Theorem 6-L'Hospital Rule:

Suppose that $f(a) = g(a) = 0$, that *f* and *g* are differentiable on an open interval *I* containing *a*, and that $\frac{dg}{dx} \neq 0$ on *I* if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{df}{dx}}{\frac{dg}{dx}},$$

assuming that the limit on the right side of this equation exists.

Using L'Hospital Rule

To find $\lim_{x\to a} \frac{f(x)}{g(x)}$

by L'Hospital's Rule, We continue to differentiate *f* and *g*, so long as we still get the form $\frac{0}{0}$ $\left(\text{or } \frac{\pm\infty}{\pm\infty}\right)$ at $x = a$.

But as soon as one or the other of these derivatives is different from zero (or infinity) at $x = a$ we stop differentiating. L'Hospital Rule does not apply when either numerator or denominator has a finite nonzero limit.

Note: L'Hospital Rule only works when we are taking the limit on a single fraction expression.

Indeterminate forms of type $(0)(\pm\infty)$ $\infty - \infty$**:**

For these types we must rewrite our expression as a fraction that satisfies the method above in order use L'Hospital Rule.

Indeterminate Powers 0^0 ∞^0 1^∞

To evaluate these types, use the procedure below:

1) if $f(x)$ is a function where $\lim_{x\to a} f(x)$ is one of the types 0^0 ∞^0 1^∞, then let $y = f(x)$
2) apply natural log on both sides: $\ln y = \ln(f(x))$
3) using the laws of logarithm, change the right hand side into a single fraction to satisfy the method described above "Using L'Hospital Rule"
4) take the limit on this modified expression (this means that we modified the expression and evaluating the limit) and obtain the limit value (let's call is α).
5) the limit value we got in step 4 is not the final answer, it is actually $\ln y = \alpha$. So by inverse function property, we get our actual answer which is $y = e^\alpha$ and thus $\lim_{x\to a} f(x) = e^\alpha$

Theorem 7-Cauchy's Mean Value Theorem

Suppose functions *f* and *g* are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $\frac{dg}{dx} \neq 0$ throughout (a,b). Then there exists a number *c* in (a,b) at which

$$\frac{\left.\frac{df}{dx}\right|_{x=c}}{\left.\frac{dg}{dx}\right|_{x=c}} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

2) $$\lim_{x\to 0} \frac{\overset{0}{\sin 5x}}{\underset{0}{x}} \overset{L}{=} \lim_{x\to 0} \frac{5\cos(5x)}{1} = 5\cos(5(0)) = 5\cos(0) = 5(1) = 5$$

$$\lim_{x\to 0} \frac{\sin 5x}{x}\left(\frac{5}{5}\right) = \lim_{x\to 0} 5\left(\frac{\sin(5x)}{(5x)}\right) = 5\left(\lim_{x\to 0} \frac{\sin(5x)}{(5x)}\right) = 5(1) = 5$$

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4) $$\lim_{x\to 1} \frac{\overset{0}{x^3-1}}{\underset{0}{4x^3-x-3}} \overset{L}{=} \lim_{x\to 1} \frac{3x^2}{12x^2-1} = \frac{3(1)^2}{12(1)^2-1} = \frac{3}{12-1} = \frac{3}{11}$$

$$\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(4x^2+4x+3)} = \lim_{x\to 1} \frac{x^2+x+1}{4x^2+4x+3} = \frac{(1)^2+(1)+1}{4(1)^2+4(1)+3} = \frac{3}{11}$$

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6) $$\lim_{x\to\infty} \frac{\overset{+\infty}{2x^2+3x}}{\underset{+\infty}{x^3+x+1}} \overset{L}{=} \lim_{x\to\infty} \frac{\overset{+\infty}{4x+3}}{\underset{+\infty}{3x^2+1}} \overset{L}{=} \lim_{x\to\infty} \frac{4}{6x} = 0$$

$$\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^3}+\frac{3x}{x^3}}{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{3}{x^2}}{1+\frac{1}{x^2}+\frac{1}{x^3}} = \frac{0+0}{1+0+0} = 0$$

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8) $$\lim_{x\to -5} \frac{\overset{0}{x^2-25}}{\underset{0}{x+5}} \overset{L}{=} \lim_{x\to -5} \frac{2x}{1} = \lim_{x\to -5} 2x = 2(-5) = -10$$

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10) $$\lim_{t\to -1} \frac{\overset{0}{3t^3+3}}{\underset{0}{4t^3-t+3}} \overset{L}{=} \lim_{t\to -1} \frac{9t^2}{12t^2-1} = \frac{9(-1)^2}{12(-1)^2-1} = \frac{9}{12-1} = \frac{9}{11}$$

---

12) $$\lim_{x\to\infty} \frac{\overset{-\infty}{x-8x^2}}{\underset{+\infty}{12x^2+5x}} \overset{L}{=} \lim_{x\to\infty} \frac{\overset{-\infty}{1-16x}}{\underset{+\infty}{24x+5}} \overset{L}{=} \lim_{x\to\infty} \frac{-16}{24} = \frac{-16}{24} = \frac{-2}{3}$$

14) $$\lim_{t\to 0}\frac{\sin 5t}{2t}\overset{L}{=}\lim_{t\to 0}\frac{5\cos(5t)}{2}=\frac{5\cos(5(0))}{2}=\frac{5(1)}{2}=\frac{5}{2}$$

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16) $$\lim_{x\to 0}\frac{\sin x-x}{x^3}\overset{L}{=}\lim_{x\to 0}\frac{\cos x-1}{3x^2}\overset{L}{=}\lim_{x\to 0}\frac{-\sin x}{6x}\overset{L}{=}\lim_{x\to 0}\frac{-\cos x}{6}$$
$$=\frac{-\cos(0)}{6}=\frac{-1}{6}$$

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18) $$\lim_{\theta\to-\frac{\pi}{3}}\frac{3\theta+\pi}{\sin(\theta+\frac{\pi}{3})}\overset{L}{=}\lim_{\theta\to-\frac{\pi}{3}}\frac{3}{\cos(\theta+\frac{\pi}{3})}=\frac{3}{\cos((\frac{-\pi}{3})+\frac{\pi}{3})}=\frac{3}{\cos(0)}=\frac{3}{1}=3$$

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20) $$\lim_{x\to 1}\frac{x-1}{\ln x-\sin\pi x}\overset{L}{=}\lim_{x\to 1}\frac{1}{\frac{1}{x}-\pi\cos(\pi x)}=\frac{1}{\frac{1}{(1)}-\pi\cos(\pi(1))}=\frac{1}{1-\pi(-1)}=\frac{1}{1+\pi}$$

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22) $$\lim_{x\to\frac{\pi}{2}}\frac{\ln(\csc x)}{(x-\frac{\pi}{2})^2}\overset{L}{=}\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\csc x}(-\csc x\cot x)}{2(x-\frac{\pi}{2})(1)}=\lim_{x\to\frac{\pi}{2}}\frac{-\cot x}{2(x-\frac{\pi}{2})}$$
$$\overset{L}{=}\lim_{x\to\frac{\pi}{2}}\frac{-[-\csc^2 x]}{2[1]}=\lim_{x\to\frac{\pi}{2}}\frac{\csc^2 x}{2}=\frac{\csc^2(\frac{\pi}{2})}{2}=\frac{(1)^2}{2}=\frac{1}{2}$$

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24) $$\lim_{t\to 0}\frac{t\sin t}{1-\cos t}\overset{L}{=}\lim_{t\to 0}\frac{(t)[\cos t]+(\sin t)[1]}{-[-\sin t]}=\lim_{t\to 0}\frac{t\cos t+\sin t}{\sin t}$$
$$\overset{L}{=}\lim_{t\to 0}\frac{\{(t)[-\sin t]+(\cos t)[1]\}+\cos t}{\cos t}=\lim_{t\to 0}\frac{-t\sin t+2\cos t}{\cos t}$$
$$=\frac{-(0)\sin(0)+2\cos(0)}{\cos(0)}=\frac{-(0)(0)+2(1)}{(1)}=\frac{2}{1}=2$$

26) $\lim_{x\to\frac{\pi}{2}^-} \left(\frac{\pi}{2}-x\right)^{0}\tan x^{+\infty} = \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\pi}{2}-x}{\cot x} \stackrel{L}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-1}{-\csc^2 x}$

$= \lim_{x\to\frac{\pi}{2}^-} \sin^2 x = \sin^2\left(\frac{\pi}{2}^-\right) = (1)^2 = 1$

---

28) $\lim_{\theta\to 0} \frac{\left(\frac{1}{2}\right)^{\theta}-1}{\theta} \stackrel{L}{=} \lim_{\theta\to 0} \frac{\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\theta}}{1} = \frac{\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{0}}{1} = \ln\left(\frac{1}{2}\right)$

$= \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$

$p = \left(\frac{1}{2}\right)^{\theta}$

$\ln p = \ln\left(\frac{1}{2}\right)^{\theta}$

$\ln p = \theta \ln\left(\frac{1}{2}\right)$

$\frac{1}{p}\frac{dp}{d\theta} = \ln\left(\frac{1}{2}\right)$

$\frac{dp}{d\theta} = \ln\left(\frac{1}{2}\right) p$

$= \ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{\theta}$

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30) $\lim_{x\to 0} \frac{3^{x}-1}{2^{x}-1} \stackrel{L}{=} \lim_{x\to 0} \frac{(\ln 3)\,3^{x}}{(\ln 2)\,2^{x}} = \frac{(\ln 3)\,3^{(0)}}{(\ln 2)\,2^{(0)}}$

$= \frac{(\ln 3)(1)}{(\ln 2)(1)}$

$= \frac{\ln 3}{\ln 2}$

$p = 3^{x}$

$\ln p = \ln 3^{x}$

$\ln p = x \ln 3$

$\frac{1}{p}\frac{dp}{dx} = \ln 3$

$\frac{dp}{dx} = (\ln 3)\, p$

$= (\ln 3)\, 3^{x}$

$q = 2^{x}$

$\ln q = \ln 2^{x}$

$\ln q = x \ln 2$

$\frac{1}{q}\frac{dq}{dx} = \ln 2$

$\frac{dq}{dx} = (\ln 2)\, q$

$= (\ln 2)\, 2^{x}$

32) $$\lim_{x\to\infty}\frac{\overset{+\infty}{\log_2 x}}{\underset{+\infty}{\log_3 (x+3)}} = \lim_{x\to\infty}\frac{\frac{\ln x}{\ln 2}}{\frac{\ln(x+3)}{\ln 3}} = \lim_{x\to\infty}\frac{(\ln 3)(\ln x)}{(\ln 2)(\ln(x+3))}$$

$$= \left(\frac{\ln 3}{\ln 2}\right)\lim_{x\to\infty}\frac{\ln x}{\ln(x+3)} \overset{L}{=} \left(\frac{\ln 3}{\ln 2}\right)\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+3}} = \left(\frac{\ln 3}{\ln 2}\right)\lim_{x\to\infty}\frac{\overset{+\infty}{x+3}}{\underset{+\infty}{x}}$$

$$\overset{L}{=} \left(\frac{\ln 3}{\ln 2}\right)\lim_{x\to\infty}\frac{1}{1} = \left(\frac{\ln 3}{\ln 2}\right)(1) = \frac{\ln 3}{\ln 2}$$

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34) $$\lim_{x\to0^+}\frac{\overset{0}{\ln(e^x-1)}}{\underset{0}{\ln x}} \overset{L}{=} \lim_{x\to0^+}\frac{\frac{1}{e^x-1}(e^x(1))}{\frac{1}{x}(1)} = \lim_{x\to0^+}\frac{\frac{e^x}{e^x-1}}{\frac{1}{x}}$$

$$= \lim_{x\to0^+}\frac{\overset{0}{x}e^x}{\underset{0}{e^x-1}} \overset{L}{=} \lim_{x\to0^+}\frac{(x)[e^x(1)]+(e^x)[1]}{e^x(1)} = \lim_{x\to0^+}\frac{xe^x+e^x}{e^x}$$

$$= \lim_{x\to0^+}\frac{e^x(x+1)}{e^x} = \lim_{x\to0^+}(x+1) = (0^+)+1 = 1$$

$a>0$

36) $$\lim_{y\to0}\frac{\overset{0}{\sqrt{ay+a^2}-a}}{\underset{0}{y}} = \lim_{y\to0}\frac{(ay+a^2)^{\frac{1}{2}}-a}{y} \overset{L}{=} \lim_{y\to0}\frac{\left[\frac{1}{2}(ay+a^2)^{-\frac{1}{2}}(a)\right]}{1}$$

$$= \lim_{y\to0}\frac{a}{2\sqrt{ay+a^2}} = \frac{a}{2\sqrt{a(0)+a^2}} = \frac{a}{2\sqrt{a^2}} = \frac{a}{2a} = \frac{1}{2}$$

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38) $$\lim_{x\to0^+}\left(\overset{(-\infty)\ -\ (-\infty)}{\ln x - \ln\sin x}\right) = \lim_{x\to0^+}\ln\left(\frac{x}{\sin x}\right) = \ln\left(\lim_{x\to0^+}\frac{\overset{0}{x}}{\underset{0}{\sin x}}\right)$$

$$\overset{L}{=} \ln\left(\lim_{x\to0^+}\frac{1}{\cos x}\right) = \ln\left(\frac{1}{\cos(0^+)}\right) = \ln\left(\frac{1}{(1)}\right) = \ln(1) = 0$$

$$40)\ \lim_{x\to 0^+}\left(\overset{(+\infty)}{\frac{3x+1}{x}}-\overset{(+\infty)}{\frac{1}{\sin x}}\right)=\lim_{x\to 0^+}\left(\left(\frac{3x+1}{x}\right)\left(\frac{\sin x}{\sin x}\right)-\frac{1}{\sin x}\left(\frac{x}{x}\right)\right)$$

$$=\lim_{x\to 0^+}\frac{(3x+1)\sin x - x}{x\sin x}\overset{L}{=}\lim_{x\to 0^+}\frac{\{(3x+1)[\cos x]+(\sin x)[3]\}-[1]}{(x)[\cos x]+(\sin x)[1]}$$

$$=\lim_{x\to 0^+}\frac{(3x+1)\cos x+3\sin x-1}{x\cos x+\sin x}\overset{L}{=}\lim_{x\to 0^+}\frac{\{(3x+1)[-\sin x]+(\cos x)[3]\}+3[\cos x]}{\{(x)[-\sin x]+(\cos x)[1]\}+[\cos x]}$$

$$=\lim_{x\to 0^+}\frac{-(3x+1)\sin x+3\cos x+3\cos x}{-x\sin x+\cos x+\cos x}=\lim_{x\to 0^+}\frac{6\cos x-(3x+1)\sin x}{2\cos x - x\sin x}$$

$$=\frac{6\cos(0^+)-(3(0^+)+1)\sin(0^+)}{2\cos(0^+)-(0^+)\sin(0^+)}=\frac{6(1)-(1)(0)}{2(1)-(0)(0)}=\frac{6}{2}=3$$

$$42)\ \lim_{x\to 0^+}\left(\overset{(+\infty)}{\csc x}-\overset{(+\infty)}{\cot x}+\overset{(1)}{\cos x}\right)=\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}+\cos x\right)$$

$$=\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)+\lim_{x\to 0^+}\cos x=\lim_{x\to 0^+}\left(\frac{1-\cos x}{\sin x}\right)+\lim_{x\to 0^+}\cos x$$

$$=(0)+\cos(0^+)=(0)+(1)=\underline{\underline{1}}$$

$$\lim_{x\to 0^+}\frac{1-\cos x}{\sin x}\overset{L}{=}\lim_{x\to 0^+}\frac{-[-\sin x]}{[\cos x]}=\lim_{x\to 0^+}\frac{\sin x}{\cos x}=\frac{\sin(0^+)}{\cos(0^+)}=\frac{0}{1}=0$$

$$44)\ \lim_{h\to 0}\frac{e^h-(1+h)}{h^2}=\lim_{h\to 0}\frac{e^h-1-h}{h^2}\overset{L}{=}\lim_{h\to 0}\frac{e^h-1}{2h}\overset{L}{=}\lim_{h\to 0}\frac{e^h}{2}$$

$$=\frac{e^0}{2}=\frac{1}{2}$$

46) $$\lim_{x\to\infty} \overset{(+\infty)(0)}{x^2 e^{-x}} = \lim_{x\to\infty} \frac{\overset{+\infty}{x^2}}{\underset{+\infty}{e^x}} \overset{L}{=} \lim_{x\to\infty} \frac{\overset{+\infty}{2x}}{\underset{+\infty}{e^x}} \overset{L}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

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48) $$\lim_{x\to 0} \frac{\overset{0}{(e^x-1)^2}}{\underset{0}{x \sin x}} \overset{L}{=} \lim_{x\to 0} \frac{2(e^x-1)'(e^x(1))}{(x)[\cos x]+(\sin x)[1]} = \lim_{x\to 0} \frac{\overset{0}{2e^{2x}-2e^x}}{\underset{0}{x\cos x+\sin x}}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{2[e^{2x}(2)]-2[e^x(1)]}{\{(x)[-\sin x]+(\cos x)[1]\}+[\cos x]} = \lim_{x\to 0} \frac{4e^{2x}-2e^x}{-x\sin x+\cos x+\cos x}$$

$$= \lim_{x\to 0} \frac{4e^{2x}-2e^x}{-x\sin x+2\cos x} = \frac{4e^{2(0)}-2e^{(0)}}{-(0)\sin(0)+2\cos(0)} = \frac{4(1)-2(1)}{-0+2(1)} = \frac{2}{2} = 1$$

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50) $$\lim_{x\to 0} \frac{\overset{0}{\sin 3x - 3x + x^2}}{\underset{0}{\sin x \sin 2x}} \overset{L}{=} \lim_{x\to 0} \frac{[\cos(3x)(3)]-3[1]+[2x]}{(\sin x)[\cos(2x)(2)]+(\sin 2x)[\cos x]}$$

$$= \lim_{x\to 0} \frac{\overset{0}{3\cos(3x)-3+2x}}{\underset{0}{2\sin x\cos(2x)+\cos x\sin(2x)}}$$

$$\overset{L}{=} \lim_{x\to 0} \frac{3[-\sin(3x)(3)]+2[1]}{\{(2\sin x)[-\sin(2x)(2)]+(\cos(2x))[2\cos x]\}+\{(\cos x)[\cos(2x)(2)]+(\sin(2x))[-\sin x]}$$

$$= \lim_{x\to 0} \frac{-9\sin(3x)+2}{-4\sin x\sin(2x)+2\cos x\cos(2x)+2\cos x\cos(2x)-\sin x\sin(2x)}$$

$$= \lim_{x\to 0} \frac{-9\sin(3x)+2}{4\cos x\cos(2x)-5\sin x\sin(2x)} = \frac{-9\sin(3(0))+2}{4\cos(0)\cos(2(0))-5\sin(0)\sin(2(0))}$$

$$= \frac{-9(0)+2}{4(1)(1)-5(0)(0)} = \frac{2}{4} = \frac{1}{2}$$

52) $\lim\limits_{x\to 1^+} x^{\frac{1}{x-1}}$ {type 1^∞}

$y = x^{\frac{1}{x-1}}$

$\ln y = \ln\left(x^{\frac{1}{x-1}}\right)$

$\ln y = \left(\frac{1}{x-1}\right)\ln x$

$\ln y = \frac{\ln x}{x-1}$

$$\lim_{x\to 1^+} \frac{\overset{0}{\ln x}}{\underset{0}{x-1}} \overset{L}{=} \lim_{x\to 1^+} \frac{\left[\frac{1}{x}(1)\right]}{[1]} = \lim_{x\to 1^+} \frac{1}{x}$$

$$= \frac{1}{(1^+)} = 1$$

$\ln y = 1 \Rightarrow y = e^1 = e \therefore \underline{\underline{\lim\limits_{x\to 1^+} x^{\frac{1}{x-1}} = e}}$

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54) $\lim\limits_{x\to e^+} (\ln x)^{\frac{1}{x-e}}$ {type 1^∞}

$y = (\ln x)^{\frac{1}{x-e}}$

$\ln y = \ln\left((\ln x)^{\frac{1}{x-e}}\right)$

$\ln y = \left(\frac{1}{x-e}\right)\ln(\ln x)$

$\ln y = \frac{\ln(\ln x)}{x-e}$

$$\lim_{x\to e^+} \frac{\overset{0}{\ln(\ln x)}}{\underset{0}{x-e}} \overset{L}{=} \lim_{x\to e^+} \frac{\left[\frac{1}{\ln x}\left(\frac{1}{x}(1)\right)\right]}{[1]}$$

$$= \lim_{x\to e^+} \frac{1}{x\ln x} = \frac{1}{(e^+)\ln(e^+)} = \frac{1}{e(1)} = \frac{1}{e}$$

$\ln y = \frac{1}{e} \Rightarrow y = e^{\frac{1}{e}} \therefore \underline{\underline{\lim\limits_{x\to e^+} (\ln x)^{\frac{1}{x-e}} = e^{\frac{1}{e}}}}$

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56) $\lim\limits_{x\to\infty} x^{\frac{1}{\ln x}}$ {type $(\infty)^0$}

$y = x^{\frac{1}{\ln x}}$

$\ln y = \ln\left(x^{\frac{1}{\ln x}}\right)$

$\ln y = \left(\frac{1}{\ln x}\right)\ln x$

$\ln y = \frac{\ln x}{\ln x}$

$\ln y = 1$

$$\lim_{x\to\infty} 1 = 1$$

$\ln y = 1 \Rightarrow y = e^1 = e \quad \therefore \underline{\underline{\lim\limits_{x\to\infty} x^{\frac{1}{\ln x}} = e}}$

58) $\lim_{x\to 0}(e^x+x)^{\frac{1}{x}}$ {type 1^∞}

$y=(e^x+x)^{\frac{1}{x}}$

$\ln y=\ln\left((e^x+x)^{\frac{1}{x}}\right)$

$\ln y=\left(\frac{1}{x}\right)\ln(e^x+x)$

$\ln y=\frac{\ln(e^x+x)}{x}$

$$\lim_{x\to 0}\frac{\overset{0}{\ln(e^x+x)}}{\underset{0}{x}} \overset{L}{=} \lim_{x\to 0}\frac{\left[\frac{1}{e^x+x}(e^x(1)+1)\right]}{[1]}$$

$$=\lim_{x\to 0}\frac{e^x+1}{e^x+x}=\frac{e^{(0)}+1}{e^{(0)}+(0)}=\frac{1+1}{1+0}=\frac{2}{1}=2$$

$\ln y=2 \Rightarrow y=e^2$ $\therefore \lim_{x\to 0}(e^x+x)^{\frac{1}{x}}=e^2$

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60) $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$ {type ∞^0}

$y=\left(1+\frac{1}{x}\right)^x$

$\ln y=\ln\left(\left(1+\frac{1}{x}\right)^x\right)$

$\ln y=x\ln\left(1+\frac{1}{x}\right)$

$\ln y=\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$$\lim_{x\to 0^+}\frac{\overset{+\infty}{\ln\left(1+\frac{1}{x}\right)}}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\ln(1+x^{-1})}{x^{-1}}$$

$$\overset{L}{=}\lim_{x\to 0^+}\frac{\left[\frac{1}{1+x^{-1}}(-1x^{-2})\right]}{[-1x^{-2}]}=\lim_{x\to 0^+}\frac{\frac{-1}{x^2\left(1+\frac{1}{x}\right)}}{\frac{-1}{x^2}}$$

$$=\lim_{x\to 0^+}\left(\frac{-1}{x^2\left(1+\frac{1}{x}\right)}\right)\left(\frac{x^2}{-1}\right)=\lim_{x\to 0^+}\frac{1}{1+\frac{1}{x}}$$

$$=\lim_{x\to 0^+}\left(\frac{\frac{1}{1}}{\frac{1}{1}+\frac{1}{x}}\right)\left(\frac{\frac{x}{1}}{\frac{x}{1}}\right)=\lim_{x\to 0^+}\frac{x}{x+1}$$

$$=\frac{(0^+)}{(0^+)+1}=0$$

$\ln y=0 \Rightarrow y=e^0=1$ $\therefore \lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x=1$

62) $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$ {type ∞^0}

$$\lim_{x\to\infty}\frac{\overset{+\infty}{x^2+1}}{\underset{+\infty}{x+2}} \overset{L}{=} \lim_{x\to\infty}\frac{2x}{1} = \lim_{x\to\infty} 2x = +\infty$$

$$y = \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$$

$$\ln y = \ln\left(\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}\right)$$

$$\ln y = \left(\frac{1}{x}\right)\ln\left(\frac{x^2+1}{x+2}\right)$$

$$\ln y = \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}$$

$$\lim_{x\to\infty}\frac{\overset{+\infty}{\ln\left(\frac{x^2+1}{x+2}\right)}}{\underset{+\infty}{x}} = \lim_{x\to\infty}\frac{\ln(x^2+1)-\ln(x+2)}{x}$$

$$\overset{L}{=}\lim_{x\to\infty}\frac{\left[\frac{1}{x^2+1}(2x)\right]-\left[\frac{1}{x+2}(1)\right]}{1}$$

$$= \lim_{x\to\infty}\left(\frac{2x}{x^2+1}-\frac{1}{x+2}\right) = \lim_{x\to\infty}\frac{2x}{x^2+1} - \lim_{x\to\infty}\frac{1}{x+2}$$

$$= (0)-(0) = 0$$

$$\lim_{x\to\infty}\frac{\overset{+\infty}{2x}}{\underset{+\infty}{x^2+1}} \overset{L}{=} \lim_{x\to\infty}\frac{2}{2x}$$

$$= \lim_{x\to\infty}\frac{1}{x} = 0$$

$$\ln y = 0 \Rightarrow y = e^0 = 1 \qquad \therefore \underline{\underline{\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}} = 1}}$$

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64) $\lim_{x\to 0^+} \overset{(0)}{x}\overset{(+\infty)}{(\ln x)^2} = \lim_{x\to 0^+}\frac{\overset{+\infty}{(\ln x)^2}}{\underset{+\infty}{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{(\ln x)^2}{x^{-1}} \overset{L}{=} \lim_{x\to 0^+}\frac{\left[2(\ln x)^1\left(\frac{1}{x}(1)\right)\right]}{[-1x^{-2}]}$

$$= \lim_{x\to 0^+}\frac{\frac{2\ln x}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\left(\frac{2\ln x}{x}\right)\left(\frac{x^2}{-1}\right) = \lim_{x\to 0^+} -2\overset{(0)}{x}\,\overset{(-\infty)}{\ln x} = \lim_{x\to 0^+}\frac{\overset{+\infty}{-2\ln x}}{\underset{+\infty}{\frac{1}{x}}}$$

$$= \lim_{x\to 0^+}\frac{-2\ln x}{x^{-1}} \overset{L}{=} \lim_{x\to 0^+}\frac{-2\left[\frac{1}{x}(1)\right]}{[-1x^{-2}]} = \lim_{x\to 0^+}\frac{\frac{-2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}\left(\frac{-2}{x}\right)\left(\frac{x^2}{-1}\right)$$

$$= \lim_{x\to 0^+} 2x = 2(0^+) = 0$$

66)

$$\lim_{x\to 0^+} (\sin x)(\ln x) \quad (0)(-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\csc x} \quad \frac{-\infty}{+\infty} \stackrel{L}{=} \lim_{x\to 0^+} \frac{\left[\frac{1}{x}(1)\right]}{[-\csc x \cot x (1)]}$$

$$= \lim_{x\to 0^+} \frac{-\sin x \tan x}{x} \quad \frac{0}{0} = \lim_{x\to 0^+} \frac{(-\sin x)[\sec^2 x] + (\tan x)[-(\cos x(1))]}{[1]}$$

$$= \lim_{x\to 0^+}(-\sin x \sec^2 x - \cos x \tan x) = -\sin(0^+)\sec^2(0^+) - \cos(0^+)\tan(0^+)$$

$$= -(0)(1)^2 - (1)(0) = 0$$

---

68)

$$\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \lim_{x\to 0^+} \sqrt{\frac{x}{\sin x}} = \sqrt{\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)}$$

$$= \sqrt{\lim_{x\to 0^+}\left(\frac{1}{\frac{\sin x}{x}}\right)} = \sqrt{\frac{1}{\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)}} = \sqrt{\frac{1}{(1)}} = \sqrt{1} = 1$$

---

70)

$$\lim_{x\to 0^+} \frac{\cot x}{\csc x} = \lim_{x\to 0^+} \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} = \lim_{x\to 0^+}\left(\frac{\cos x}{\sin x}\right)\left(\frac{\sin x}{1}\right)$$

$$= \lim_{x\to 0^+} \cos x = \cos(0^+) = 1$$

---

72)

$$\lim_{x\to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x\to -\infty} \frac{\frac{2^x}{2^x} + \frac{4^x}{2^x}}{\frac{5^x}{2^x} - \frac{2^x}{2^x}} = \lim_{x\to -\infty} \frac{1 + \frac{4^x}{2^x}}{\frac{5^x}{2^x} - 1}$$

$$= \lim_{x\to -\infty} \frac{1 + \left(\frac{4}{2}\right)^x}{\left(\frac{5}{2}\right)^x - 1} = \lim_{x\to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \frac{1+0}{0-1} = \frac{1}{-1} = -1$$

74) $\lim\limits_{x\to 0^+} \dfrac{\overset{0}{x}}{\underset{0}{e^{-\frac{1}{x}}}} = \overset{(0)(\infty)}{\lim\limits_{x\to 0^+} x\, e^{\frac{1}{x}}} = \lim\limits_{x\to 0^+} \dfrac{\overset{+\infty}{e^{\frac{1}{x}}}}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{e^{x^{-1}}}{x^{-1}}$

$\overset{L}{=} \lim\limits_{x\to 0^+} \dfrac{[e^{x^{-1}}(-1x^{-2})]}{[-1x^{-2}]} = \lim\limits_{x\to 0} \dfrac{\frac{-e^{\frac{1}{x}}}{x^2}}{\frac{-1}{x^2}} = \lim\limits_{x\to 0} \left(\dfrac{-e^{\frac{1}{x}}}{x^2}\right)\left(\dfrac{x^2}{-1}\right)$

$= \lim\limits_{x\to 0^+} e^{\frac{1}{x}} = +\infty$

---

76) a) $\lim\limits_{x\to 0} \dfrac{\overset{0}{x^2-2x}}{\underset{0}{x^2-\sin x}} \overset{L}{=} \lim\limits_{x\to 0} \dfrac{\overset{-2}{2x-2}}{\underset{-1}{2x-\cos x}} = \lim\limits_{x\to 0} \dfrac{2}{2+\sin x} = \dfrac{2}{2+0} = 1$

not correct

b) $\lim\limits_{x\to 0} \dfrac{\overset{0}{x^2-2x}}{\underset{0}{x^2-\sin x}} \overset{L}{=} \lim\limits_{x\to 0} \dfrac{\overset{-2}{2x-2}}{\underset{-1}{2x-\cos x}} = \dfrac{-2}{0-1} = 2$

---

78) a, b $\quad f(x) = x \qquad g(x) = x^2$

$\dfrac{df}{dx} = 1 \qquad \dfrac{dg}{dx} = 2x$

a) $(a,b) = (-2,0)$

$f(0) = (0) = 0 \qquad f(-2) = (-2) = -2 \qquad \left.\dfrac{df}{dx}\right|_{x=c} = 1$

$g(0) = (0)^2 = 0 \qquad g(-2) = (-2)^2 = 4 \qquad \left.\dfrac{dg}{dx}\right|_{x=c} = 2(c) = 2c$

$$\frac{1 = \left.\frac{df}{dx}\right|_{x=c}}{2c = \left.\frac{dg}{dx}\right|_{x=c}} = \frac{f(0)-f(-2)}{g(0)-g(-2)} = \frac{(0)-(-2)}{(0)-(4)} = \frac{2}{-4} = \frac{-1}{2} = \frac{1}{-2}$$

$\Downarrow$

$\dfrac{1}{2c} = \dfrac{1}{-2} \Rightarrow 2c = -2$

$\underline{\underline{c = -1}}$

78) continued

b) (a, b) arbitrary

$f(b) = (b) = b \qquad f(a) = (a) = a \qquad \left.\frac{df}{dx}\right|_{x=c} = 1$

$g(b) = (b)^2 = b^2 \qquad g(a) = (a)^2 = a^2 \qquad \left.\frac{dg}{dx}\right|_{x=c} = 2(c) = 2c$

$$\frac{1 = \left.\frac{df}{dx}\right|_{x=c}}{2c = \left.\frac{dg}{dx}\right|_{x=c}} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b)-(a)}{(b^2)-(a^2)} = \frac{b-a}{b^2-a^2} = \frac{b-a}{(b+a)(b-a)} = \frac{1}{b+a}$$

$$\frac{1}{2c} = \frac{1}{b+a} \Rightarrow \begin{array}{l} 2c = b+a \\ c = \frac{b+a}{2} \end{array}$$

---

c) $f(x) = \frac{x^3}{3} - 4x \qquad g(x) = x^2 \qquad (a, b) = (0, 3)$

$\frac{df}{dx} = \frac{1}{3}[3x^2] - 4[1] = x^2 - 4 \qquad \frac{dg}{dx} = [2x] = 2x$

$f(3) = \frac{(3)^3}{3} - 4(3) = 9 - 12 = -3 \qquad f(0) = \frac{(0)^3}{3} - 4(0) = 0 \qquad \left.\frac{df}{dx}\right|_{x=c} = (c)^2 - 4 = c^2 - 4$

$g(3) = (3)^2 = 9 \qquad g(0) = (0)^2 = 0 \qquad \left.\frac{dg}{dx}\right|_{x=c} = 2(c) = 2c$

$$\frac{c^2 - 4 = \left.\frac{df}{dx}\right|_{x=c}}{2c = \left.\frac{dg}{dx}\right|_{x=c}} = \frac{f(3) - f(0)}{g(3) - g(0)} = \frac{(-3)-(0)}{(9)-(0)} = \frac{-3}{9} = \frac{-1}{3}$$

$\frac{c^2 - 4}{2c} = \frac{-1}{3}$

$3(c^2 - 4) = -1(2c)$

$3c^2 - 12 = -2c$

$3c^2 + 2c - 12 = 0$

$c = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-12)}}{2(3)}$

$c = \frac{-2 \pm \sqrt{4(1+36)}}{2(3)}$

$c = \frac{-2 \pm 2\sqrt{37}}{2(3)} = \frac{2(-1 \pm \sqrt{37})}{2(3)}$

$c = \frac{-1 - \sqrt{37}}{3}$ discard

$\underline{\underline{c = \frac{-1 + \sqrt{37}}{3}}}$

86) a) $y = x^{\frac{1}{x}}$

$\ln y = \ln\left(x^{\frac{1}{x}}\right)$

$\ln y = \frac{\ln x}{x}$

$\left[\frac{1}{y}\frac{dy}{dx}\right] = \frac{(x)\left[\frac{1}{x}(1)\right] - (\ln x)[1]}{x^2}$

$\frac{dy}{dx} = \left\{\frac{1-\ln x}{x^2}\right\} y = \left\{\frac{1-\ln x}{x^2}\right\} x^{\frac{1}{x}}$

critical points

$0 = \frac{dy}{dx} = \left\{\frac{1-\ln x}{x^2}\right\} x^{\frac{1}{x}}$

$0 \neq x^{\frac{1}{x}}$ discard | $\left\{\frac{1-\ln x}{x^2}\right\} = 0$

$1 - \ln x = 0$

$1 = \ln x$

$\Downarrow$

$x = e^1 = e$

$\frac{dy}{dx}$ $0 ($ INC. $\underset{e}{|}$ dec.

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \left\{\frac{1-\ln(1)}{(1)^2}\right\}(1)^{\frac{1}{(1)}} = \left\{\frac{1-0}{1}\right\}(1) > 0$ INC

at $x=3$: $\left.\frac{dy}{dx}\right|_{x=3} = \left\{\frac{1-\ln(3)}{(3)^2}\right\}(3)^{\frac{1}{(3)}} < 0$ dec.

max value:

$y|_{x=e} = (e)^{\frac{1}{(e)}} = e^{\frac{1}{e}}$

---

b) $y = x^{\frac{1}{x^2}}$

$\ln y = \ln\left(x^{\frac{1}{x^2}}\right)$

$\ln y = \frac{\ln x}{x^2}$

$\left[\frac{1}{y}\frac{dy}{dx}\right] = \frac{(x^2)\left[\frac{1}{x}(1)\right] - (\ln x)[2x]}{(x^2)^2}$

$\frac{1}{y}\frac{dy}{dx} = \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$

$\frac{dy}{dx} = \left\{\frac{1-2\ln x}{x^3}\right\} y = \left\{\frac{1-2\ln x}{x^3}\right\} x^{\frac{1}{x^2}}$

critical points

$0 = \frac{dy}{dx} = \left\{\frac{1-2\ln x}{x^3}\right\} x^{\frac{1}{x^2}}$

$0 \neq x^{\frac{1}{x^2}}$ discard | $\left\{\frac{1-2\ln x}{x^3}\right\} = 0$

$1 - 2\ln x = 0$

$1 = 2\ln x$

$\frac{1}{2} = \ln x$

$\Downarrow$

$x = e^{\frac{1}{2}} = \sqrt{e}$

$\frac{dy}{dx}$ $0 ($ INC. $\underset{\sqrt{e}}{|}$ dec.

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \left\{\frac{1-2\ln(1)}{(1)^3}\right\}(1)^{\frac{1}{(1)^2}} > 0$ INC

at $x=2$: $\left.\frac{dy}{dx}\right|_{x=2} = \left\{\frac{1-2\ln(2)}{(2)^3}\right\}(2)^{\frac{1}{(2)^2}} < 0$ dec.

max value:

$y|_{x=\sqrt{e}} = (\sqrt{e})^{\frac{1}{(\sqrt{e})^2}} = (\sqrt{e})^{\frac{1}{e}} = e^{\frac{1}{2e}}$

86) continued

c) $y = x^{\frac{1}{x^n}}$

$\ln y = \ln\left(x^{\frac{1}{x^n}}\right)$

$\ln y = \dfrac{\ln x}{x^n}$

$\left[\dfrac{1}{y}\dfrac{dy}{dx}\right] = \dfrac{(x^n)\left[\frac{1}{x}(1)\right] - (\ln x)\left[n x^{n-1}\right]}{(x^n)^2}$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x^{n-1} - n x^{n-1}\ln x}{x^{2n}}$

$\dfrac{dy}{dx} = \left\{\dfrac{x^{n-1}(1-n\ln x)}{x^{2n}}\right\} y = \left\{\dfrac{1-n\ln x}{x^{n+1}}\right\} x^{\frac{1}{x^n}}$

critical points

$0 = \dfrac{dy}{dx} = \left\{\dfrac{1-n\ln x}{x^{n+1}}\right\} x^{\frac{1}{x^n}}$

$0 \neq x^{\frac{1}{x^n}}$ discard

$\left\{\dfrac{1-n\ln x}{x^{n+1}}\right\} = 0$

$1 - n\ln x = 0$

$1 = n\ln x$

$\dfrac{1}{n} = \ln x$

$\Downarrow$

$x = e^{\frac{1}{n}} = \sqrt[n]{e}$

$\dfrac{dy}{dx}$ 0 (INC $\sqrt[n]{e}$ dec.

max value:

$y\big|_{x=\sqrt[n]{e}} = (\sqrt[n]{e})^{\frac{1}{(\sqrt[n]{e})^n}} = (\sqrt[n]{e})^{\frac{1}{e}}$

$= e^{\frac{1}{ne}}$

at $x=1$: $\left.\dfrac{dy}{dx}\right|_{x=1} = \left\{\dfrac{1-n\ln(1)}{(1)^{n+1}}\right\}(1)^{\frac{1}{(1)^n}} > 0$ INC.

at $x=2$: $\left.\dfrac{dy}{dx}\right|_{x=2} = \left\{\dfrac{1-n\ln(2)}{(2)^{n+1}}\right\}(2)^{\frac{1}{(2)^n}} < 0$ dec.

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d) $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x^n}}$ $\quad$ n - positive integer

$\ln y = \dfrac{\ln x}{x^n}$

$\displaystyle\lim_{x\to\infty} \frac{\overset{+\infty}{\ln x}}{\underset{+\infty}{x^n}} \overset{L}{=} \lim_{x\to\infty}\frac{\left[\frac{1}{x}(1)\right]}{\left[n x^{n-1}\right]} = \lim_{x\to\infty}\frac{1}{n(x^{n-1})(x)}$

$\displaystyle = \lim_{x\to\infty}\frac{1}{n x^n} = 0$

$\ln y = 0 \Rightarrow y = e^0 = 1$

$\therefore \displaystyle\lim_{x\to\infty} x^{\frac{1}{x^n}} = 1$ for n - positive integer