Definition:

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number J is the **definite integral of f over** $[a,b]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon .$$

A Formula for the Riemann Sum with Equal-Width Subintervals

(Equation 1) $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\Delta x\right)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

Theorem 1-Integrability of Continuous Functions

If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Theorem 2

When f and g are integrable over the interval $[a,b]$, othe definite integral satisfies the rules listed in Table 5.6.

Table 5.6 Rules satisfied by definite integrals

1. Order of Integration: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ {a definition}

2. Zero Width Interval: $\int_a^a f(x)\,dx = 0$ {a definition when $f(a)$ exists}

3. Constant Multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ {any constant k}

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. Max-Min Inequality: If f has maximum value max f and minimum value min f on $[a,b]$, then $(\min f)(b-a) \le \int_a^b f(x)\,dx \le (\max f)(b-a)$.

7. Domination: If $f(x) \ge g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

If $f(x) \ge 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \ge 0$. {special case}

Definition
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **over** $[a,b]$ is the integral of *f* from *a* to *b*,

$$A = \int_a^b f(x)\,dx\,.$$

(Equation 2) $$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad a < b$$

(Equation 3) $$\int_a^b c\,dx = c(b-a) \qquad c \text{ any constant}$$

(Equation 4) $$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} \qquad a < b$$

Definition
If *f* is integrable on $[a,b]$, then its **average value on** $[a,b]$, which is also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The textbook have exercises in this section where we need to evaluate the definite integral. So they list 3 formulas (Equation 2), (Equation 3), and (Equation 4) in order for us to be able to evaluate.

Instead, I believe that it is more efficient if we first cover the section 5.4 and use the Fundamental Theorem of Calculus part 2 to solve the integration exercises given in this section.

Therefore, on my examples, the part of the exercises in this section using Equations 2-4 will be shown using the Fundamental Theorem of Calculus part 2.

2) $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} 2c_k^3 \Delta x_k$, $[-1,0] \Rightarrow \int_{-1}^{0} 2x^3 dx$

4) $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, $[1,4] \Rightarrow \int_{1}^{4} \frac{1}{x} dx$

6) $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} \sqrt{4-c_k^2}\, \Delta x_k$, $[0,1] \Rightarrow \int_{0}^{1} \sqrt{4-x^2}\, dx$

8) $\lim\limits_{\|P\|\to 0} \sum\limits_{k=1}^{n} (\tan c_k) \Delta x_k$, $[0,\frac{\pi}{4}] \Rightarrow \int_{0}^{\frac{\pi}{4}} \tan x \, dx$

---

10) $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, $\int_7^9 h(x)\,dx = 4$

a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

b) $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = (5)+(4) = 9$

c) $\int_7^9 [2f(x)-3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5)-3(4) = 10-12 = -2$

d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = (-1)-(5) = -6$

f) $\int_9^7 [h(x)-f(x)]\,dx = -\int_7^9 [h(x)-f(x)]\,dx = -\left\{\int_7^9 h(x)\,dx - \int_7^9 f(x)\,dx\right\}$

$= -\{(4)-(5)\} = -\{-1\} = 1$

12) $\int_{-3}^{0} g(t)\,dt = \sqrt{2}$

a) $\int_{0}^{-3} g(t)\,dt = -\int_{-3}^{0} g(t)\,dt = -(\sqrt{2}) = -\sqrt{2}$

b) $\int_{-3}^{0} g(u)\,du = \int_{-3}^{0} g(t)\,dt = (\sqrt{2}) = \sqrt{2}$

c) $\int_{-3}^{0} [-g(x)]\,dx = -\int_{-3}^{0} g(x)\,dx = -\int_{-3}^{0} g(t)\,dt = -(\sqrt{2}) = -\sqrt{2}$

d) $\int_{-3}^{0} \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^{0} g(r)\,dr = \frac{1}{\sqrt{2}}\int_{-3}^{0} g(t)\,dt = \frac{1}{\sqrt{2}}(\sqrt{2}) = 1$

---

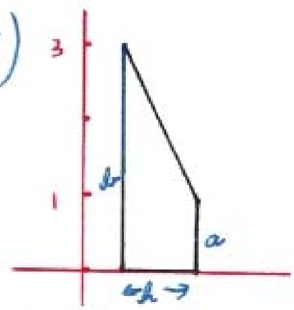
14) $\int_{-1}^{1} h(r)\,dr = 0, \quad \int_{-1}^{3} h(r)\,dr = 6$

a) $\int_{1}^{3} h(r)\,dr = \int_{-1}^{3} h(r)\,dr - \int_{-1}^{1} h(r)\,dr = (6) - (0) = 6$

b) $-\int_{3}^{1} h(u)\,du = -\left\{-\int_{1}^{3} h(u)\,du\right\} = \int_{1}^{3} h(u)\,du = \int_{1}^{3} h(r)\,dr = (6) = 6$

---

16)



$A = \frac{1}{2}h(a+b)$

$\int_{\frac{1}{2}}^{\frac{3}{2}} (-2x+4)\,dx$ generates a trapezoid

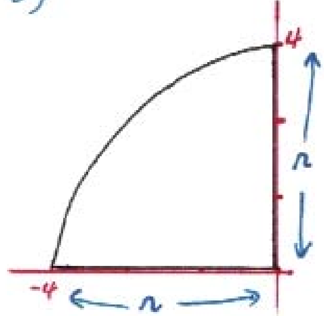
$f(x) = -2x+4 \qquad a = f(\tfrac{3}{2}) = -2(\tfrac{3}{2})+4 = -3+4 = 1$

$h = (\tfrac{3}{2}) - (\tfrac{1}{2}) = \tfrac{2}{2} = 1 \qquad b = f(\tfrac{1}{2}) = -2(\tfrac{1}{2})+4 = -1+4 = 3$

$A = \frac{1}{2}(1)((1)+(3)) = \frac{1}{2}(1)(4) = 2$ units^2

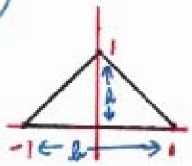
---

18) $\int_{-4}^{0} \sqrt{16-x^2}\,dx = \int_{4}^{0} \sqrt{(4)^2-x^2}\,dx$ generates a quarter of a circle



$r = 4 \qquad A = \frac{\pi r^2}{4}$

$A = \frac{\pi(4)^2}{4} = 4\pi$ units^2

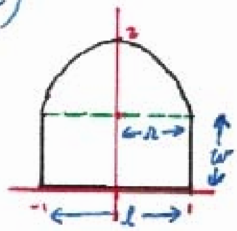
20) 

$\int_{-1}^{1}(1-|x|)\,dx$ generates a triangle

$b=2 \quad h=1 \qquad A=\frac{1}{2}hb$

$A=\frac{1}{2}(1)(2)=1$ units^2

---

22) 

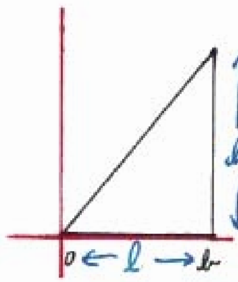
$\int_{-1}^{1}(1+\sqrt{1-x^2})\,dx$ generates a semi circle on top of a rectangle

$l=2,\ w=1\ ;\ r=1$

$A_R=lw=(2)(1)=2 \qquad A_{SC}=\frac{\pi r^2}{2}=\frac{\pi(1)^2}{2}=\frac{\pi}{2}$

$A=A_R+A_{SC}=(2)+(\frac{\pi}{2})=(2+\frac{\pi}{2})$ units^2

---

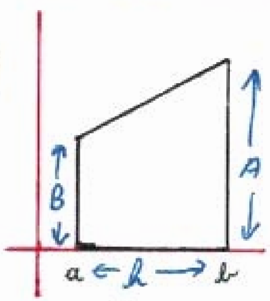
24) 

$\int_0^b 4x\,dx$, $b>0$ generates a triangle above x-axis

$l=(b)-(0)=b \qquad f(x)=4x \quad h=f(b)=4b$

$\int_0^b 4x\,dx=\frac{1}{2}lh=\frac{1}{2}(b)(4b)=2b^2$ units^2

---

26) 

$\int_a^b 3t\,dt,\ 0<a<b$ generates a trapezoid above x-axis

$f(t)=3t \qquad A=f(a)=3a \qquad B=f(b)=3b$

$h=(b)-(a)=(b-a)$

$\int_a^b 3t\,dt=\frac{1}{2}h(A+B)=\frac{1}{2}(b-a)((3a)+(3b))=\frac{1}{2}(b-a)(3(a+b))=\frac{3}{2}(b-a)(b+a)$

$=\frac{3}{2}(b^2-a^2)$ units^2

---

28) $f(x)=3x+\sqrt{1-x^2}$

a) $[-1,0]$

triangle below x-axis

quarter circle

$\int_{-1}^{0}(3x+\sqrt{1-x^2})\,dx=\int_{-1}^{0}3x\,dx+\int_{-1}^{0}\sqrt{1-x^2}\,dx$

$=-\left\{\frac{1}{2}(1)(3)\right\}+\left\{\frac{\pi(1)^2}{4}\right\}=\frac{-3}{2}+\frac{\pi}{4}=\frac{\pi}{4}-\frac{3}{2}$ units^2

28) continued

b) $[-1,1]$

triangle below x-axis — triangle above x-axis — quarter circle

$$\int_{-1}^{1}(3x+\sqrt{1-x^2})\,dx = \int_{-1}^{1}3x\,dx + \int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-1}^{0}3x\,dx + \int_{0}^{1}3x\,dx + \int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$= -\left\{\tfrac{1}{2}(1)(3)\right\} + \left\{\tfrac{1}{2}(1)(3)\right\} + \left\{\frac{\pi(1)^2}{2}\right\} = \frac{-3}{2} + \frac{3}{2} + \frac{\pi}{2} = \frac{\pi}{2} \text{ units}^2$$

---

for examples 30 to 50, examples written in blue are shown via method of section 5.4; examples written in black are shown using equation of this section (5.3).

30) $\displaystyle\int_{0.5}^{2.5} x\,dx = \int_{\frac{1}{2}}^{\frac{5}{2}} x\,dx = \left[\frac{x^2}{2}+C\right]_{\frac{1}{2}}^{\frac{5}{2}} = \left[\frac{(\frac{5}{2})^2}{2}+C\right] - \left[\frac{(\frac{1}{2})^2}{2}+C\right]$

$$= \left[\frac{25}{8}\right] - \left[\frac{1}{8}\right] = \frac{24}{8} = 3$$

$$\int_{0.5}^{2.5} x\,dx = \frac{(2.5)^2}{2} - \frac{(0.5)^2}{2} = \frac{(\frac{5}{2})^2}{2} - \frac{(\frac{1}{2})^2}{2} = \frac{25}{8} - \frac{1}{8} = \frac{24}{8} = 3$$

---

32) $\displaystyle\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \left[\frac{r^2}{2}+C\right]_{\sqrt{2}}^{5\sqrt{2}} = \left[\frac{(5\sqrt{2})^2}{2}+C\right] - \left[\frac{(\sqrt{2})^2}{2}+C\right]$

$$= \left[\frac{25(2)}{2}\right] - \left[\frac{2}{2}\right] = 25 - 1 = 24$$

$$\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \frac{(5\sqrt{2})^2}{2} - \frac{(\sqrt{2})^2}{2} = \frac{25(2)}{2} - \frac{2}{2} = 25 - 1 = 24$$

34) $\int_0^{0.3} s^2\,ds = \int_0^{\frac{3}{10}} s^2\,ds = \left[\frac{s^3}{3}+C\right]_0^{\frac{3}{10}} = \left[\frac{(\frac{3}{10})^3}{3}+C\right]-\left[\frac{(0)^3}{3}+C\right]$

$= \left[\frac{9}{1000}\right]-\left[0\right] = \frac{9}{1000}$

$\int_0^{0.3} s^2\,ds = \frac{(0.3)^3}{3} - \frac{(0)^3}{3} = 0.009 - 0 = 0.009$

---

36) $\int_0^{\frac{\pi}{2}} \theta^2\,d\theta = \left[\frac{\theta^3}{3}+C\right]_0^{\frac{\pi}{2}} = \left[\frac{(\frac{\pi}{2})^3}{3}+C\right]-\left[\frac{(0)^3}{3}+C\right]$

$= \left[\frac{\pi^3}{24}\right]-\left[0\right] = \frac{\pi^3}{24}$

$\int_0^{\frac{\pi}{2}} \theta^2\,d\theta = \frac{(\frac{\pi}{2})^3}{3} - \frac{(0)^3}{3} = \frac{\pi^3}{24} - 0 = \frac{\pi^3}{24}$

---

38) $\int_a^{\sqrt{3}} x\,dx = \left[\frac{x^2}{2}+C\right]_a^{\sqrt{3}} = \left[\frac{(\sqrt{3})^2}{2}+C\right]-\left[\frac{(a)^2}{2}+C\right]$

$= \left[\frac{3}{2}\right]-\left[\frac{a^2}{2}\right] = \frac{3-a^2}{2}$

$\int_a^{\sqrt{3}} x\,dx = \frac{(\sqrt{3})^2}{2} - \frac{(a)^2}{2} = \frac{3}{2} - \frac{a^2}{2} = \frac{3-a^2}{2}$

---

40) $\int_0^{3b} x^2\,dx = \left[\frac{x^3}{3}+C\right]_0^{3b} = \left[\frac{(3b)^3}{3}+C\right]-\left[\frac{(0)^3}{3}+C\right]$

$= \left[9b^3\right]-\left[0\right] = 9b^3$

$\int_0^{3b} x^2\,dx = \frac{(3b)^3}{3} - \frac{(0)^3}{3} = 9b^3$

42) $\int_0^2 5x\,dx = \left[\frac{5}{2}x^2+C\right]_0^2 = \left[\frac{5}{2}(2)^2+C\right]-\left[\frac{5}{2}(0)^2+C\right]$

$= [10]-[0] = 10$

$\int_0^2 5x\,dx = 5\int_0^2 x\,dx = 5\left\{\frac{(2)^2}{2}-\frac{(0)^2}{2}\right\} = 5\{2\} = 10$

---

44) $\int_0^{\sqrt{2}} (t-\sqrt{2})\,dt = \left[\frac{t^2}{2}-\sqrt{2}\,t+C\right]_0^{\sqrt{2}} = \left[\frac{(\sqrt{2})^2}{2}-\sqrt{2}(\sqrt{2})+C\right]-\left[\frac{(0)^2}{2}-\sqrt{2}(0)+C\right]$

$= [1-2]-[0] = -1$

$\int_0^{\sqrt{2}} (t-\sqrt{2})\,dt = \int_0^{\sqrt{2}} t\,dt - \int_0^{\sqrt{2}} \sqrt{2}\,dt = \left\{\frac{(\sqrt{2})^2}{2}-\frac{(0)^2}{2}\right\}-\left\{\sqrt{2}\left((\sqrt{2})-(0)\right)\right\}$

$= \{1\}-\{2\} = -1$

---

46) $\int_3^0 (2z-3)\,dz = \left[z^2-3z+C\right]_3^0 = \left[(0)^2-3(0)+C\right]-\left[(3)^2-3(3)+C\right]$

$= [0]-[9-9] = 0$

$\int_3^0 (2z-3)\,dz = -\int_0^3 (2z-3)\,dz = -\left\{2\int_0^3 z\,dz - \int_0^3 3\,dz\right\}$

$= -\left\{2\left(\frac{(3)^2}{2}-\frac{(0)^2}{2}\right)-3\left((3)-(0)\right)\right\} = -\{9-9\} = -\{0\} = 0$

---

48) $\int_{\frac{1}{2}}^1 24u^2\,du = \left[8u^3+C\right]_{\frac{1}{2}}^1 = \left[8(1)^3+C\right]-\left[8(\tfrac{1}{2})^3+C\right] = [8]-[1] = 7$

$\int_{\frac{1}{2}}^1 24u^2\,du = 24\int_{\frac{1}{2}}^1 u^2\,du = 24\left\{\frac{(1)^3}{3}-\frac{(\frac{1}{2})^3}{3}\right\} = 24\left\{\frac{1}{3}-\frac{1}{24}\right\}$

$= 24\left\{\frac{8}{24}-\frac{1}{24}\right\} = 24\left\{\frac{7}{24}\right\} = 7$

50) $\int_1^0 (3x^2+x-5)\,dx = \left[x^3+\frac{x^2}{2}-5x+C\right]_1^0$

$= \left[(0)^3+\frac{(0)^2}{2}-5(0)+C\right]-\left[(1)^3+\frac{(1)^2}{2}-5(1)+C\right]$

$= [0]-\left[1+\frac{1}{2}-5\right] = 0-\left[\frac{-7}{2}\right] = \frac{7}{2}$

$\int_1^0 (3x^2+x-5)\,dx = -\int_0^1 (3x^2+x-5)\,dx = -\left\{3\int_0^1 x^2\,dx+\int_0^1 x\,dx-\int_0^1 5\,dx\right\}$

$= -\left\{3\left(\frac{(1)^3}{3}-\frac{(0)^3}{3}\right)+\left(\frac{(1)^2}{2}-\frac{(0)^2}{2}\right)-5\big((1)-(0)\big)\right\}$

$= -\left\{3\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)-5(1)\right\} = -\left\{1+\frac{1}{2}-5\right\}$

$= -\left\{\frac{-7}{2}\right\} = \frac{7}{2}$

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52) $y = \pi x^2$ $\quad [0, b]$ $\quad f(x) = y = \pi x^2$

$\Delta x = \frac{(b)-(0)}{n} = \frac{b}{n}$ $\qquad x_k = a + k\Delta x = (0)+k\left(\frac{b}{n}\right) = \frac{bk}{n}$

$f(x_k) = \pi\left(\frac{bk}{n}\right)^2 = \frac{\pi b^2}{n^2}k^2$

$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{\pi b^2}{n^2}k^2\right)\left(\frac{b}{n}\right) = \frac{\pi b^3}{n^3}\sum_{k=1}^{n}k^2$

$= \frac{\pi b^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{\pi b^3}{6}\left(\frac{n(n+1)(2n+1)}{n^3}\right) = \frac{\pi b^3}{6}\left(\frac{n(2n^2+3n+1)}{n^3}\right)$

$= \frac{\pi b^3}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right) = \frac{\pi b^3}{6}\left(\frac{2n^3}{n^3}+\frac{3n^2}{n^3}+\frac{n}{n^3}\right) = \frac{\pi b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$\int_0^b \pi x^2\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\left(\frac{\pi b^2}{n^2}k^2\right)\left(\frac{b}{n}\right)$

$= \lim_{n\to\infty}\left(\frac{\pi b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right) = \frac{\pi b^3}{6}(2+0+0) = \frac{\pi b^3}{3}$

54) $y = \frac{x}{2} + 1 \qquad [0, b] \qquad f(x) = \frac{x}{2} + 1$

$\Delta x = \frac{(b)-(0)}{n} = \frac{b}{n} \qquad x_k = a + k\Delta x = (0) + k\left(\frac{b}{n}\right) = \frac{bk}{n}$

$f(x_k) = \frac{\left(\frac{bk}{n}\right)}{2} + 1 = \frac{b}{2n}k + 1$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{b}{2n}k + 1\right)\left(\frac{b}{n}\right) = \sum_{k=1}^{n}\left(\frac{b^2}{2n^2}k + \frac{b}{n}\right)$$

$$= \sum_{k=1}^{n}\frac{b^2}{2n^2}k + \sum_{k=1}^{n}\frac{b}{n} = \frac{b^2}{2n^2}\sum_{k=1}^{n}k + \frac{b}{n}\sum_{k=1}^{n}1 = \frac{b^2}{2n^2}\left(\frac{n(n+1)}{2}\right) + \frac{b}{n}(n)$$

$$= \frac{b^2}{2(2)}\left(\frac{n(n+1)}{n^2}\right) + b = \frac{b^2}{4}\left(\frac{n^2+n}{n^2}\right) + b = \frac{b^2}{4}\left(\frac{n^2}{n^2} + \frac{n}{n^2}\right) + b$$

$$= \frac{b^2}{4}\left(1 + \frac{1}{n}\right) + b$$

$$\int_0^b \left(\frac{x}{2} + 1\right)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty}\left(\frac{b}{2n}k + 1\right)\left(\frac{b}{n}\right) = \lim_{n\to\infty}\left(\frac{b^2}{4}\left(1 + \frac{1}{n}\right) + b\right)$$

$$= \frac{b^2}{4}(1 + 0) + b = \frac{b^2}{4} + b$$

---

56) $f(x) = \frac{-x^2}{2} \qquad [0, 3] \qquad av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$av(f) = \frac{1}{(3)-(0)}\int_0^3 \frac{-x^2}{2}\,dx = \frac{1}{3}\left[\frac{-x^3}{6} + C\right]_0^3 = \frac{1}{3}\left\{\left[\frac{-(3)^3}{6} + C\right] - \left[\frac{-(0)^3}{6} + C\right]\right\}$$

$$= \frac{1}{3}\left\{\left[\frac{-(3)^3}{6}\right] - [0]\right\} = \frac{1}{3}\left\{\frac{-(3)^3}{6}\right\} = \frac{-3}{2}$$

58) $f(x) = 3x^2 - 3 \qquad [0,1]$

$$av(f) = \frac{1}{(1)-(0)} \int_0^1 (3x^2-3)\,dx = \frac{1}{1}\left[x^3 - 3x + C\right]_0^1$$
$$= \frac{1}{1}\left\{\left[(1)^3 - 3(1) + C\right] - \left[(0)^3 - 3(0) + C\right]\right\} = \frac{1}{1}\left\{[1-3]-[0]\right\} = \frac{1}{1}\{-2\} = -2$$

---

60) $f(t) = t^2 - t \qquad [-2,1]$

$$av(f) = \frac{1}{(1)-(-2)} \int_{-2}^1 (t^2 - t)\,dt = \frac{1}{1+2}\left[\frac{t^3}{3} - \frac{t^2}{2} + C\right]_{-2}^1$$
$$= \frac{1}{3}\left\{\left[\frac{(1)^3}{3} - \frac{(1)^2}{2} + C\right] - \left[\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + C\right]\right\} = \frac{1}{3}\left\{\left[\frac{1}{3} - \frac{1}{2}\right] - \left[\frac{-8}{3} - 2\right]\right\}$$
$$= \frac{1}{3}\left\{\frac{1}{3} - \frac{1}{2} + \frac{8}{3} + 2\right\} = \frac{1}{3}\left\{\frac{9}{3} - \frac{1}{2} + 2\right\} = \frac{1}{3}\left\{3 - \frac{1}{2} + 2\right\} = \frac{1}{3}\left\{5 - \frac{1}{2}\right\}$$
$$= \frac{1}{3}\left\{\frac{10}{2} - \frac{1}{2}\right\} = \frac{1}{3}\left\{\frac{9}{2}\right\} = \frac{3}{2}$$

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62) $h(x) = -|x|$

a) $[-1, 0]$

$$av(h) = \frac{1}{(0)-(-1)} \int_{-1}^0 -|x|\,dx = \frac{1}{(0)-(-1)} \int_{-1}^0 -(-x)\,dx = \frac{1}{0+1} \int_{-1}^0 x\,dx = \frac{1}{1}\left[\frac{x^2}{2} + C\right]_{-1}^0$$
$$= \frac{1}{1}\left\{\left[\frac{(0)^2}{2} + C\right] - \left[\frac{(-1)^2}{2} + C\right]\right\} = \frac{1}{1}\left\{[0] - \left[\frac{1}{2}\right]\right\} = \frac{1}{1}\left\{\frac{-1}{2}\right\} = \frac{-1}{2}$$

b) $[0, 1]$

$$av(h) = \frac{1}{(1)-(0)} \int_0^1 -|x|\,dx = \frac{1}{(1)-(0)} \int_0^1 -(x)\,dx = \frac{1}{1-0} \int_0^1 -x\,dx = \frac{1}{1}\left[\frac{-x^2}{2} + C\right]_0^1$$
$$= \frac{1}{1}\left\{\left[\frac{-(1)^2}{2} + C\right] - \left[\frac{-(0)^2}{2} + C\right]\right\} = \frac{1}{1}\left\{\left[\frac{-1}{2}\right] - [0]\right\} = \frac{1}{1}\left\{\frac{-1}{2}\right\} = \frac{-1}{2}$$

c) $[-1, 1]$ using results of part a and b

$$av(h) = \frac{1}{(1)-(-1)} \int_{-1}^1 -|x|\,dx = \frac{1}{(1)-(-1)}\left(\int_{-1}^0 -|x|\,dx + \int_0^1 -|x|\,dx\right) = \frac{1}{2}\left(\left\{\frac{-1}{2}\right\} + \left\{\frac{-1}{2}\right\}\right) = \frac{1}{2}(-1) = \frac{-1}{2}$$

64) $\int_0^2 (2x+1)\,dx \Rightarrow f(x)=2x+1 \qquad [0,2]$

$\Delta x = \frac{(2)-(0)}{n} = \frac{2}{n} \qquad x_k = a + k\Delta x = (0) + k\left(\frac{2}{n}\right) = \frac{2k}{n}$

$f(x_k) = 2\left(\frac{2k}{n}\right) + 1 = \frac{4}{n}k + 1$

$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{4}{n}k+1\right)\left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{8}{n^2}k + \frac{2}{n}\right) = \frac{8}{n^2}\sum_{k=1}^{n} k + \frac{2}{n}\sum_{k=1}^{n} 1$

$= \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n}(n) = \frac{8}{2}\left(\frac{n(n+1)}{n^2}\right) + 2 = 4\left(\frac{n^2+n}{n^2}\right) + 2$

$= 4\left(\frac{n^2}{n^2} + \frac{n}{n^2}\right) + 2 = 4\left(1 + \frac{1}{n}\right) + 2 = 4 + \frac{4}{n} + 2 = 6 + \frac{4}{n}$

$\int_0^2 (2x+1)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{x\to\infty}\sum_{k=1}^{n}\left(\frac{4}{n}k+1\right)\left(\frac{2}{n}\right)$

$= \lim_{n\to\infty}\left(6 + \frac{4}{n}\right) = 6 + 0 = 6$

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66) $\int_{-1}^{0} (x - x^2)\,dx \Rightarrow f(x) = x - x^2 \qquad [-1,0]$

$\Delta x = \frac{(0)-(-1)}{n} = \frac{1}{n} \qquad x_k = a + k\Delta x = (-1) + k\left(\frac{1}{n}\right) = -1 + \frac{1}{n}k = \frac{1}{n}k - 1$

$f(x_k) = \left(\frac{1}{n}k - 1\right) - \left(\frac{1}{n}k - 1\right)^2 = \left(\frac{1}{n}k - 1\right) - \left(\frac{1}{n^2}k^2 - \frac{2}{n}k + 1\right) = \frac{-1}{n^2}k^2 + \frac{3}{n}k - 2$

$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{-1}{n^2}k^2 + \frac{3}{n}k - 2\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(\frac{-1}{n^3}k^2 + \frac{3}{n^2}k - \frac{2}{n}\right)$

$= \frac{-1}{n^3}\sum_{k=1}^{n} k^2 + \frac{3}{n^2}\sum_{k=1}^{n} k - \frac{2}{n}\sum_{k=1}^{n} 1 = \frac{-1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{2}{n}(n)$

$= \frac{-1}{6}\left(\frac{n(2n^2+3n+1)}{n^3}\right) + \frac{3}{2}\left(\frac{n^2+n}{n^2}\right) - 2 = \frac{-1}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right) + \frac{3}{2}\left(\frac{n^2+n}{n^2}\right) - 2$

$= \frac{-1}{6}\left(\frac{2n^3}{n^3} + \frac{3n^2}{n^3} + \frac{n}{n^3}\right) + \frac{3}{2}\left(\frac{n^2}{n^2} + \frac{n}{n^2}\right) - 2 = \frac{-1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + \frac{3}{2}\left(1 + \frac{1}{n}\right) - 2$

66) continued

$$\int_{-1}^{0}(x-x^2)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} f(x_k)\Delta x = \lim_{n\to\infty}\left(\frac{-1}{n^2}k^2+\frac{3}{n}k-2\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-1}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+\frac{3}{2}\left(1+\frac{1}{n}\right)-2\right)$$

$$= \frac{-1}{6}(2+0+0)+\frac{3}{2}(1+0)-2 = \frac{-2}{6}+\frac{3}{2}-2 = \frac{-2}{6}+\frac{9}{6}-\frac{12}{6} = \frac{-5}{6}$$

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68) $\int_{-1}^{1} x^3\,dx \Rightarrow f(x)=x^3 \quad [-1,1]$

$$\Delta x = \frac{(1)-(-1)}{n} = \frac{2}{n} \qquad x_k = a+k\Delta x = (-1)+k\left(\frac{2}{n}\right) = \frac{2}{n}k-1$$

$$f(x_k) = \left(\frac{2}{n}k-1\right)^3 = \left(\frac{2}{n}k-1\right)\left(\frac{4}{n^2}k^2-\frac{4}{n}k+1\right)$$

$$= \frac{8}{n^3}k^3-\frac{8}{n^2}k^2+\frac{2}{n}k-\frac{4}{n^2}k^2+\frac{4}{n}k-1 = \frac{8}{n^3}k^3-\frac{12}{n^2}k^2+\frac{6}{n}k-1$$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{8}{n^3}k^3-\frac{12}{n^2}k^2+\frac{6}{n}k-1\right)\left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{16}{n^4}k^3-\frac{24}{n^3}k^2+\frac{12}{n^2}k-\frac{2}{n}\right)$$

$$= \frac{16}{n^4}\sum_{k=1}^{n}k^3-\frac{24}{n^3}\sum_{k=1}^{n}k^2+\frac{12}{n^2}\sum_{k=1}^{n}k-\frac{2}{n}\sum_{k=1}^{n}1$$

$$= \frac{16}{n^4}\left(\left(\frac{n(n+1)}{2}\right)^2\right)-\frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{12}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{2}{n}(n)$$

$$= \frac{16}{n^4}\left(\left(\frac{n^2+n}{2}\right)^2\right)-\frac{24}{n^3}\left(\frac{n(2n^2+3n+1)}{6}\right)+\frac{12}{n^2}\left(\frac{n^2+n}{2}\right)-2$$

$$= \frac{16}{n^4}\left(\frac{n^4+2n^3+n^2}{4}\right)-\frac{24}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)+\frac{12}{n^2}\left(\frac{n^2+n}{2}\right)-2$$

$$= \frac{16}{4}\left(\frac{n^4+2n^3+n^2}{n^4}\right)-\frac{24}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right)+\frac{12}{2}\left(\frac{n^2+n}{n^2}\right)-2$$

$$= 4\left(\frac{n^4}{n^4}+\frac{2n^3}{n^4}+\frac{n^2}{n^4}\right)-4\left(\frac{2n^3}{n^3}+\frac{3n^2}{n^3}+\frac{n}{n^3}\right)+6\left(\frac{n^2}{n^2}+\frac{n}{n^2}\right)-2$$

$$= 4\left(1+\frac{2}{n}+\frac{1}{n^2}\right)-4\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+6\left(1+\frac{1}{n}\right)-2$$

68) continued

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{8}{n^3}k^3 - \frac{12}{n^2}k^2 + \frac{6}{n}k - 1\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\left(4\left(1+\frac{2}{n}+\frac{1}{n^2}\right) - 4\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 6\left(1+\frac{1}{n}\right) - 2\right)$$

$$= 4(1+0+0) - 4(2+0+0) + 6(1+0) - 2 = 4 - 8 + 6 - 2 = 0$$

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70) $\int_0^1 (3x - x^3)\,dx \Rightarrow f(x) = 3x - x^3 \quad [0,1]$

$$\Delta x = \frac{(1)-(0)}{n} = \frac{1}{n} \qquad x_k = a + k\Delta x = (0) + k\left(\frac{1}{n}\right) = \frac{1}{n}k$$

$$f(x_k) = 3\left(\frac{1}{n}k\right) - \left(\frac{1}{n}k\right)^3 = \frac{3}{n}k - \frac{1}{n^3}k^3$$

$$R_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{3}{n}k - \frac{1}{n^3}k^3\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(\frac{3}{n^2}k - \frac{1}{n^4}k^3\right)$$

$$= \frac{3}{n^2}\sum_{k=1}^{n} k - \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^4}\left(\left(\frac{n(n+1)}{2}\right)^2\right)$$

$$= \frac{3}{n^2}\left(\frac{n^2+n}{2}\right) - \frac{1}{n^4}\left(\frac{n^2(n^2+2n+1)}{4}\right) = \frac{3}{2}\left(\frac{n^2+n}{n^2}\right) - \frac{1}{4}\left(\frac{n^4+2n^3+n^2}{n^4}\right)$$

$$= \frac{3}{2}\left(\frac{n^2}{n^2} + \frac{n}{n^2}\right) - \frac{1}{4}\left(\frac{n^4}{n^4} + \frac{2n^3}{n^4} + \frac{n^2}{n^4}\right) = \frac{3}{2}\left(1+\frac{1}{n}\right) - \frac{1}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$\int_0^1 (3x - x^3)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3}{n}k - \frac{1}{n^3}k^3\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{2}\left(1+\frac{1}{n}\right) - \frac{1}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\right)$$

$$= \frac{3}{2}(1+0) - \frac{1}{4}(1+0+0) = \frac{3}{2} - \frac{1}{4} = \frac{6}{4} - \frac{1}{4} = \frac{5}{4}$$

80) $\sec x \geq 1 + \frac{x^2}{2}$ on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\Downarrow$$

$\sec x - \left(1 + \frac{x^2}{2}\right) \geq 0$ on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\Downarrow$$

$\int_0^1 \left\{\sec x - \left(1 + \frac{x^2}{2}\right)\right\} dx \geq 0$ since $[0,1]$ is contained in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\Downarrow$$

$\int_0^1 \sec x \, dx - \int_0^1 \left(1 + \frac{x^2}{2}\right) dx \geq 0$

$$\Downarrow$$

$\int_0^1 \sec x \, dx \geq \int_0^1 \left(1 + \frac{x^2}{2}\right) dx$

$$\Downarrow$$

$\int_0^1 \sec x \, dx \geq \frac{7}{6}$

$$\boxed{\begin{aligned} &\int_0^1 \left(1 + \frac{x^2}{2}\right) dx \\ &= \left[x + \frac{x^3}{6} + C\right]_0^1 \\ &= \left[(1) + \frac{(1)^3}{6} + C\right] - \left[(0) + \frac{(0)^3}{6} + C\right] \\ &= \left[1 + \frac{1}{6}\right] - [0] \\ &= \frac{6}{6} + \frac{1}{6} = \frac{7}{6} \end{aligned}}$$

So a lower bound is $\frac{7}{6}$.