi \cos p \, dp = \pi [\sin p] + C$

$dp = \pi \, dt$ $\qquad = \pi \sin(\pi t) + C$

$8 = v(0) = \pi \sin(\pi(0)) + C$

$8 = \pi(0) + C$

$8 = C$ $\qquad v(t) = \frac{ds}{dt} = \pi \sin(\pi t) + (8)$

$= \pi \sin(\pi t) + 8$

$s(t) = \int (\pi \sin(\pi t) + 8) \, dt = \int \pi \sin(\pi t) \, dt + \int 8 \, dt$

$q = \pi t$ $\qquad = \int \sin(\pi t)(\pi \, dt) + \int 8 \, dt$

$dq = \pi \, dt$ $\qquad = \int \sin q \, dq + \int 8 \, dt$

$= [-\cos q] + 8t + D$

$= -\cos(\pi t) + 8t + D$

$0 = s(0) = -\cos(\pi(0)) + 8(0) + D$

$0 = -(1) + 0 + D$ $\qquad s(t) = -\cos(\pi t) + 8t + (1)$

$0 = -1 + D$ $\qquad s(t) = -\cos(\pi t) + 8t + 1$

$1 = D$

$s(1) = -\cos(\pi(1)) + 8(1) + 1 = -\cos(\pi) + 9$

$= -(-1) + 9 = 10$ m