Theorem 3 – The Mean Value Theorem for Definite Integrals

If $f(x)$ is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx .$$

Theorem 4 – The fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

(Equation 2) $$f(x)=F'(x)=\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt$$

Theorem 4 (continued) – The fundamental Theorem of Calculus, Part 2

If f is continuous on $[a,b]$, then F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=F(b)-F(a) .$$

Theorem 5 – The Net Change Theorem

The net change is differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b)-F(a)=\int_a^b F'(x)\,dx .$$

Total Area Summary:

To Find the area between the graph of $y=f(x)$ and the x-axis over the interval $[a,b]$:

1. Subdivide $[a,b]$ at the zeros of f.

2. Integrate f over each subinterval.

3. Add the absolute values of the integrals.

For some of the exercises in this section it is more efficient if we use the Substitution techniques given in section 5.5.

2) $\int_{-1}^{1}(x^2-2x+3)\,dx = \left[\frac{x^3}{3} - x^2 + 3x + C\right]_{-1}^{1}$

$= \left[\frac{(1)^3}{3} - (1)^2 + 3(1) + C\right] - \left[\frac{(-1)^3}{3} - (-1)^2 + 3(-1) + C\right]$

$= \left[\frac{1}{3} - 1 + 3\right] - \left[\frac{-1}{3} - 1 - 3\right] = \left[\frac{1}{3} + 2\right] - \left[\frac{-1}{3} - 4\right] = \frac{2}{3} + 6 = \frac{2}{3} + \frac{18}{3} = \frac{20}{3}$

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4) $\int_{-1}^{1} x^{299}\,dx = \left[\frac{x^{300}}{300} + C\right]_{-1}^{1} = \left[\frac{(1)^{300}}{300} + C\right] - \left[\frac{(-1)^{300}}{300} + C\right]$

$= \left[\frac{1}{300}\right] - \left[\frac{1}{300}\right] = 0$

---

6) $\int_{-2}^{3} (x^3-2x+3)\,dx = \left[\frac{x^4}{4} - x^2 + 3x + C\right]_{-2}^{3}$

$= \left[\frac{(3)^4}{4} - (3)^2 + 3(3) + C\right] - \left[\frac{(-2)^4}{4} - (-2)^2 + 3(-2) + C\right]$

$= \left[\frac{81}{4} - 9 + 9\right] - \left[4 - 4 - 6\right] = \left[\frac{81}{4}\right] - \left[-6\right] = \frac{81}{4} + 6 = \frac{81}{4} + \frac{24}{4} = \frac{105}{4}$

---

8) $\int_{1}^{32} x^{-6/5}\,dx = \left[\frac{x^{-\frac{1}{5}}}{-\frac{1}{5}} + C\right]_{1}^{32} = \left[\frac{-5}{\sqrt[5]{x}} + C\right]_{1}^{32} = \left[\frac{-5}{\sqrt[5]{(32)}} + C\right] - \left[\frac{-5}{\sqrt[5]{(1)}} + C\right]$

$= \left[\frac{-5}{2}\right] - \left[\frac{-5}{1}\right] = \frac{-5}{2} + 5 = \frac{-5}{2} + \frac{10}{2} = \frac{5}{2}$

---

10) $\int_{0}^{\pi} (1 + \cos x)\,dx = \left[x + \sin x + C\right]_{0}^{\pi}$

$= \left[(\pi) + \sin(\pi) + C\right] - \left[(0) + \sin(0) + C\right] = \left[\pi + 0\right] - \left[0\right] = \pi$

12) $\int 4\frac{\sin u}{\cos^2 u}\,du = \int 4\frac{1}{(\cos u)^2}(\sin u\,du) = \int 4\frac{1}{p^2}(-1\,dp)$

$p = \cos u$ $\quad = \int \frac{-4}{p^2}\,dp = \int -4p^{-2}\,dp = -4\left[\frac{p^{-1}}{-1}\right] + C$

$dp = -\sin u\,du$ $\quad = \frac{4}{p} + C = \frac{4}{\cos u} + C$ [technique from section 5.5]

$-1\,dp = \sin u\,du$

$$\int_0^{\frac{\pi}{3}} 4\frac{\sin u}{\cos^2 u}\,du = \left[\frac{4}{\cos u} + C\right]_0^{\frac{\pi}{3}} = \left[\frac{4}{\cos(\frac{\pi}{3})} + C\right] - \left[\frac{4}{\cos(0)} + C\right]$$

$$= \left[\frac{4}{(\frac{1}{2})}\right] - \left[\frac{4}{(1)}\right] = 8 - 4 = 4$$

---

14) $\int \sin^2 t\,dt = \int\left(\frac{1-\cos(2t)}{2}\right)dt = \int\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)dt$

$= \int \frac{1}{2}\,dt - \int \frac{1}{2}\cos(2t)\,dt = \frac{1}{2}t - \frac{1}{4}\cos(2t) + C$

$\int \frac{1}{2}\cos(2t)\,dt = \int \frac{1}{2}\cos p\left(\frac{1}{2}dp\right) = \int \frac{1}{4}\cos p\,dp = \frac{1}{4}\sin p + C$

$p = 2t$ $\quad = \frac{1}{4}\sin(2t) + C$

$dp = 2\,dt$

$\frac{1}{2}dp = dt$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2 t\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2t) + C\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4}\sin\left(2\left(\frac{\pi}{3}\right)\right) + C\right] - \left[\frac{1}{2}\left(\frac{-\pi}{3}\right) - \frac{1}{4}\sin\left(2\left(\frac{-\pi}{3}\right)\right) + C\right]$$

$$= \left[\frac{\pi}{6} - \frac{1}{4}\sin\left(\frac{2\pi}{3}\right)\right] - \left[\frac{-\pi}{6} - \frac{1}{4}\sin\left(\frac{-2\pi}{3}\right)\right] = \left[\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{-\pi}{6} - \frac{1}{4}\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \left[\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right] - \left[\frac{-\pi}{6} + \frac{\sqrt{3}}{8}\right] = \frac{2\pi}{6} - \frac{2\sqrt{3}}{8} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

16) $(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$

$\cos^2\theta + \sin^2\theta = 1$ $\quad = \sec^2 x + 2\sec x \tan x + (\sec^2 x - 1)$

$1 + \tan^2\theta = \sec^2\theta$ $\quad = 2\sec^2 x + 2\sec x \tan x - 1$

$\tan^2\theta = \sec^2\theta - 1$

$$\int_0^{\frac{\pi}{6}} (\sec x + \tan x)^2\,dx = \int_0^{\frac{\pi}{6}} (2\sec^2 x + 2\sec x \tan x - 1)\,dx$$

$$= \left[2\tan x + 2\sec x - x + C\right]_0^{\frac{\pi}{6}}$$

$$= \left[2\tan\left(\tfrac{\pi}{6}\right) + 2\sec\left(\tfrac{\pi}{6}\right) - \left(\tfrac{\pi}{6}\right) + C\right] - \left[2\tan(0) + 2\sec(0) - (0) + C\right]$$

$$= \left[2\left(\tfrac{1}{\sqrt{3}}\right) + 2\left(\tfrac{2}{\sqrt{3}}\right) - \tfrac{\pi}{6}\right] - \left[2(0) + 2(1) - 0\right] = \left[\tfrac{2}{\sqrt{3}} + \tfrac{4}{\sqrt{3}} - \tfrac{\pi}{6}\right] - [2]$$

$$= \frac{6}{\sqrt{3}} - \frac{\pi}{6} - 2 = \frac{6\sqrt{3}}{3} - \frac{\pi}{6} - 2 = 2\sqrt{3} - \frac{\pi}{6} - 2$$

---

18) $$\int_{-\frac{\pi}{3}}^{-\frac{\pi}{4}} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\frac{\pi}{3}}^{-\frac{\pi}{4}} (4\sec^2 t + \pi t^{-2})\,dt$$

$$= \left[4\tan t + \pi\left[\frac{t^{-1}}{-1}\right] + C\right]_{-\frac{\pi}{3}}^{-\frac{\pi}{4}} = \left[4\tan t - \frac{\pi}{t} + C\right]_{-\frac{\pi}{3}}^{-\frac{\pi}{4}}$$

$$= \left[4\tan\left(\tfrac{-\pi}{4}\right) - \frac{\pi}{\left(\frac{-\pi}{4}\right)} + C\right] - \left[4\tan\left(\tfrac{-\pi}{3}\right) - \frac{\pi}{\left(\frac{-\pi}{3}\right)} + C\right]$$

$$= \left[4(-1) + 4\right] - \left[4\left(\tfrac{-\sqrt{3}}{1}\right) + 3\right] = [0] - \left[-4\sqrt{3} + 3\right]$$

$$= 4\sqrt{3} - 3$$

20) $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^3+t^2+4t+4)\,dt$

$= \left[\frac{t^4}{4} + \frac{t^3}{3} + 2t^2 + 4t + C\right]_{-\sqrt{3}}^{\sqrt{3}}$

$= \left[\frac{(\sqrt{3})^4}{4} + \frac{(\sqrt{3})^3}{3} + 2(\sqrt{3})^2 + 4(\sqrt{3}) + C\right] - \left[\frac{(-\sqrt{3})^4}{4} + \frac{(-\sqrt{3})^3}{3} + 2(-\sqrt{3})^2 + 4(-\sqrt{3}) + C\right]$

$= \left[\frac{9}{4} + \sqrt{3} + 6 + 4\sqrt{3}\right] - \left[\frac{9}{4} - \sqrt{3} + 6 - 4\sqrt{3}\right] = 2\sqrt{3} + 8\sqrt{3} = 10\sqrt{3}$

---

22) $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\,dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^2}\right) dy = \int_{-3}^{-1} \left(y^2 - \frac{2}{y^2}\right) dy$

$= \int_{-3}^{-1} (y^2 - 2y^{-2})\,dy = \left[\frac{y^3}{3} - 2\left(\frac{y^{-1}}{-1}\right) + C\right]_{-3}^{-1} = \left[\frac{y^3}{3} + \frac{2}{y} + C\right]_{-3}^{-1}$

$= \left[\frac{(-1)^3}{3} + \frac{2}{(-1)} + C\right] - \left[\frac{(-3)^3}{3} + \frac{2}{(-3)} + C\right] = \left[\frac{-1}{3} - 2\right] - \left[-9 - \frac{2}{3}\right]$

$= \frac{-1}{3} - 2 + 9 + \frac{2}{3} = 7 + \frac{1}{3} = \frac{21}{3} + \frac{1}{3} = \frac{22}{3}$

---

24) $\int_1^8 \frac{(x^{\frac{1}{3}}+1)(2-x^{2/3})}{x^{\frac{1}{3}}}\,dx = \int_1^8 \left(\frac{2 + 2x^{\frac{1}{3}} - x^{\frac{2}{3}} - x^{\frac{3}{3}}}{x^{\frac{1}{3}}}\right) dx$

$= \int_1^8 \left(\frac{2}{x^{\frac{1}{3}}} + \frac{2x^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}} - \frac{x^{\frac{3}{3}}}{x^{\frac{1}{3}}}\right) dx = \int_1^8 \left(\frac{2}{x^{\frac{1}{3}}} + 2 - x^{\frac{1}{3}} - x^{\frac{2}{3}}\right) dx$

$= \int_1^8 \left(2x^{-\frac{1}{3}} + 2 - x^{\frac{1}{3}} - x^{\frac{2}{3}}\right) dx = \left[2\left(\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right) + 2x - \left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) - \left(\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\right) + C\right]_1^8$

$= \left[3(\sqrt[3]{x})^2 + 2x - \frac{3}{4}(\sqrt[3]{x})^4 - \frac{3}{5}(\sqrt[3]{x})^5 + C\right]_1^8$

$= \left[3(\sqrt[3]{(8)})^2 + 2(8) - \frac{3}{4}(\sqrt[3]{(8)})^4 - \frac{3}{5}(\sqrt[3]{(8)})^5 + C\right] - \left[3(\sqrt[3]{(1)})^2 + 2(1) - \frac{3}{4}(\sqrt[3]{(1)})^4 - \frac{3}{5}(\sqrt[3]{(1)})^5 + C\right]$

24) continued

$= \left[3(2)^2 + 16 - \frac{3}{4}(2)^4 - \frac{3}{5}(2)^5\right] - \left[3(1)^2 + 2 - \frac{3}{4}(1)^4 - \frac{3}{5}(1)^5\right]$

$= \left[12 + 16 - 12 - \frac{3}{5}(32)\right] - \left[3 + 2 - \frac{3}{4} - \frac{3}{5}\right] = \left[16 - \frac{96}{5}\right] - \left[5 - \frac{3}{4} - \frac{3}{5}\right]$

$= 11 - \frac{96}{5} + \frac{3}{4} + \frac{3}{5} = \frac{220}{20} - \frac{384}{20} + \frac{15}{20} + \frac{12}{20} = \frac{-137}{20}$

---

26) $(\cos x + \sec x)^2 = \cos^2 x + 2\cos x \sec x + \sec^2 x$

$= \frac{1+\cos(2x)}{2} + 2\cos x\left(\frac{1}{\cos x}\right) + \sec^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) + 2 + \sec^2 x$

$= \frac{1}{2}\cos(2x) + \frac{5}{2} + \sec^2 x$

$\int \left(\frac{1}{2}\cos(2x) + \frac{5}{2} + \sec^2 x\right)dx = \int \frac{1}{2}\cos(2x)\,dx + \int \frac{5}{2}\,dx + \int \sec^2 x\,dx$

$\int \frac{1}{2}\cos(2x)\,dx = \int \frac{1}{2}\cos p\left(\frac{1}{2}dp\right)$

$p = 2x$ $\quad = \frac{1}{4}\sin p + C$

$dp = 2dx \Rightarrow \frac{1}{2}dp = dx$ $\quad = \frac{1}{4}\sin(2x) + C$

$= \frac{1}{4}\sin(2x) + \frac{5}{2}x + \tan x + C$

$\int_0^{\frac{\pi}{3}} (\cos x + \sec x)^2\,dx = \left[\frac{1}{4}\sin(2x) + \frac{5}{2}x + \tan x + C\right]_0^{\frac{\pi}{3}}$

$= \left[\frac{1}{4}\sin\left(2\left(\frac{\pi}{3}\right)\right) + \frac{5}{2}\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) + C\right] - \left[\frac{1}{4}\sin(2(0)) + \frac{5}{2}(0) + \tan(0) + C\right]$

$= \left[\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) + \frac{5\pi}{6} + \left(\frac{\sqrt{3}}{1}\right)\right] - \left[\frac{1}{4}(0) + 0 + 0\right] = \frac{\sqrt{3}}{8} + \frac{5\pi}{6} + \sqrt{3}$

$= \frac{\sqrt{3}}{8} + \frac{5\pi}{6} + \frac{8\sqrt{3}}{8} = \frac{5\pi}{6} + \frac{9\sqrt{3}}{8}$

28) $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|)dx$ since $\cos x$ is positive on $(0, \frac{\pi}{2})$ and negative on $(\frac{\pi}{2}, \pi)$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos x + (+\cos x))dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(\cos x + (-\cos x))dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos x + \cos x)dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(0)dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos x)dx$

$= \int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x + C]_0^{\frac{\pi}{2}} = [\sin(\frac{\pi}{2}) + C] - [\sin(0) + C] = [1] - [0] = 1$

---

30) $\int (\frac{1}{x} - e^{-x})dx = \int \frac{1}{x}dx - \int e^{-x}dx$

$\int e^{-x}dx = \int e^{p}(-1dp)$
$= -e^{p} + C$
$= -e^{-x} + C$

$p = -x$
$dp = -1dx$
$-1dp = dx$

$= \ln|x| - (-e^{-x}) + C$

$= \ln|x| + e^{-x} + C = \ln|x| + \frac{1}{e^{x}} + C$

$\int_1^2 (\frac{1}{x} - e^{-x})dx = [\ln|x| + \frac{1}{e^{x}} + C]_1^2$

$= [\ln|(2)| + \frac{1}{e^{(2)}} + C] - [\ln|(1)| + \frac{1}{e^{(1)}} + C]$

$= [\ln 2 + \frac{1}{e^2}] - [0 + \frac{1}{e}] = \ln 2 + \frac{1}{e^2} - \frac{1}{e}$

---

32) $\int \frac{dx}{1+4x^2} = \int \frac{1}{1+(2x)^2}dx = \int \frac{1}{1+p^2}(\frac{1}{2}dp) = \frac{1}{2}\int \frac{1}{(1)^2+p^2}dp$

$p = 2x$
$dp = 2dx$
$\frac{1}{2}dp = dx$

$= \frac{1}{2}(\frac{1}{1}\tan^{-1}(\frac{p}{1})) + C = \frac{1}{2}\tan^{-1}(\frac{2x}{1}) + C$

$= \frac{1}{2}\tan^{-1}(2x) + C$

32) continued

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+4x^2} = \left[\frac{1}{2}\tan^{-1}(2x)+C\right]_0^{\frac{1}{\sqrt{3}}} = \left[\frac{1}{2}\tan^{-1}\left(2\left(\frac{1}{\sqrt{3}}\right)\right)+C\right] - \left[\frac{1}{2}\tan^{-1}(2(0))+C\right]$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)\right] - \left[\frac{1}{2}(0)\right] = \frac{1}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

---

34) $\int \pi^{x-1}\,dp = \int \pi^{p}\,dp = \frac{1}{\ln \pi}\pi^{p} + C$

$p = x-1$ $\qquad = \frac{1}{\ln \pi}\pi^{x-1} + C$

$dp = dx$

$$\int_{-1}^{0} \pi^{x-1}\,dx = \left[\frac{1}{\ln \pi}\pi^{x-1}+C\right]_{-1}^{0} = \left[\frac{1}{\ln \pi}\pi^{(0)-1}+C\right] - \left[\frac{1}{\ln \pi}\pi^{(-1)-1}+C\right]$$

$$= \left[\frac{1}{\ln \pi}\left(\frac{1}{\pi}\right)\right] - \left[\frac{1}{\ln \pi}\left(\frac{1}{\pi^2}\right)\right] = \frac{1}{\ln \pi}\left\{\frac{1}{\pi} - \frac{1}{\pi^2}\right\} = \frac{1}{\ln \pi}\left\{\frac{\pi - 1}{\pi^2}\right\}$$

---

36) $\int \frac{\ln x}{x}\,dx = \int \ln x\left(\frac{1}{x}dx\right) = \int p\,dp = \frac{p^2}{2} + C$

$p = \ln x$ $\qquad = \frac{(\ln x)^2}{2} + C = \frac{1}{2}(\ln x)^2 + C$

$dp = \frac{1}{x}dx$

$$\int_1^2 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2 + C\right]_1^2 = \left[\frac{1}{2}(\ln(2))^2 + C\right] - \left[\frac{1}{2}(\ln(1))^2 + C\right]$$

$$= \left[\frac{1}{2}(\ln 2)^2\right] - \left[\frac{1}{2}(0)^2\right] = \frac{1}{2}(\ln 2)^2$$

---

38) $\int \sin^2 x \cos x\,dx = \int p^2\,dp = \frac{p^3}{3} + C$

$p = \sin x$ $\qquad = \frac{\sin^3 x}{3} + C = \frac{1}{3}\sin^3 x + C$

$dp = \cos x\,dx$

38) continued

$$\int_0^{\frac{\pi}{3}} \sin^2 x \cos x \, dx = \left[\frac{1}{3}\sin^3 x + C\right]_0^{\frac{\pi}{3}} = \left[\frac{1}{3}\sin^3\left(\frac{\pi}{3}\right) + C\right] - \left[\frac{1}{3}\sin^3(0) + C\right]$$

$$= \left[\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3\right] - \left[\frac{1}{3}(0)^3\right] = \frac{1}{3}\left(\frac{3\sqrt{3}}{8}\right) = \frac{\sqrt{3}}{8}$$

40-a) $\int_1^{\sin x} 3t^2 \, dt = \left[t^3 + C\right]_1^{\sin x} = \left[(\sin x)^3 + C\right] - \left[(1)^3 + C\right] = \sin^3 x - 1$

$$\frac{d}{dx}\left(\int_1^{\sin x} 3t^2 \, dt\right) = \frac{d}{dx}\left(\sin^3 x - 1\right) = \left[3\sin^2 x\left(\cos x(1)\right)\right] - \left[0\right] = 3\sin^2 x \cos x$$

40-b) $\frac{d}{dx}(\sin x) = \cos x(1) = \cos x$

$$\frac{d}{dx}\left(\int_1^{\sin x} 3t^2 \, dt\right) = 3(\sin x)^2 \frac{d}{dx}(\sin x) = 3\sin^2 x \cos x$$

42-a) $\int_0^{\tan\theta} \sec^2 y \, dy = \left[\tan y + C\right]_0^{\tan\theta} = \left[\tan(\tan\theta) + C\right] - \left[\tan(0) + C\right]$

$$= \tan(\tan\theta) - 0 = \tan(\tan\theta)$$

$$\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y \, dy\right) = \frac{d}{d\theta}\left(\tan(\tan\theta)\right) = \sec^2(\tan\theta)\left(\sec^2\theta(1)\right)$$

$$= \sec^2(\tan\theta)\sec^2\theta$$

42-b) $\frac{d}{d\theta}(\tan\theta) = \sec^2\theta(1) = \sec^2\theta$

$$\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y \, dy\right) = \sec^2(\tan\theta)\frac{d}{d\theta}(\tan\theta) = \sec^2(\tan\theta)\sec^2\theta$$

44-a) $\int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx = \left[\frac{x^5}{5} + 3\left(\sin^{-1}\left(\frac{x}{1}\right)\right) + C\right]_0^{\sqrt{t}}$

$$= \left[\frac{(\sqrt{t})^5}{5} + 3\sin^{-1}(\sqrt{t}) + C\right] - \left[\frac{(0)^5}{5} + 3\sin^{-1}(0) + C\right]$$

$$= \frac{1}{5} t^{\frac{5}{2}} + 3\sin^{-1}(\sqrt{t}) - 0 = \frac{1}{5}(\sqrt{t})^5 + 3\sin^{-1}(\sqrt{t})$$

$$\frac{d}{dt}\left(\int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx\right) = \frac{d}{dt}\left(\frac{1}{5} t^{\frac{5}{2}} + 3\sin^{-1}(\sqrt{t})\right)$$

$$= \frac{d}{dt}\left(\frac{1}{5} t^{\frac{5}{2}}\right) + 3\frac{d}{dt}\left(\sin^{-1}(\sqrt{t})\right)$$

$$= \frac{1}{5}\left[\frac{5}{2} t^{\frac{3}{2}}\right] + 3\left[\frac{1}{2\sqrt{t}\sqrt{1-t}}\right]$$

$$= \frac{1}{2}(\sqrt{t})^3 + \frac{3}{2\sqrt{t}\sqrt{1-t}}$$

$p = \sin^{-1}\sqrt{t}$

$\Downarrow$

$\sin p = \sqrt{t} = t^{\frac{1}{2}}$

$\cos p \frac{dp}{dt} = \frac{1}{2} t^{-\frac{1}{2}}$

$\frac{dp}{dt} = \frac{1}{2\sqrt{t}\cos p} = \frac{1}{2\sqrt{t}\left(\frac{\sqrt{(1)^2 - (\sqrt{t})^2}}{1}\right)}$

$= \frac{1}{2\sqrt{t}\sqrt{1-t}}$

44-b) $\frac{d}{dt}(\sqrt{t}) = \frac{d}{dt}\left(t^{\frac{1}{2}}\right) = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

$$\frac{d}{dt}\left(\int_0^{\sqrt{t}} \left(x^4 + \frac{3}{\sqrt{1-x^2}}\right) dx\right) = \left((\sqrt{t})^4 + \frac{3}{\sqrt{1-(\sqrt{t})^2}}\right)\frac{d}{dt}(\sqrt{t})$$

$$= \left(t^2 + \frac{3}{\sqrt{1-t}}\right)\left(\frac{1}{2\sqrt{t}}\right)$$

$$= \frac{t^2}{2\sqrt{t}} + \frac{3}{2\sqrt{t}\sqrt{1-t}} = \frac{(\sqrt{t})^4}{2\sqrt{t}} + \frac{3}{2\sqrt{t}\sqrt{1-t}}$$

$$= \frac{1}{2}(\sqrt{t})^3 + \frac{3}{2\sqrt{t}\sqrt{1-t}}$$

46) $y = \int_1^x \frac{1}{t}\,dt \quad x>0$

$\frac{d}{dx}(x) = 1$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{(x)}\frac{d}{dx}(x) = \frac{1}{x}(1) = \frac{1}{x}$$

---

48) $y = x\int_2^{x^2} \sin(t^3)\,dt \qquad \frac{d}{dx}(x^2) = 2x$

product rule needed here

$$\frac{dy}{dx} = (x)\left[\frac{d}{dx}\left(\int_2^{x^2} \sin(t^3)\,dt\right)\right] + \left(\int_2^{x^2} \sin(t^3)\,dt\right)[1]$$

$$= (x)\left[\sin((x^2)^3)\frac{d}{dx}(x^2)\right] + \left(\int_2^{x^2} \sin(t^3)\,dt\right)[1]$$

$$= (x)\left[\sin(x^6)(2x)\right] + \int_2^{x^2} \sin(t^3)\,dt$$

$$= 2x^2\sin(x^6) + \int_2^{x^2} \sin(t^3)\,dt$$

note: $\int_2^{x^2} \sin(t^3)\,dt$ is not integratable

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50) $y = \left(\int_0^x (t^3+1)^{10}\,dt\right)^3 \qquad \frac{d}{dx}(x) = 1$

$$\frac{dy}{dx} = 3\left(\int_0^x (t^3+1)^{10}\,dt\right)^2\left(\frac{d}{dx}\int_0^x (t^3+1)^{10}\,dt\right)$$

$$= 3\left(\int_0^x (t^3+1)^{10}\,dt\right)^2\left((x^3+1)^{10}\frac{d}{dx}(x)\right)$$

$$= 3\left(\int_0^x (t^3+1)^{10}\,dt\right)^2\left((x^3+1)^{10}(1)\right)$$

$$= 3(x^3+1)^{10}\left(\int_0^x (t^3+1)^{10}\,dt\right)$$

52) $y=\int_{\tan x}^{0}\frac{dt}{1+t^2}=-\int_{0}^{\tan x}\frac{1}{1+t^2}\,dt$ $\qquad \frac{d}{dx}(\tan x)=\sec^2 x(1)=\sec^2 x$

$$\frac{dy}{dx}=-\frac{d}{dx}\left(\int_0^{\tan x}\frac{1}{1+t^2}\,dt\right)=-\left(\frac{1}{1+(\tan x)^2}\right)\frac{d}{dx}(\tan x)$$

$$=\frac{-1}{1+\tan^2 x}(\sec^2 x)=\frac{-1}{\sec^2 x}(\sec^2 x)=-1$$

---

54) $y=\int_{2^x}^{1}\sqrt[3]{t}\,dt=-\int_{1}^{2^x}\sqrt[3]{t}\,dt$

$p=2^x$ $\qquad \frac{d}{dx}(2^x)$

$\ln p=\ln 2^x$ $\qquad =(\ln 2)2^x$

$\ln p=x\ln 2$

$\frac{1}{p}\frac{dp}{dx}=\ln 2$

$\frac{dp}{dx}=(\ln 2)p=(\ln 2)2^x$

$$\frac{dy}{dx}=-\frac{d}{dx}\left(\int_1^{2^x}\sqrt[3]{t}\,dt\right)$$

$$=-\left(\sqrt[3]{(2^x)}\right)\frac{d}{dx}(2^x)=-\left(\sqrt[3]{2^x}\right)\left((\ln 2)2^x\right)$$

$$=-\left(\sqrt[3]{2^x}\right)(\ln 2)\left(\sqrt[3]{2^x}\right)^3=-(\ln 2)\left(\sqrt[3]{2^x}\right)^4$$

---

56) $y=\int_{-1}^{x^{\frac{1}{\pi}}}\sin^{-1}t\,dt$ $\qquad \frac{d}{dx}\left(x^{\frac{1}{\pi}}\right)=\frac{1}{\pi}x^{\frac{1}{\pi}-1}$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\int_{-1}^{x^{\frac{1}{\pi}}}\sin^{-1}t\,dt\right)=\left(\sin^{-1}\left(x^{\frac{1}{\pi}}\right)\right)\frac{d}{dx}\left(x^{\frac{1}{\pi}}\right)$$

$$=\left(\sin^{-1}\left(x^{\frac{1}{\pi}}\right)\right)\left(\frac{1}{\pi}x^{\frac{1}{\pi}-1}\right)=\frac{1}{\pi}\left(x^{\frac{1}{\pi}-1}\right)\sin^{-1}\left(x^{\frac{1}{\pi}}\right)$$

58) $y = 3x^2 - 3 \qquad -2 \le x \le 2$

$0 = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$0 = 3(x+1)(x-1)$

$x + 1 = 0 \mid x - 1 = 0$

$x = -1 \mid x = 1$

$\int (3x^2 - 3)\,dx = x^3 - 3x + C$

$A_L = \int_{-2}^{-1} (3x^2-3)\,dx = \left[x^3 - 3x + C\right]_{-2}^{-1} = \left[(-1)^3 - 3(-1) + C\right] - \left[(-2)^3 - 3(-2) + C\right]$

$= [-1+3] - [-8+6] = [2] - [-2] = 2 + 2 = 4$

$A_M = -\int_{-1}^{1} (3x^2-3)\,dx = -\left[x^3 - 3x + C\right]_{-1}^{1} = -\left\{\left[(1)^3 - 3(1) + C\right] - \left[(-1)^3 - 3(-1) + C\right]\right\}$

$= -\left\{[1-3] - [-1+3]\right\} = -\left\{[-2] - [2]\right\} = -\{-4\} = 4$

$A_R = \int_{1}^{2} (3x^2-3)\,dx = \left[x^3 - 3x + C\right]_{1}^{2} = \left[(2)^3 - 3(2) + C\right] - \left[(1)^3 - 3(1) + C\right]$

$= [8-6] - [1-3] = [2] - [-2] = 4$

$A = A_L + A_M + A_R = (4) + (4) + (4) = 12 \text{ unit}^2$

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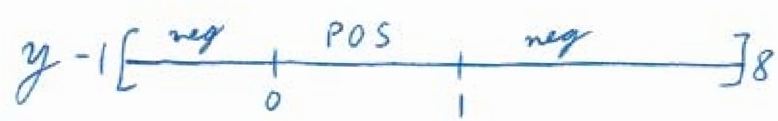
60) $y = x^{\frac{1}{3}} - x = \sqrt[3]{x} - x \qquad -1 \le x \le 8$



$0 = \sqrt[3]{x} - x$

$0 = \sqrt[3]{x} - (\sqrt[3]{x})^3$

$0 = \sqrt[3]{x}\left(1 - (\sqrt[3]{x})^2\right)$

$0 = \sqrt[3]{x}\,(1 + \sqrt[3]{x})(1 - \sqrt[3]{x})$

$\sqrt[3]{x} = 0 \mid 1 + \sqrt[3]{x} = 0 \mid 1 - \sqrt[3]{x} = 0$

$x = 0 \mid \sqrt[3]{x} = -1 \mid \sqrt[3]{x} = 1$

$\quad \mid x = -1 \mid x = 1$

$\int (x^{\frac{1}{3}} - x)\,dx = \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} - \dfrac{x^2}{2} + C$

$= \frac{3}{4} x^{\frac{4}{3}} - \frac{1}{2} x^2 + C$

$= \frac{3}{4} (\sqrt[3]{x})^4 - \frac{1}{2} x^2 + C$

60) continued

$$A_L = \int_{-1}^{0} -(x^{\frac{1}{3}} - x)\,dx = -\left[\frac{3}{4}(\sqrt[3]{x})^4 - \frac{1}{2}x^2 + C\right]_{-1}^{0}$$

$$= -\left\{\left[\frac{3}{4}(\sqrt[3]{(0)})^4 - \frac{1}{2}(0)^2 + C\right] - \left[\frac{3}{4}(\sqrt[3]{(-1)})^4 - \frac{1}{2}(-1)^2 + C\right]\right\}$$

$$= -\left\{[0] - \left[\frac{3}{4} - \frac{1}{2}\right]\right\} = -\left\{\frac{-1}{4}\right\} = \frac{1}{4}$$

$$A_M = \int_0^1 (x^{\frac{1}{3}} - x)\,dx = \left[\frac{3}{4}(\sqrt[3]{x})^4 - \frac{1}{2}x^2 + C\right]_0^1$$

$$= \left[\frac{3}{4}(\sqrt[3]{(1)})^4 - \frac{1}{2}(1)^2 + C\right] - \left[\frac{3}{4}(\sqrt[3]{(0)})^4 - \frac{1}{2}(0)^2 + C\right]$$

$$= \left[\frac{3}{4} - \frac{1}{2}\right] - [0] = \frac{1}{4}$$

$$A_R = \int_1^8 -(x^{\frac{1}{3}} - x)\,dx = -\left[\frac{3}{4}(\sqrt[3]{x})^4 - \frac{1}{2}x^2 + C\right]_1^8$$

$$= -\left\{\left[\frac{3}{4}(\sqrt[3]{(8)})^4 - \frac{1}{2}(8)^2 + C\right] - \left[\frac{3}{4}(\sqrt[3]{(1)})^4 - \frac{1}{2}(1)^2\right]\right\}$$

$$= -\left\{\left[\frac{3}{4}(2)^4 - (8)(4)\right] - \left[\frac{3}{4} - \frac{1}{2}\right]\right\} = -\left\{[12 - 32] - \left[\frac{1}{4}\right]\right\}$$

$$= -\left\{-20 - \frac{1}{4}\right\} = 20 + \frac{1}{4}$$

$$A = A_L + A_M + A_R = \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right) + \left(20 + \frac{1}{4}\right) = 20 + \frac{3}{4} = \frac{80}{4} + \frac{3}{4} = \frac{83}{4} \text{ unit}^2$$

---

62) curve: $A_C = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x\,dx = \left[-\cos x + C\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \left[-\cos\left(\frac{5\pi}{6}\right) + C\right] - \left[-\cos\left(\frac{\pi}{6}\right) + C\right]$

$$= \left[-\left(\frac{-\sqrt{3}}{2}\right)\right] - \left[-\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

rectangle: $w = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $l = \left(\frac{5\pi}{6}\right) - \left(\frac{\pi}{6}\right) = \frac{4\pi}{6} = \frac{2\pi}{3}$

$$A_R = lw = \left(\frac{2\pi}{3}\right)\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$A = A_C - A_R = (\sqrt{3}) - \left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3} \text{ unit}^2$$

64) rectangle: $w=2$, $l=(1)-(\frac{-\pi}{4})=(1+\frac{\pi}{4})$ $A_R = lw = (1+\frac{\pi}{4})(2) = 2+\frac{\pi}{2}$

curve: $A_T = \int_{-\frac{\pi}{4}}^{0} \sec^2 t\, dt = [\tan t + C]_{-\frac{\pi}{4}}^{0} = [\tan(0)+C]-[\tan(\frac{-\pi}{4})+C]$

$= [0]-[-1] = 1$

$A_P = \int_0^1 (1-t^2)dt = [t - \frac{t^3}{3} + C]_0^1 = [(1)-\frac{(1)^3}{3}+C]-[(0)-\frac{(0)^3}{3}+C]$

$= [1-\frac{1}{3}]-[0] = 1-\frac{1}{3} = \frac{2}{3}$

$A = A_R - A_T - A_P = (2+\frac{\pi}{2})-(1)-(\frac{2}{3}) = \frac{1}{3}+\frac{\pi}{2}$ units^2

---

66) $y' = \sec x$, $y(-1)=4$

$y = \int_{-1}^{x} \sec t\, dt \Rightarrow \frac{dy}{dx} = \sec x$ and $y(-1) = \int_{-1}^{-1} \sec t\, dt + 4 = 0+4 = 4$

equation c is a solution

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68) $y' = \frac{1}{x}$, $y(1) = -3$

$y = \int_1^x \frac{1}{t}\, dt \Rightarrow \frac{dy}{dx} = \frac{1}{x}$ and $y(1) = \int_1^1 \frac{1}{t}\, dt - 3 = 0-3 = -3$

equation a is a solution

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70) $\frac{dy}{dx} = \sqrt{1+x^2}$, $y(1) = -2$

$y = \int_1^x \sqrt{1+t^2}\, dt - 2$