Solving Applied Optimization Problems

1) Read the problem. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?

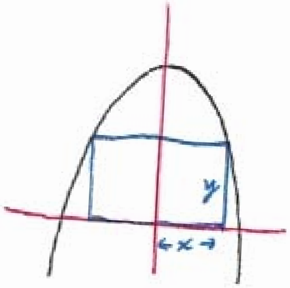
2) Draw a picture. Label any part that may be important to the problem.

3) Introduce variables. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

4) Write an equation for the unknown quantity. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. If you decide to set up two equations, then one should equal to a constant value, called constraint equation, and the other equal to another unknown, called objective equation. This may require considerable manipulation.

5) Test the critical points and endpoints in the domain of the unknown. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

4)



$y = 12 - x^2$

$A = (2x)y = (2x)(12 - x^2)$

$A = 24x - 2x^3 \qquad 0 \le x \le \sqrt{12} = 2\sqrt{3}$

$\frac{dA}{dx} = 24[1] - 2[3x^2] = 24 - 6x^2$

$\frac{d^2A}{dx^2} = -6[2x] = -12x$

critical points

$0 = \frac{dA}{dx} = 24 - 6x^2$

$0 = 24 - 6x^2$

$0 = 6(4 - x^2)$

$0 = 6(2+x)(2-x)$

$2+x=0$	$2-x=0$
$x=-2$	$x=2$
discard	

at $x=2$: $\left.\frac{d^2A}{dx^2}\right|_{x=2} = -12(2) < 0$ C.D. Local Max

$A|_{x=2} = (2(2))(12-(2)^2) = (4)(8) = 32$ units^2

endpoints:

at $x=0$: $A|_{x=0} = (2(0))(12-(0)^2) = 0$

at $x=\sqrt{12}$: $A|_{x=\sqrt{12}} = (2(\sqrt{12}))(12-(\sqrt{12})^2) = 0$

We have maximum area of $A = 32$ units^2 when $x = 2$ units and the corresponding $y = 12-(2)^2 = 8$ units.

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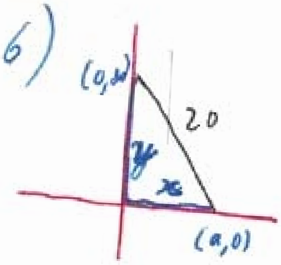
6)



$x^2 + y^2 = (20)^2$

$y^2 = (20)^2 - x^2$

$y = ^{+}_{-}\sqrt{(20)^2 - x^2}$ discard negative $\Rightarrow y = ^{+}\sqrt{(20)^2 - x^2}$

$A = \frac{1}{2}xy = \frac{1}{2}x\left(\sqrt{(20)^2 - x^2}\right)$

$0 \le x \le 20$

$$\frac{dA}{dx} = \frac{1}{2}\left\{(x)\left(\frac{1}{2}((20)^2-x^2)^{-\frac{1}{2}}(-2x)\right) + \left(((20)^2-x^2)^{\frac{1}{2}}\right)[1]\right\} = \frac{1}{2}\left\{\frac{-x^2}{\sqrt{(20)^2-x^2}} + \sqrt{(20)^2-x^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{-x^2}{\sqrt{(20)^2-x^2}} + \frac{\sqrt{(20)^2-x^2}}{1}\left(\frac{\sqrt{(20)^2-x^2}}{\sqrt{(20)^2-x^2}}\right)\right\} = \frac{1}{2}\left\{\frac{-x^2 + ((20)^2 - x^2)}{\sqrt{(20)^2-x^2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{(20)^2 - 2x^2}{\sqrt{(20)^2-x^2}}\right\} = \frac{1}{2}\left\{\frac{400 - 2x^2}{\sqrt{(20)^2-x^2}}\right\} = \frac{1}{2}\left\{\frac{2(200 - x^2)}{\sqrt{(20)^2-x^2}}\right\}$$

$$= \frac{200 - x^2}{\sqrt{(20)^2-x^2}} = \frac{200 - x^2}{((20)^2-x^2)^{\frac{1}{2}}}$$

6) continued

$$\frac{d^2A}{dx^2} = \frac{\left(\left((20)^2-x^2\right)^{\frac{1}{2}}\right)\left[-2x\right] - \left(200-x^2\right)\left[\frac{1}{2}\left((20)^2-x^2\right)^{-\frac{1}{2}}(-2x)\right]}{\left(\left((20)^2-x^2\right)^{\frac{1}{2}}\right)^2}$$

$$= \frac{-2x\sqrt{(20)^2-x^2} + \frac{(200-x^2)x}{\sqrt{(20)^2-x^2}}}{\left(\sqrt{(20)^2-x^2}\right)^2} = \frac{\left(\frac{-2x\sqrt{(20)^2-x^2}}{1}\right)\left(\frac{\sqrt{(20)^2-x^2}}{\sqrt{(20)^2-x^2}}\right) + \frac{(200-x^2)x}{\sqrt{(20)^2-x^2}}}{\left(\sqrt{(20)^2-x^2}\right)^2}$$

$$= \frac{-2x\left((20)^2-x^2\right) + (200-x^2)x}{\left(\sqrt{(20)^2-x^2}\right)^3} = \frac{-2x(400-x^2) + (200-x^2)x}{\left(\sqrt{(20)^2-x^2}\right)^3}$$

$$= \frac{-800x + 2x^3 + 200x - x^3}{\left(\sqrt{(20)^2-x^2}\right)^3} = \frac{x^3 - 600x}{\left(\sqrt{(20)^2-x^2}\right)^3} = \frac{x(x^2-600)}{\left(\sqrt{(20)^2-x^2}\right)^3}$$

critical point

$$0 = \frac{dA}{dx} = \frac{200-x^2}{\sqrt{(20)^2-x^2}}$$

$$0 = 200 - x^2$$

$$0 = (10\sqrt{2} + x)(10\sqrt{2} - x)$$

$10\sqrt{2} + x = 0$	$10\sqrt{2} - x = 0$
$x = -10\sqrt{2}$	$x = 10\sqrt{2}$
discard	

at $x = 10\sqrt{2}$: $\left.\frac{d^2A}{dx^2}\right|_{x=10\sqrt{2}} = \frac{(10\sqrt{2})\left((10\sqrt{2})^2-600\right)}{\sqrt{(20)^2-(10\sqrt{2})^2}}$

$$= \frac{(10\sqrt{2})(200-600)}{\sqrt{400-200}} < 0 \quad \text{C.D.}$$

Local Max

$$A\big|_{x=10\sqrt{2}} = \frac{1}{2}(10\sqrt{2})\left(\sqrt{(20)^2-(10\sqrt{2})^2}\right)$$

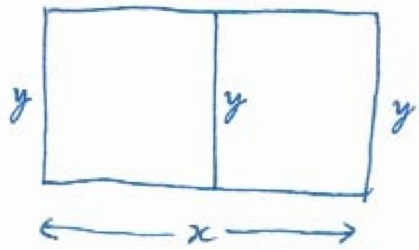
$$= \frac{1}{2}(10\sqrt{2})\left(\sqrt{400-200}\right)$$

$$= \frac{1}{2}(10\sqrt{2})\left(\sqrt{200}\right) = \frac{1}{2}(10\sqrt{2})(10\sqrt{2})$$

$$= \frac{1}{2}(200) = 100 \text{ units}^2$$

The Maximum area of 100 units^2 occurs when $x = 10\sqrt{2}$ units and $y = \sqrt{(20)^2-(10\sqrt{2})^2} = 10\sqrt{2}$ units. This means $x = y$ which implies that $a = b$.

8)



$$A = 216 = xy \Rightarrow y = \frac{216}{x}$$

$$P = x + x + y + y + y = 2x + 3y = 2x + 3\left(\frac{216}{x}\right)$$

$$P = 2x + 3(216)x^{-1}$$

$$\frac{dP}{dx} = 2[1] + 3(216)[-1x^{-2}] = 2 - 3(216)x^{-2} = 2 - \frac{3(216)}{x^2}$$

$$\frac{d^2P}{dx^2} = -3(216)[-2x^{-3}] = \frac{6(216)}{x^3}$$

critical point

$$0 = \frac{dP}{dx} = 2 - \frac{3(216)}{x^2}$$

$$0 = 2 - \frac{3(216)}{x^2}$$

$$0 = 2\left(1 - \frac{3(108)}{x^2}\right)$$

$$0 = 1 - \frac{3(108)}{x^2}$$

$$\frac{3(108)}{x^2} = 1$$

$$3(108) = x^2$$

$$x = \pm\sqrt{3(108)}$$

$$x = \pm\sqrt{3(3(36))}$$

$$x = \pm 3(6)$$

$x = -18$ discard | $x = 18$

at $x = 18$, $\left.\frac{d^2P}{dx^2}\right|_{x=18} = \frac{6(216)}{(18)^3} > 0$ C.U.

local min

$$P\Big|_{x=18} = 2(18) + 3\left(\frac{216}{(18)}\right)$$

$$= 2(18) + \frac{216}{6}$$

$$= 36 + 36$$

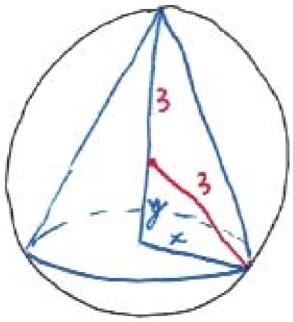
$$= 72 \text{ m}$$

$$y = \frac{216}{(18)} = 12 \text{ m}$$

The dimension of outer rectangle is $x = 18$ m by $y = 12$ m.

The smallest amount of fencing needed is $P = 72$ m.

12)



$V = \frac{1}{3}\pi r^2 h \qquad r = x = \sqrt{(3)^2 - y^2} \qquad h = 3+y$

$V = \frac{1}{3}\pi\left(\sqrt{(3)^2-y^2}\right)^2(3+y) = \frac{1}{3}\pi(9-y^2)(3+y)$

$V = \frac{\pi}{3}(27 + 9y - 3y^2 - y^3)$

$$\frac{dV}{dy} = \frac{\pi}{3}\left([0] + 9[1] - 3[2y] - [3y^2]\right) = \frac{\pi}{3}(9 - 6y - 3y^2)$$

$$= \frac{\pi}{3}\left(3(3-2y-y^2)\right) = \pi(3-2y-y^2)$$

$$\frac{d^2V}{dy^2} = \pi\left([0] - 2[1] - [2y]\right) = \pi(-2-2y)$$

critical point

$0 = \frac{dV}{dy} = \pi(3-2y-y^2)$

$0 = \pi(3-2y-y^2)$

$0 = 3-2y-y^2$

$y^2+2y-3=0$

$(y+3)(y-1)=0$

$y+3=0$	$y-1=0$
$y=-3$	$y=1$
discard	

at $y=1$: $\left.\frac{d^2V}{dy^2}\right|_{y=1} = \pi(-2-2(1)) < 0$ C.D.

Local Max

$\left.V\right|_{y=1} = \frac{\pi}{3}\left(27+9(1)-3(1)^2-(1)^3\right)$

$= \frac{\pi}{3}(27+9-3-1) = \frac{\pi}{3}(27+5)$

$= \frac{32\pi}{3}$ units^3

The largest volume of the right circular cone is $V = \frac{32\pi}{3}$ units^3 where $h = 3+(1) = 4$ units and $r = x = \sqrt{(3)^2-(1)^2} = \sqrt{9-1} = \sqrt{8}$ units.

14)

$$V = 1000\,cm^3 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

$$S = \underbrace{(\pi r^2)}_{\text{bottom part}} + \underbrace{((2\pi r)\,h)}_{\text{side wall}} = \pi r^2 + 2\pi r h = \pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= \pi r^2 + \frac{2000}{r} = \pi r^2 + 2000 r^{-1}$$

$$\frac{dS}{dr} = \pi[2r] + 2000[-1r^{-2}] = 2\pi r - 2000r^{-2} = 2\pi r - \frac{2000}{r^2}$$

$$= \frac{2\pi r}{1}\left(\frac{r^2}{r^2}\right) - \frac{2000}{r^2} = \frac{2\pi r^3 - 2000}{r^2}$$

$$\frac{d^2S}{dr^2} = -2000[-2r^{-3}] = \frac{4000}{r^3}$$

critical point

$$0 = \frac{dS}{dr} = \frac{2\pi r^3 - 2000}{r^2}$$

$$0 = \frac{2\pi r^3 - 2000}{r^2}$$

$$0 = 2\pi r^3 - 2000$$

$$0 = 2(\pi r^3 - 1000)$$

$$0 = 2\left((\sqrt[3]{\pi}\, r)^3 - (10)^3\right)$$

$$0 = 2(\sqrt[3]{\pi}\, r - 10)\underbrace{\left((\sqrt[3]{\pi}\, r)^2 + (\sqrt[3]{\pi}\, r)(10) + (10)^2\right)}_{\text{no real \# solution}}$$

$$\sqrt[3]{\pi}\, r - 10 = 0$$

$$r = \frac{10}{\sqrt[3]{\pi}}\ cm$$

at $r = \frac{10}{\sqrt[3]{\pi}}$: $\left.\frac{d^2S}{dr^2}\right|_{r=\frac{10}{\sqrt[3]{\pi}}} = \frac{4000}{\left(\frac{10}{\sqrt[3]{\pi}}\right)^3} > 0$ C.U.

local min

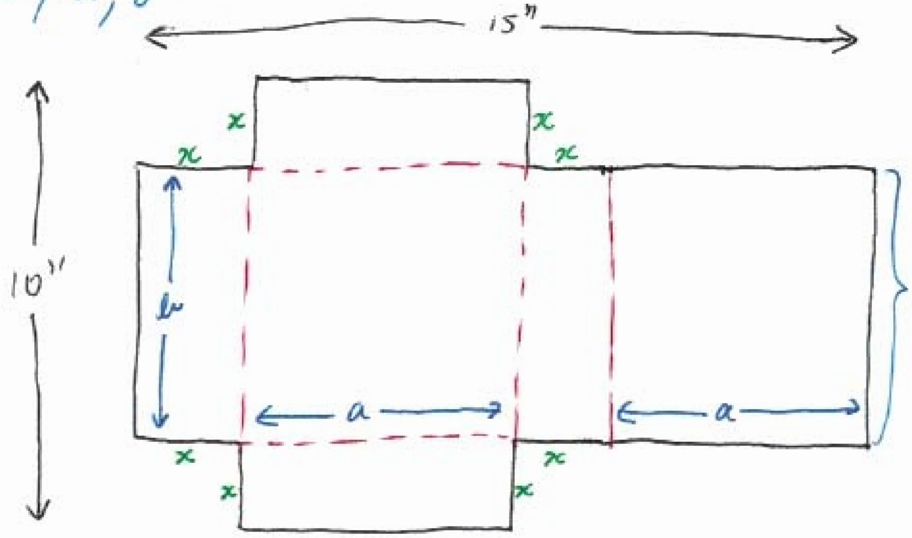
$$\boxed{h = \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{\pi}}\right)^2} = \frac{1000}{(\sqrt[3]{\pi})^3\left(\frac{100}{(\sqrt[3]{\pi})^2}\right)} = \frac{10}{\sqrt[3]{\pi}}\ cm}$$

$$\left.S\right|_{r=\frac{10}{\sqrt[3]{\pi}}} = \pi\left(\frac{10}{\sqrt[3]{\pi}}\right)^2 + \frac{2000}{\left(\frac{10}{\sqrt[3]{\pi}}\right)}$$

$$= 100\sqrt[3]{\pi} + 200\sqrt[3]{\pi} = 300\sqrt[3]{\pi}\ cm^2$$

The lightest can will have $r = \frac{10}{\sqrt[3]{\pi}}$ cm and $h = \frac{10}{\sqrt[3]{\pi}}$ cm using $S = 300\sqrt[3]{\pi}\ cm^2$ of materials to make this can.

16) a, d



$2a = 15 - 2x \rightarrow 2x < 15$

$a = \dfrac{15-2x}{2} \qquad x < 7.5$

$10 - 2x \rightarrow 2x < 10 \Rightarrow \underline{\underline{x < 5}}$

$V = (a)(b)(x)$

$V = \left(\dfrac{15-2x}{2}\right)(10-2x)(x)$

$$V(x) = V = (x)(15-2x)(5-x) = x(75 - 25x + 2x^2)$$

$$= 2x^3 - 25x^2 + 75x$$

also $\underline{\underline{x > 0}}$ so $0 < x < 5$

$$\frac{dV}{dx} = 2[3x^2] - 25[2x] + 75[1] = 6x^2 - 50x + 75$$

$$\frac{d^2V}{dx^2} = 6[2x] - 50[1] + [0] = 12x - 50$$

critical point

$$0 = \frac{dV}{dx} = 6x^2 - 50x + 75$$

$$0 = 6x^2 - 50x + 75$$

$$x = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)}$$

$$x = \frac{50 \pm \sqrt{4(25)\{(5)^2 - (6)(3)\}}}{12}$$

$$x = \frac{50 \pm 2(5)\sqrt{25-18}}{12}$$

$$x = \frac{2(25 \pm 5\sqrt{7})}{12} = \frac{25 \pm 5\sqrt{7}}{6}$$

$x = \dfrac{25 - 5\sqrt{7}}{6} \quad \Big| \quad x = \dfrac{25 + 5\sqrt{7}}{6} > 5$ discard

$$\left.\frac{d^2V}{dx^2}\right|_{x = \frac{25-5\sqrt{7}}{6}} = 12\left(\frac{25-5\sqrt{7}}{6}\right) - 50 = 2(25 - 5\sqrt{7}) - 50 < 0$$

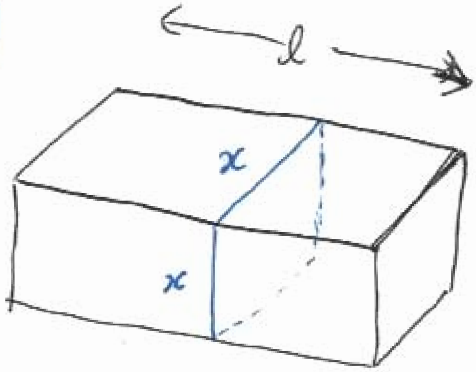
C.D. Local Max.

Maximum volume occurs when

$x = \dfrac{25 - 5\sqrt{7}}{6}$ in.

$$V = 2\left(\frac{25-5\sqrt{7}}{6}\right)^3 - 25\left(\frac{25-5\sqrt{7}}{6}\right)^2 + 75\left(\frac{25-5\sqrt{7}}{6}\right) \text{ in}^3$$

20-a)



length of Girth = $4x$

$108\text{ in} = 4x + l \Rightarrow l = 108 - 4x$

$x > 0$, $\quad 4x < 108$, $\quad x < 27$

$0 < x < 27$

$V = (x)(x)(l) = x^2 l = x^2(108-4x) = 108x^2 - 4x^3 = 4(27x^2 - x^3)$

$\frac{dV}{dx} = 4(27[2x] - [3x^2]) = 4(27(2)x - 3x^2) = 4(3)(9(2)x - x^2)$

$= 12(18x - x^2)$

$\frac{d^2V}{dx^2} = 12(18[1] - [2x]) = 12(18 - 2x) = 12(2(9-x)) = 24(9-x)$

critical points

$0 = \frac{dV}{dx} = 12(18x - x^2)$

$0 = 12(18x - x^2)$

$0 = 12x(18 - x)$

$12x = 0$	$18 - x = 0$
$x = 0$	$x = 18$
discard	

at $x = 18$: $\left.\frac{d^2V}{dx^2}\right|_{x=18} = 24(9-(18)) < 0$ C.D.

Local Max

$V|_{x=18} = 4(27(18)^2 - (18)^3) = 4(18)^2(27-18)$

$= 4(9)(18)^2 = 4(9)(324) = 4(2916)$

$= 11664\text{ in}^3$

$l = 108 - 4(18) = 108 - 72 = 36\text{ in}$

The dimension for maximum volume is $x = 18$ in by $x = 18$ in by $l = 36$ in.

36) $f(x) = x^3 + ax^2 + bx$

$$\frac{df}{dx} = [3x^2] + a[2x] + b[1] = 3x^2 + 2ax + b$$

$$\frac{d^2f}{dx^2} = 3[2x] + 2a[1] + [0] = 6x + 2a$$

a) local min at $x = -1$, Local Max at $x = 3$

at $x = -1$: $0 = \left.\frac{df}{dx}\right|_{x=-1} = 3(-1)^2 + 2a(-1) + b = 3 - 2a + b \Rightarrow 0 = 3 - 2a + b$

at $x = 3$: $0 = \left.\frac{df}{dx}\right|_{x=3} = 3(3)^2 + 2a(3) + b = 27 + 6a + b \Rightarrow 0 = 27 + 6a + b$

$0 = 3 - 2a + b \Rightarrow (2a - 3) = b$

$0 = 27 + 6a + (2a - 3)$ $\qquad -24 = 8a \qquad b = 2(-3) - 3 = -6 - 3 = -9$

$0 = 24 + 8a \qquad \Longrightarrow \qquad -3 = a \qquad \underline{\underline{a = -3,\ b = -9}}$

b) local min at $x = 4$, IP at $x = 1$

at $x = 4$: $0 = \left.\frac{df}{dx}\right|_{x=4} = 3(4)^2 + 2a(4) + b = 48 + 8a + b \Rightarrow 0 = 48 + 8a + b$

at $x = 1$: $0 = \left.\frac{d^2f}{dx^2}\right|_{x=1} = 6(1) + 2a = 6 + 2a \Rightarrow 0 = 6 + 2a$

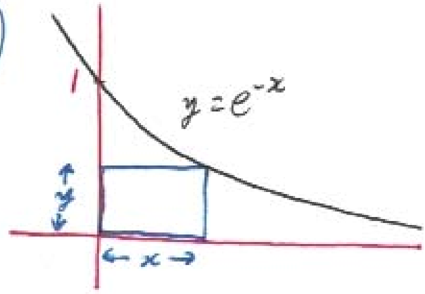
$0 = 6 + 2a \qquad 0 = 48 + 8(-3) + b$

$-6 = 2a \qquad 0 = 48 - 24 + b$

$-3 = a \qquad 0 = 24 + b \qquad \underline{\underline{a = -3,\ b = -24}}$

$-24 = b$

38)



$$A=(x)(y)=x(e^{-x})=xe^{-x}=\frac{x}{e^x}$$

$$0<x<\infty$$

$$\frac{dA}{dx}=(x)[e^{-x}(-1)]+(e^{-x})[1]=-xe^{-x}+e^{-x}=e^{-x}(1-x)$$
$$=\frac{1-x}{e^x}=(1-x)e^{-x}$$

$$\frac{d^2A}{dx^2}=(1-x)[e^{-x}(-1)]+(e^{-x})[-1]=e^{-x}\{(1-x)[-1]+(1)[-1]\}$$
$$=e^{-x}\{-1+x-1\}=e^{-x}\{x-2\}=\frac{x-2}{e^x}$$

critical point

$$0=\frac{dA}{dx}=\frac{1-x}{e^x}$$

$$0=\frac{1-x}{e^x}$$

$$0=1-x$$

$$x=1$$

at $x=1$: $\frac{d^2A}{dx^2}=\frac{(1)-2}{e^{(1)}}<0$ C.D.

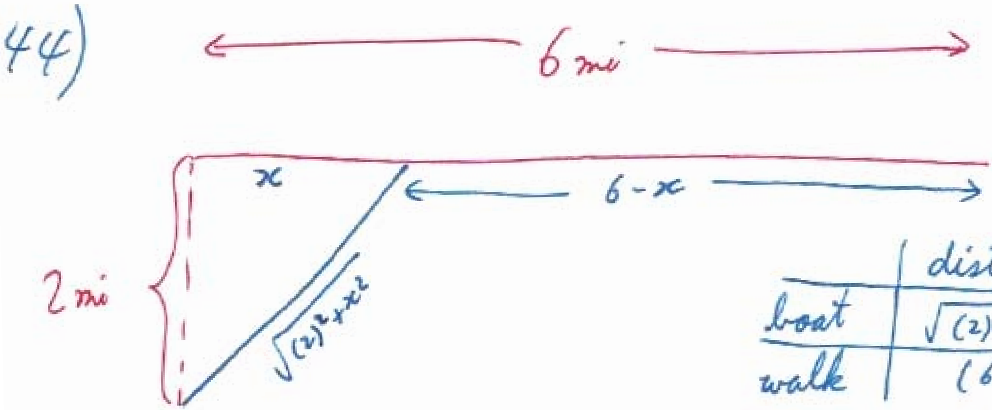
Local Max

$$A\Big|_{x=1}=\frac{(1)}{e^{(1)}}=\frac{1}{e}\text{ unit}^2$$

$y=e^{-x}=\frac{1}{e^x}$ at $x=1$: $y=\frac{1}{e^{(1)}}=\frac{1}{e}$

The maximum area of the rectangle is obtained when $x=1$ and $y=e^{-(1)}=\frac{1}{e}$ with the area $A=\frac{1}{e}$ units^2.

44)



let x be the landing point

	distance	speed	time
boat	$\sqrt{(2)^2+x^2}$	2 mph	$\frac{\sqrt{(2)^2+x^2}}{2}$
walk	$(6-x)$	5 mph	$\frac{6-x}{5}$

$$t(x) = \frac{\sqrt{(2)^2+x^2}}{2} + \frac{6-x}{5} = \frac{1}{2}\left(4+x^2\right)^{\frac{1}{2}} + \frac{6}{5} - \frac{1}{5}x \qquad 0 \le x \le 6$$

$$\frac{dt}{dx} = \frac{1}{2}\left[\frac{1}{2}\left(4+x^2\right)^{-\frac{1}{2}}(2x)\right] + [0] - \frac{1}{5}[1] = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$$

$$= \frac{1}{2}\left(\frac{x}{\sqrt{4+x^2}}\right) - \frac{1}{5} = \frac{1}{2}\left(\frac{x}{(4+x^2)^{\frac{1}{2}}}\right) - \frac{1}{5}$$

$$\frac{d^2t}{dx^2} = \frac{1}{2}\left(\frac{((4+x^2)^{\frac{1}{2}})[1] - (x)\left[\frac{1}{2}(4+x^2)^{-\frac{1}{2}}(2x)\right]}{((4+x^2)^{\frac{1}{2}})^2}\right) = \frac{1}{2}\left(\frac{\sqrt{4+x^2} - \frac{x^2}{\sqrt{4+x^2}}}{(\sqrt{4+x^2})^2}\right)$$

$$= \frac{1}{2}\left(\frac{\frac{\sqrt{4+x^2}}{1}\left(\frac{\sqrt{4+x^2}}{\sqrt{4+x^2}}\right) - \frac{x^2}{\sqrt{4+x^2}}}{(\sqrt{4+x^2})^2}\right) = \frac{1}{2}\left(\frac{(4+x^2) - x^2}{(\sqrt{4+x^2})^3}\right) = \frac{2}{(\sqrt{4+x^2})^3}$$

critical point

$$0 = \frac{dt}{dx} = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$$

$$0 = \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5}$$

$$\frac{1}{5} = \frac{x}{2\sqrt{4+x^2}}$$

$$2\sqrt{4+x^2} = 5x$$

$$4(4+x^2) = 25x^2$$

$$16 + 4x^2 = 25x^2$$

$$0 = 21x^2 - 16$$

$$0 = (\sqrt{21}x+4)(\sqrt{21}x-4)$$

$\sqrt{21}x+4=0$	$\sqrt{21}x-4=0$
$x = \frac{-4}{\sqrt{21}}$ discard	$x = \frac{4}{\sqrt{21}}$

at $x = \frac{4}{\sqrt{21}}$: $\left.\frac{d^2t}{dx^2}\right|_{x=\frac{4}{\sqrt{21}}} = \frac{2}{\left(\sqrt{4+\left(\frac{4}{\sqrt{21}}\right)^2}\right)^3} > 0$

C.V. local min

Jane should land on $x = \frac{4}{\sqrt{21}}$ miles.

68-a) $\ell(x) = \dfrac{x}{\sqrt{a^2+x^2}} = \dfrac{x}{(a^2+x^2)^{\frac{1}{2}}}$

$$\frac{d\ell}{dx} = \frac{((a^2+x^2)^{\frac{1}{2}})[1] - (x)[\frac{1}{2}(a^2+x^2)^{\frac{-1}{2}}(2x)]}{((a^2+x^2)^{\frac{1}{2}})^2} = \frac{\sqrt{a^2+x^2} - \frac{x^2}{\sqrt{a^2+x^2}}}{(\sqrt{a^2+x^2})^2}$$

$$= \frac{\frac{\sqrt{a^2+x^2}}{1}\left(\frac{\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\right) - \frac{x^2}{\sqrt{a^2+x^2}}}{(\sqrt{a^2+x^2})^2} = \frac{(a^2+x^2)-x^2}{(\sqrt{a^2+x^2})^3} = \frac{a^2}{(\sqrt{a^2+x^2})^3} > 0 \quad \text{INC}$$

so $\ell(x)$ is Increasing function of x.

68-b) $g(x) = \dfrac{d-x}{\sqrt{b^2+(d-x)^2}} = \dfrac{d-x}{(b^2+(d-x)^2)^{\frac{1}{2}}}$

$$\frac{dg}{dx} = \frac{((b^2+(d-x)^2)^{\frac{1}{2}})[-1] - (d-x)\left[\frac{1}{2}(b^2+(d-x)^2)^{-\frac{1}{2}}(2(d-x)(-1))\right]}{((b^2+(d-x)^2)^{\frac{1}{2}})^2}$$

$$= \frac{-\sqrt{b^2+(d-x)^2} + \frac{(d-x)^2}{\sqrt{b^2+(d-x)^2}}}{(\sqrt{b^2+(d-x)^2})^2} = \frac{\frac{-\sqrt{b^2+(d-x)^2}}{1}\left(\frac{\sqrt{b^2+(d-x)^2}}{\sqrt{b^2+(d-x)^2}}\right) + \frac{(d-x)^2}{\sqrt{b^2+(d-x)^2}}}{(\sqrt{b^2+(d-x)^2})^2}$$

$$= \frac{-(b^2+(d-x)^2) + (d-x)^2}{(\sqrt{b^2+(d-x)^2})^3} = \frac{-b^2-(d-x)^2+(d-x)^2}{(\sqrt{b^2+(d-x)^2})^3} = \frac{-b^2}{(\sqrt{b^2+(d-x)^2})^3} < 0 \ \text{dec.}$$

so $g(x)$ is decreasing function of x.

68-c) $\dfrac{dt}{dx} = \dfrac{x}{c_1\sqrt{a^2+x^2}} - \dfrac{d-x}{c_2\sqrt{b^2+(d-x)^2}} = \dfrac{1}{c_1}\ell(x) - \dfrac{1}{c_2}g(x)$

$\dfrac{d^2t}{dx^2} = \dfrac{1}{c_1}\dfrac{d\ell}{dx} - \dfrac{1}{c_2}\dfrac{dg}{dx} > 0$ because $\dfrac{d\ell}{dx} > 0$ and $\dfrac{dg}{dx} < 0$ and since $c_1, c_2 > 0$

$\dfrac{dt}{dx}$ is an increasing function of x.