Definition:

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval *I* if $\frac{df}{dx}$ is increasing on *I*.

(b) concave down on an open interval *I* if $\frac{df}{dx}$ is decreasing on *I*.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval *I*.

1. If $\frac{d^2 f}{dx^2} > 0$ on *I*, the graph of $f(x)$ over *I* is concave up.

2. If $\frac{d^2 f}{dx^2} < 0$ on *I*, the graph of $f(x)$ over *I* is concave down.

Definition:

A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $\left.\frac{d^2 f}{dx^2}\right|_{x=c} = 0$ or $\left.\frac{d^2 f}{dx^2}\right|_{x=c}$ fails to exist.

Theorem 5 – Second Derivative Test for Local Extrema

Suppose $\frac{d^2 f}{dx^2}$ is continuous on an open interval that contains $x = c$.

1. If $\left.\frac{df}{dx}\right|_{x=c} = 0$ and $\left.\frac{d^2 f}{dx^2}\right|_{x=c} < 0$, then $f(x)$ has a local maximum at $x = c$.

2. If $\left.\frac{df}{dx}\right|_{x=c} = 0$ and $\left.\frac{d^2 f}{dx^2}\right|_{x=c} > 0$, then $f(x)$ has a local minimum at $x = c$.

3. If $\left.\frac{df}{dx}\right|_{x=c} = 0$ and $\left.\frac{d^2 f}{dx^2}\right|_{x=c} = 0$, then the test fails. The function $f(x)$ may have a local maximum, a local minimum, or neither at $x = c$.

Instead of using the Procedure for Graphing $y = f(x)$, on page 248 of your text, I'll be using more detailed procedure shown on next page.

Check the figure that summarizes how the first derivative and second derivative affect the shape of a graph on page 251 of your textbook.

The step by step procedure below is for regular rational and polynomial functions. If a function contains radical or trigonometric term, then proceed carefully because the steps below must be modified.

Step 1:
Determine if the function is rational or polynomial. If the function is a polynomial then the domain is $(-\infty,\infty)$ and skip to step 5

Step 2:
Determine if the rational function is proper or improper. If it is proper (or improper with numerator and denominator of the same degree) then apply limit as $x \to \pm\infty$ to the function to find the Horizontal Asymptote, and go to step 4.

Step 3:
For the improper fraction with degree of numerator greater than degree of denominator, do a long polynomial division (numerator divided by denominator) to break up the rational expression into a simple polynomial and a fraction at the end. The simple polynomial is your oblique asymptote (not vertical).

Step 4:
Find the vertical asymptotes. If you have a real number solution, then the solution(s)s is/are the vertical asymptote. The domain is all real numbers with values of the asymptotes removed.

Step 5:
Find *y*-intercept (set $x = 0$), if exists.

Step 6:
Find the first and second derivatives if necessary.

Step 7:
Draw 2 lines one for first derivative and the other for second derivative. Each line must be the same amount as the intervals of the domain.

Step 8:
Compute the Critical Points (by setting first derivative equal to 0) and Inflection Points (by setting second derivative equal to 0). If you have real number solution, then label on the lines created in step 7.

Step 9:
Only when a critical number is unique (the value is not a solution of inflection points), apply the second derivative test on the value. With this information we can find if it is a local maximum or minimum and predict the behavior surrounding this critical value. Otherwise, use a full test (take a test point on the interval and compute the value of its first and second derivatives to find the behavior).

Step 10:
Compute the *y* value of all critical and inflection points and sketch the graph.

To illustrate the steps above, a curve sketching of a polynomial and rational functions are show on following pages (page 2 to page 6).

Example 1: $y = 2 + 3x^2 - x^3$

Step 1: Function is polynomial. Domain: $(-\infty, \infty)$

Step 5: y-intercept: $y = 2 + 3(0)^2 - (0)^3 = 0 \quad \Rightarrow \quad y = 0$

Step 6:

$$\frac{dy}{dx} = 0 + 3[2x] - [3x^2] = 6x - 3x^2$$

$$\frac{d^2y}{dx^2} = 6[1] - 3[2x] = 6 - 6x$$

Step 7:

$\frac{dy}{dx}$

$\frac{d^2y}{dx^2}$

Step 8:

Critical points

$$0 = \frac{dy}{dx} = 6x - 3x^2 \quad \Rightarrow \quad 0 = 3x(2-x) \quad \Rightarrow \quad \begin{array}{ll} 3x = 0 & 2 - x = 0 \\ x = 0 & x = 2 \end{array}$$

$$0 = 6x - 3x^2$$

Inflection points

$$0 = \frac{d^2y}{dx^2} = 6 - 6x \quad \Rightarrow \quad 0 = 6(1-x) \quad \Rightarrow \quad \begin{array}{l} 1 - x = 0 \\ x = 1 \end{array}$$

$\frac{dy}{dx}$: 0, 2

$\frac{d^2y}{dx^2}$: 1

Step 9:

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6 - 6(0) > 0 \quad \text{C.U.} \quad \text{local min.} \qquad \left.\frac{d^2y}{dx^2}\right|_{x=2} = 6 - 6(2) < 0 \quad \text{C.D.} \quad \text{local max.}$$

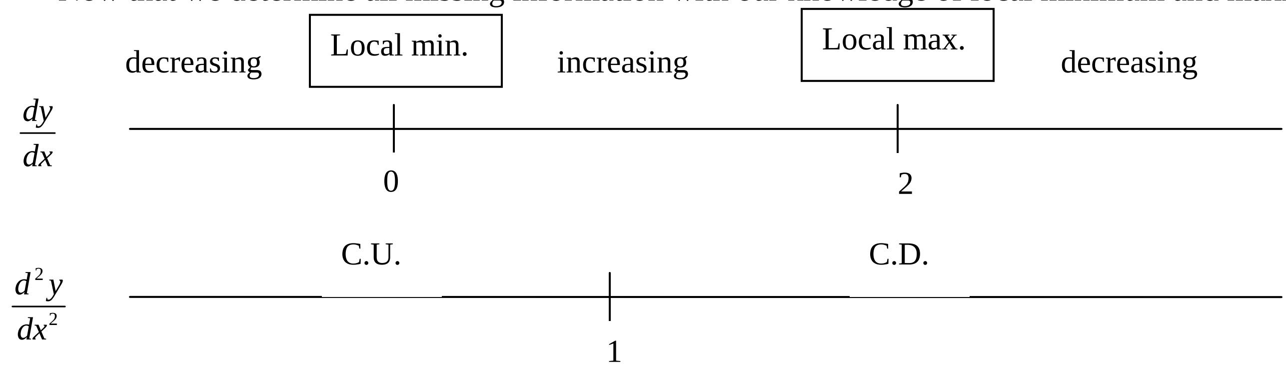
Local min. Local max.

$\frac{dy}{dx}$: 0, 2

C.U. C.D.

$\frac{d^2y}{dx^2}$: 1

Now that we determine all missing information with our knowledge of local minimum and maximum.

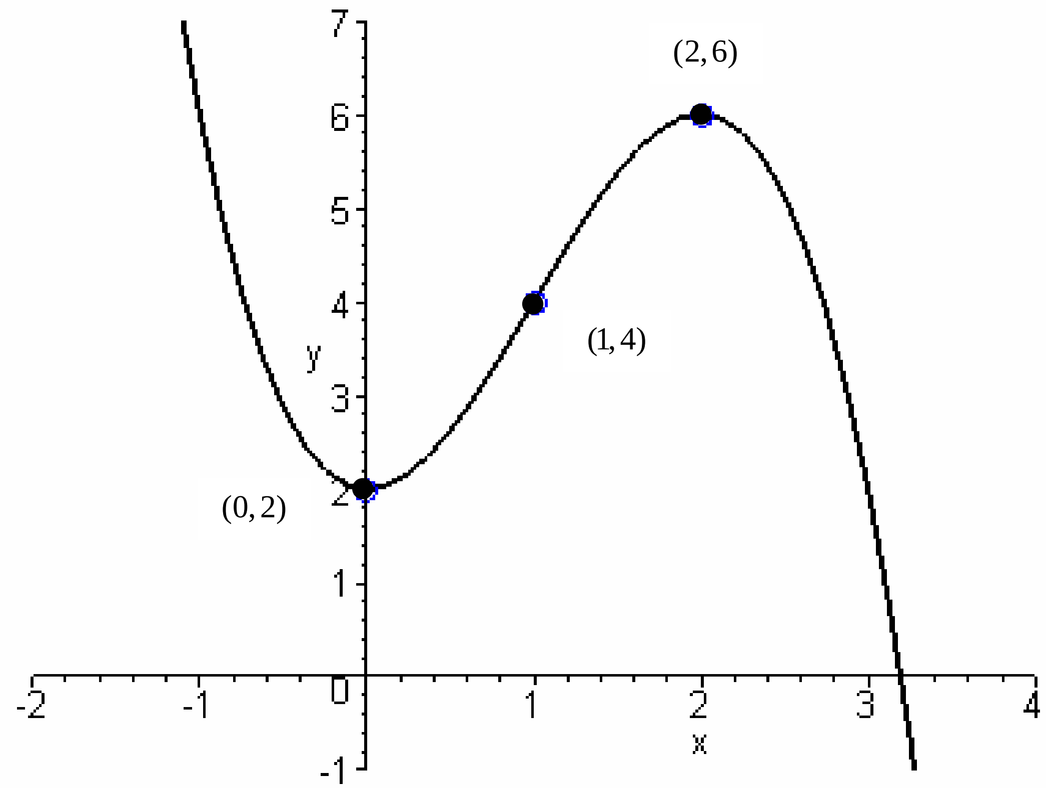


Step 10:

$y\big|_{x=0} = 2+3(0)^2-(0)^3 = 2 \quad (0,2)$

$y\big|_{x=2} = 2+3(2)^2-(2)^3 = 2+12-8 = 6 \quad (2,6)$

$y\big|_{x=1} = 2+3(1)^2-(1)^3 = 2+3-1 = 4 \quad (1,4)$



Example 2: $y = \dfrac{x^2}{x^2+9}$

Steps 1 and 2: Improper rational function with degree of numerator same as degree of denominator.

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{x^2}{x^2+9} = \lim_{x\to\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{9}{x^2}} = \frac{1}{1+0} = \frac{1}{1} = 1$$

We have a horizontal asymptote of $y=1$

Step 4: $x^2+9=0$ no solution Domain: $(-\infty,\infty)$

Step 5: $y=\dfrac{(0)^2}{(0)^2+9}=0$

$$\frac{dy}{dx}=\frac{(x^2+9)[2x]-(x^2)[2x]}{(x^2+9)^2}=\frac{2x\left\{(x^2+9)[1]-(x^2)[1]\right\}}{(x^2+9)^2}=\frac{2x\{9\}}{(x^2+9)^2}=\frac{18x}{(x^2+9)^2}$$

Step 6: $$\frac{d^2y}{dx^2}=\frac{\left\{\left((x^2+9)^2\right)[18]-(18x)\left[2(x^2+9)^1(2x)\right]\right\}}{\left((x^2+9)^2\right)^2}=\frac{18(x^2+9)\left\{(x^2+9)[1]-(x)\left[4x\right]\right\}}{(x^2+9)^4}$$

$$=\frac{18\{x^2+9-4x^2\}}{(x^2+9)^3}=\frac{18\{9-3x^2\}}{(x^2+9)^3}$$

Step 7:

$\dfrac{dy}{dx}$

$\dfrac{d^2y}{dx^2}$

Step 8:

Critical points

$$0=\frac{dy}{dx}=\frac{18x}{(x^2+9)^2} \Rightarrow 0=\frac{18x}{(x^2+9)^2} \Rightarrow \begin{aligned}18x&=0\\ x&=0\end{aligned}$$

Inflection points

$$0=\frac{d^2y}{dx^2}=\frac{18\{9-3x^2\}}{(x^2+9)^3}$$

$$0=\frac{18\{9-3x^2\}}{(x^2+9)^3}$$

$$\Rightarrow \begin{aligned}0&=18\{9-3x^2\}\\ 0&=18(3)(3-x^2)\\ 0&=18(3)\left(\sqrt{3}+x\right)\left(\sqrt{3}-x\right)\end{aligned} \Rightarrow \begin{aligned}\sqrt{3}+x&=0 & \sqrt{3}-x&=0\\ x&=-\sqrt{3} & x&=\sqrt{3}\end{aligned}$$

$\dfrac{dy}{dx}$

0

$\dfrac{d^2y}{dx^2}$

$-\sqrt{3}$ $\sqrt{3}$

Step 9:

$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=\frac{18\{9-3(0)^2\}}{((0)^2+9)^3}>0$$ C.U. local min.

$\boxed{\text{Local min.}}$

$\frac{dy}{dx}$ 0

C.U.

$\frac{d^2y}{dx^2}$ $-\sqrt{3}$ $\sqrt{3}$

Now that we determine all missing information with our knowledge of local minimum and maximum.

decreasing $\boxed{\text{Local min.}}$ increasing

$\frac{dy}{dx}$ 0

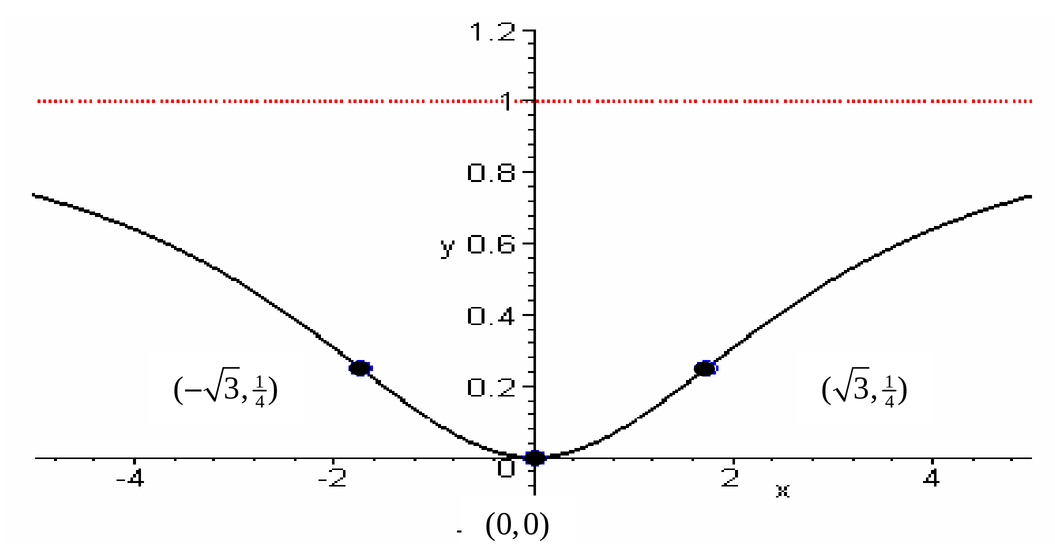
C.D. C.U. C.D.

$\frac{d^2y}{dx^2}$ $-\sqrt{3}$ $\sqrt{3}$

Step 10:

$$y\big|_{x=0} = \frac{(0)^2}{(0)^2+9} = 0 \quad (0,0)$$

$$y\big|_{x=-\sqrt{3}} = \frac{\left(-\sqrt{3}\right)^2}{\left(-\sqrt{3}\right)^2+9} = \frac{3}{3+9} = \frac{3}{12} = \frac{1}{4} \quad (-\sqrt{3}, \tfrac{1}{4}) \qquad y\big|_{x=\sqrt{3}} = \frac{\left(\sqrt{3}\right)^2}{\left(\sqrt{3}\right)^2+9} = \frac{3}{3+9} = \frac{3}{12} = \frac{1}{4} \quad (\sqrt{3}, \tfrac{1}{4})$$



2) $y = \frac{1}{4}x^4 - 2x^2 + 4$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = \frac{1}{4}[4x^3] - 2[2x] + [0] = x^3 - 4x$$

$$\frac{d^2y}{dx^2} = [3x^2] - 4[1] = 3x^2 - 4$$

critical points

$$0 = \frac{dy}{dx} = x^3 - 4x$$

$$0 = x^3 - 4x$$

$$0 = x(x^2 - 4)$$

$$0 = (x+2)(x)(x-2)$$

inflection points

$$0 = \frac{d^2y}{dx^2} = 3x^2 - 4$$

$$0 = 3x^2 - 4$$

$$0 = (\sqrt{3}x + 2)(\sqrt{3}x - 2)$$

$\sqrt{3}x + 2 = 0$, $x = \frac{-2}{\sqrt{3}}$, $\left(\frac{-2}{\sqrt{3}}, \frac{16}{9}\right)$

$\sqrt{3}x - 2 = 0$, $x = \frac{2}{\sqrt{3}}$, $\left(\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$

$x + 2 = 0$

$x = -2$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 3(-2)^2 - 4 > 0$$

C.U. {Concave Up}

local min $(-2, 0)$

$x = 0$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 3(0)^2 - 4 < 0$$

C.D. {Concave Down}

Local Max

$(0, 4)$

$x - 2 = 0$

$x = 2$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 3(2)^2 - 4 > 0$$

C.U.

see pg 11 for calculation

local min $(2, 0)$

$\frac{dy}{dx}$: dec. | -2 local min | INC | 0 Local Max | dec | 2 local min | INC

$\frac{d^2y}{dx^2}$: C.U. | $\frac{-2}{\sqrt{3}}$ | C.D. | $\frac{2}{\sqrt{3}}$ | C.U.

Increasing: $(-2, 0), (2, \infty)$

decreasing: $(-\infty, -2), (0, 2)$

Concave Up: $\left(-\infty, \frac{-2}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \infty\right)$

Concave Down: $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

4) $y = \frac{9}{14}x^{\frac{1}{3}}(x^2-7) = \frac{9}{14}\left(x^{\frac{7}{3}} - 7x^{\frac{1}{3}}\right)$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = \frac{9}{14}\left(\left[\frac{7}{3}x^{\frac{4}{3}}\right] - 7\left[\frac{1}{3}x^{-\frac{2}{3}}\right]\right) = \frac{9}{14}\left(\frac{7}{3}x^{\frac{4}{3}} - \frac{7}{3}x^{-2/3}\right) = \frac{3}{2}\left(x^{4/3} - x^{-2/3}\right)$$

$$= \frac{9}{14}\left(\frac{7}{3}\right)\left(\left(\sqrt[3]{x}\right)^4 - \frac{1}{(\sqrt[3]{x})^2}\right) = \frac{3}{2}\left(\frac{(\sqrt[3]{x})^4}{1}\left(\frac{(\sqrt[3]{x})^2}{(\sqrt[3]{x})^2}\right) - \frac{1}{(\sqrt[3]{x})^2}\right) = \frac{3}{2}\left(\frac{(\sqrt[3]{x})^6 - 1}{(\sqrt[3]{x})^2}\right)$$

$$= \frac{3}{2}\left(\frac{x^2-1}{(\sqrt[3]{x})^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}\left(\left[\frac{4}{3}x^{\frac{1}{3}}\right] - \left[\frac{-2}{3}x^{-\frac{5}{3}}\right]\right) = \frac{3}{2}\left(\frac{4(\sqrt[3]{x})}{3} + \frac{2}{3(\sqrt[3]{x})^5}\right)$$

$$= \frac{3}{2}\left(\frac{4(\sqrt[3]{x})}{3}\left(\frac{(\sqrt[3]{x})^5}{(\sqrt[3]{x})^5}\right) + \frac{2}{3(\sqrt[3]{x})^5}\right) = \frac{3}{2}\left(\frac{4(\sqrt[3]{x})^6 + 2}{3(\sqrt[3]{x})^5}\right) = \frac{2x^2+1}{(\sqrt[3]{x})^5}$$

critical points

$0 = \frac{dy}{dx} = \frac{3}{2}\left(\frac{x^2-1}{(\sqrt[3]{x})^2}\right)$

$0 = \frac{3}{2}\left(\frac{x^2-1}{(\sqrt[3]{x})^2}\right)$

$0 = x^2 - 1$

$0 = (x+1)(x-1)$

inflection points

$0 = \frac{d^2y}{dx^2} = \frac{2x^2+1}{(\sqrt[3]{x})^5}$

$0 = \frac{2x^2+1}{(\sqrt[3]{x})^5}$

$0 = 2x^2+1$

no solution

denominator $= 0$

$(\sqrt[3]{x})^5 = 0$

$\sqrt[3]{x} = 0$

$x = 0$ $(0,0)$

see pg 12 for calculation

$x + 1 = 0$

$x = -1$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{2(-1)^2+1}{(\sqrt[3]{(-1)})^5} < 0$

C.D.

Local Max $\left(-1, \frac{27}{7}\right)$

$x - 1 = 0$

$x = 1$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2(1)^2+1}{(\sqrt[3]{(1)})^5} > 0$

C.U.

local min $\left(1, -\frac{27}{7}\right)$

$\frac{dy}{dx}$: INC | -1 | dec | 0 | dec | 1 | INC

$\frac{d^2y}{dx^2}$: C.D. | 0 | C.U.

inflection point: $(0,0)$

Increasing: $(-\infty, -1), (1, \infty)$

decreasing: $(-1, 0) \cup (0, 1)$

Concave Up: $(0, \infty)$

Concave Down: $(-\infty, 0)$

6) $y = \tan x - 4x$ $\qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\frac{dy}{dx} = [\sec^2 x(1)] - 4[1] = \sec^2 x - 4$$

$$\frac{d^2y}{dx^2} = [2\sec x(\sec x \tan x(1))] - [0] = 2\sec^2 x \tan x$$

critical points

$$0 = \frac{dy}{dx} = \sec^2 x - 4$$

$$0 = \sec^2 x - 4$$

$$0 = (\sec x + 2)(\sec x - 2)$$

inflection points

$$0 = \frac{d^2y}{dx^2} = 2\sec^2 x \tan x$$

$$0 = 2\sec^2 x \tan x$$

$2\sec^2 x = 0$ discard | $\tan x = 0$, $x = 0$ $\quad (0, 0)$

$\sec x + 2 = 0$

$\sec x = -2$

$\frac{1}{\cos x} = -2$

$\Downarrow$

$\cos x = \frac{-1}{2}$

discard

not in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\sec x - 2 = 0$

$\sec x = 2$

$\frac{1}{\cos x} = 2$

$\Downarrow$

$\cos x = \frac{1}{2}$

QI: $x = \frac{\pi}{3}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{3}} = 2\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 2(2)^2(\sqrt{3}) > 0$$

C.U. local min $\left(\frac{\pi}{3}, \sqrt{3} - \frac{4\pi}{3}\right)$

QIV: $x = \frac{-\pi}{3}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{-\pi}{3}} = 2\sec^2\left(\frac{-\pi}{3}\right)\tan\left(\frac{-\pi}{3}\right) = 2(2)^2(-\sqrt{3}) < 0$$

see pg 12 for calculation

C.D. Local Max $\left(\frac{-\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$

$\frac{dy}{dx}$ $\quad \frac{-\pi}{2}$ (INC. | Local Max at $\frac{-\pi}{3}$ | dec. | local min at $\frac{\pi}{3}$ | INC.) $\frac{\pi}{2}$

$\frac{d^2y}{dx^2}$ $\quad \frac{-\pi}{2}$ (C.D. | 0 | C.U.) $\frac{\pi}{2}$

Increasing: $\left(\frac{-\pi}{2}, \frac{-\pi}{3}\right), \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ $\qquad$ Concave Up: $\left(0, \frac{\pi}{2}\right)$

decreasing: $\left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ $\qquad$ Concave Down: $\left(\frac{-\pi}{2}, 0\right)$

8) $y = 2\cos x - \sqrt{2}\,x \qquad -\pi \le x \le \frac{3\pi}{2}$

$\frac{dy}{dx} = 2[-\sin x(1)] - \sqrt{2}[1] = -2\sin x - \sqrt{2}$

$\frac{d^2y}{dx^2} = -2[\cos x(1)] = -2\cos x$

critical point

$0 = \frac{dy}{dx} = -2\sin x - \sqrt{2}$

$0 = -2\sin x - \sqrt{2}$

$2\sin x = -\sqrt{2}$

$\sin x = \frac{-\sqrt{2}}{2} = \frac{-1}{\sqrt{2}}$

$x = \frac{-3\pi}{4}$

inflection point

$0 = \frac{d^2y}{dx^2} = -2\cos x \Rightarrow 0 = -2\cos x \Rightarrow \cos x = 0$

$x = \frac{-\pi}{2}$: $y\big|_{x=\frac{-\pi}{2}} = 2\cos\left(\frac{-\pi}{2}\right) - \sqrt{2}\left(\frac{-\pi}{2}\right) = 2(0) + \pi\sqrt{2} = \pi\sqrt{2}$
$\left(\frac{-\pi}{2}, \pi\sqrt{2}\right)$

$x = \frac{\pi}{2}$: $y\big|_{x=\frac{\pi}{2}} = 2\cos\left(\frac{\pi}{2}\right) - \sqrt{2}\left(\frac{\pi}{2}\right) = 2(0) - \pi\sqrt{2} = -\pi\sqrt{2}$ $\left(\frac{\pi}{2}, -\pi\sqrt{2}\right)$

$x = \frac{3\pi}{2}$: since $\frac{3\pi}{2}$ is an endpoint we will not get a concavity change here.

$\frac{d^2y}{dx^2}\Big|_{x=\frac{-3\pi}{4}} = -2\cos\left(\frac{-3\pi}{4}\right) = -2\left(\frac{-1}{\sqrt{2}}\right) > 0$ C.U. local min

$y\big|_{x=\frac{-3\pi}{4}} = 2\cos\left(\frac{-3\pi}{4}\right) - \sqrt{2}\left(\frac{-3\pi}{4}\right) = 2\left(\frac{-1}{\sqrt{2}}\right) + \frac{3\pi\sqrt{2}}{4} = -\sqrt{2} + \frac{3\pi\sqrt{2}}{4} = \frac{-4\sqrt{2}}{4} + \frac{3\pi\sqrt{2}}{4}$
$= \frac{3\pi\sqrt{2} - 4\sqrt{2}}{4} \qquad \left(\frac{-3\pi}{4}, \frac{3\pi\sqrt{2} - 4\sqrt{2}}{4}\right)$

$x = \frac{-\pi}{4}$

$\frac{d^2y}{dx^2}\Big|_{x=\frac{-\pi}{4}} = -2\cos\left(\frac{-\pi}{4}\right) = -2\left(\frac{1}{\sqrt{2}}\right) < 0$ C.D. Local Max $\left(\frac{-\pi}{4}, \frac{4\sqrt{2} + \pi\sqrt{2}}{4}\right)$

$y\big|_{x=\frac{-\pi}{4}} = 2\cos\left(\frac{-\pi}{4}\right) - \sqrt{2}\left(\frac{-\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi\sqrt{2}}{4} = \sqrt{2} + \frac{\pi\sqrt{2}}{4} = \frac{4\sqrt{2}}{4} + \frac{\pi\sqrt{2}}{4} = \frac{4\sqrt{2} + \pi\sqrt{2}}{4}$

$x = \frac{5\pi}{4}$

$\frac{d^2y}{dx^2}\Big|_{x=\frac{5\pi}{4}} = -2\cos\left(\frac{5\pi}{4}\right) = -2\left(\frac{-1}{\sqrt{2}}\right) > 0$ C.U. local min $\left(\frac{5\pi}{4}, \frac{-4\sqrt{2} - 5\pi\sqrt{2}}{4}\right)$

$y\big|_{x=\frac{5\pi}{4}} = 2\cos\left(\frac{5\pi}{4}\right) - \sqrt{2}\left(\frac{5\pi}{4}\right) = 2\left(\frac{-1}{\sqrt{2}}\right) - \frac{5\pi\sqrt{2}}{4} = -\sqrt{2} - \frac{5\pi\sqrt{2}}{4} = \frac{-4\sqrt{2} - 5\pi\sqrt{2}}{4}$

8) continued

$\frac{dy}{dx}$: $-\pi$ [dec. | local min $-\frac{3\pi}{4}$ | INC. | Local Max $-\frac{\pi}{4}$ | dec. | local min $\frac{5\pi}{4}$ | INC] $\frac{3\pi}{2}$

$\frac{d^2y}{dx^2}$: $-\pi$ [C.U. | $-\frac{\pi}{2}$ | C.D. | $\frac{\pi}{2}$ | C.U.] $\frac{3\pi}{2}$

Since this function is evaluated on a closed interval we must check the endpoints for extrema.

$y|_{x=-\pi} = 2\cos(-\pi) - \sqrt{2}(-\pi) = 2(-1) + \pi\sqrt{2} = \pi\sqrt{2} - 2$ $\quad(-\pi, \pi\sqrt{2}-2)$ Local Max

$y|_{x=\frac{3\pi}{2}} = 2\cos\left(\frac{3\pi}{2}\right) - \sqrt{2}\left(\frac{3\pi}{2}\right) = 2(0) - 3\pi\sqrt{2} = -3\pi\sqrt{2}$ $\quad\left(\frac{3\pi}{2}, -3\pi\sqrt{2}\right)$ local min

Increasing: $\left(\frac{-3\pi}{4}, \frac{-\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ $\quad$ Concave Up: $\left(-\pi, \frac{-\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

decreasing: $\left(-\pi, \frac{-3\pi}{4}\right), \left(\frac{-\pi}{4}, \frac{5\pi}{4}\right)$ $\quad$ Concave Down: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

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2) continued

$y|_{x=-2} = \frac{1}{4}(-2)^4 - 2(-2)^2 + 4 = 4 - 8 + 4 = 0$ $\quad(-2, 0)$

$y|_{x=0} = \frac{1}{4}(0)^4 - 2(0)^2 + 4 = 4$ $\quad(0, 4)$

$y|_{x=2} = \frac{1}{4}(2)^4 - 2(2)^2 + 4 = 4 - 8 + 4 = 0$ $\quad(2, 0)$

$y|_{x=\frac{-2}{\sqrt{3}}} = \frac{1}{4}\left(\frac{-2}{\sqrt{3}}\right)^4 - 2\left(\frac{-2}{\sqrt{3}}\right)^2 + 4 = \frac{4}{9} - \frac{8}{3} + 4 = \frac{4}{9} - \frac{24}{9} + \frac{36}{9} = \frac{16}{9}$ $\quad\left(\frac{-2}{\sqrt{3}}, \frac{16}{9}\right)$

$y|_{x=\frac{2}{\sqrt{3}}} = \frac{1}{4}\left(\frac{2}{\sqrt{3}}\right)^4 - 2\left(\frac{2}{\sqrt{3}}\right)^2 + 4 = \frac{4}{9} - \frac{8}{3} + 4 = \frac{4}{9} - \frac{24}{9} + \frac{36}{9} = \frac{16}{9}$ $\quad\left(\frac{2}{\sqrt{3}}, \frac{16}{9}\right)$

4) continued

$$y = \frac{9}{14} x^{\frac{1}{3}} (x^2 - 7) = \frac{9}{14}(\sqrt[3]{x})(x^2-7)$$

$$y\Big|_{x=0} = \frac{9}{14}(\sqrt[3]{(0)})((0)^2-7) = 0 \quad (0,0)$$

$$y\Big|_{x=-1} = \frac{9}{14}(\sqrt[3]{(-1)})((-1)^2-7) = \frac{9}{14}(-1)(-6) = \frac{27}{7} \quad \left(-1, \frac{27}{7}\right)$$

$$y\Big|_{x=1} = \frac{9}{14}(\sqrt[3]{(1)})((1)^2-7) = \frac{9}{14}(1)(-6) = \frac{-27}{7} \quad \left(1, \frac{-27}{7}\right)$$

---

6) continued

$$y\Big|_0 = \tan(0) - 4(0) = (0) - 0 = 0 \quad (0,0)$$

$$y\Big|_{x=\frac{\pi}{3}} = \tan\left(\frac{\pi}{3}\right) - 4\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{1}\right) - \frac{4\pi}{3} = \sqrt{3} - \frac{4\pi}{3} \quad \left(\frac{\pi}{3}, \sqrt{3} - \frac{4\pi}{3}\right)$$

$$y\Big|_{x=-\frac{\pi}{3}} = \tan\left(\frac{-\pi}{3}\right) - 4\left(\frac{-\pi}{3}\right) = \left(\frac{-\sqrt{3}}{1}\right) + \frac{4\pi}{3} = \frac{4\pi}{3} - \sqrt{3} \quad \left(\frac{-\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$$

---

10) $y = 6 - 2x - x^2$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = [0] - 2[1] - [2x] = -2 - 2x$$

$$\frac{d^2y}{dx^2} = [0] - 2[1] = -2$$

		Local Max	
$\frac{dy}{dx}$	INC	-1	dec
$\frac{d^2y}{dx^2}$		C.D.	

critical points

$0 = \frac{dy}{dx} = -2 - 2x$

$0 = -2 - 2x$

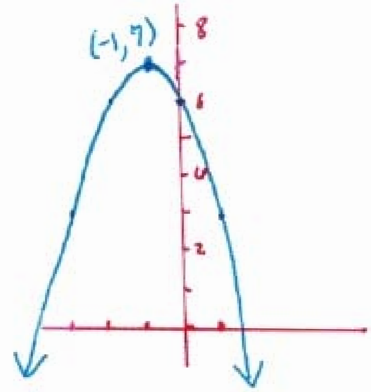
$2x = -2$

$x = -1$

inflection point

$0 = \frac{d^2y}{dx^2} \neq -2$

none



$$\frac{d^2y}{dx^2}\Big|_{x=-1} = -2 < 0 \quad \text{C.D. Local Max} \quad y\Big|_{x=-1} = 6 - 2(-1) - (-1)^2 = 6 + 2 - 1 = 7 \quad (-1, 7)$$

12) $y = x(6-2x)^2$ domain: $(-\infty, \infty)$

$y = x(36 - 24x + 4x^2) = 36x - 24x^2 + 4x^3$

$\frac{dy}{dx} = 36[1] - 24[2x] + 4[3x^2] = 36 - 48x + 12x^2 = 12(3 - 4x + x^2)$

$\frac{d^2y}{dx^2} = [0] - 48[1] + 12[2x] = -48 + 24x = 24(x-2)$

inflection point

$0 = \frac{d^2y}{dx^2} = 24(x-2)$

$0 = 24(x-2)$

$0 = x-2$

$x = 2$

$y|_{x=2} = (2)(6-2(2))^2 = 2(6-4)^2 = 2(2)^2$

$= 8$ $(2, 8)$

critical points

$0 = \frac{dy}{dx} = 12(3-4x+x^2)$

$0 = 12(3-4x+x^2)$

$0 = 3-4x+x^2$

$0 = (1-x)(3-x)$

$1-x=0 \Rightarrow x=1$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 24((1)-2) = 24(-1) < 0$ C.D.

Local Max $(1, 16)$

$y|_{x=1} = (1)(6-2(1))^2 = (1)(4)^2 = 16$

$3-x=0 \Rightarrow x=3$

$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 24((3)-2) = 24(1) > 0$ C.U.

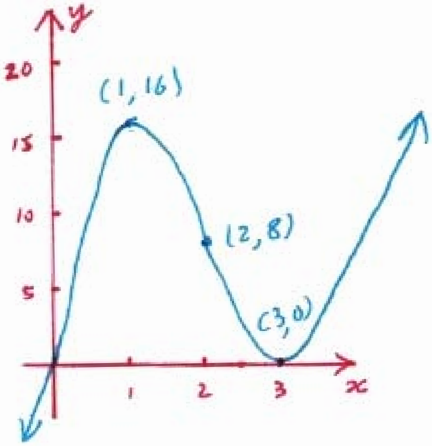
local min $(3, 0)$

$y|_{x=3} = (3)(6-2(3))^2 = (3)(0)^2 = 0$

12) continued

$\frac{dy}{dx}$: INC — Local Max at 1 — dec — local min at 3 — INC

$\frac{d^2y}{dx^2}$: C.D — 2 — C.U.



14) $y = 1 - 9x - 6x^2 - x^3$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = [0] - 9[1] - 6[2x] - [3x^2] = -9 - 12x - 3x^2 = -3(x^2 + 4x + 3)$$

$$\frac{d^2y}{dx^2} = [0] - 12[1] - 3[2x] = -12 - 6x = -6(x+2)$$

inflection point

$$0 = \frac{d^2y}{dx^2} = -6(x+2) \qquad y\big|_{x=-2} = 1 - 9(-2) - 6(-2)^2 - (-2)^3$$

$$0 = -6(x+2) \qquad = 1 + 18 - 24 + 8 = 3$$

$$0 = x+2 \Rightarrow x = -2 \qquad (-2, 3)$$

critical points

$$0 = \frac{dy}{dx} = -3(x^2+4x+3)$$

$$0 = -3(x^2 + 4x + 3)$$

$$0 = x^2 + 4x + 3$$

$$0 = (x+3)(x+1)$$

$x + 3 = 0 \Rightarrow x = -3$

$$\frac{d^2y}{dx^2}\bigg|_{x=-3} = -6((-3)+2) = -6(-1) > 0 \quad \text{C.U.}$$

$(-3, 1)$

local min

$$y\big|_{x=-3} = 1 - 9(-3) - 6(-3)^2 - (-3)^3 = 1 + 27 - 54 + 27 = 1$$

$x + 1 = 0 \Rightarrow x = -1$

$$\frac{d^2y}{dx^2}\bigg|_{x=-1} = -6((-1)+2) = -6(1) < 0 \quad \text{C.D.}$$

$(-1, 5)$

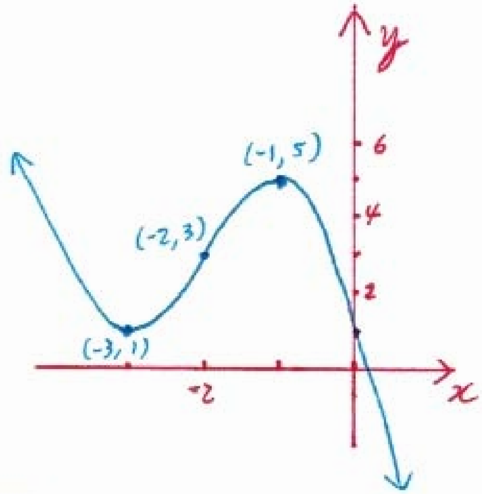
Local Max

$$y\big|_{x=-1} = 1 - 9(-1) - 6(-1)^2 - (-1)^3 = 1 + 9 - 6 + 1 = 5$$

14) continued

$\frac{dy}{dx}$: dec — local min at -3 — INC — Local Max at -1 — dec

$\frac{d^2y}{dx^2}$: C.U. — -2 — C.D.



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16) $y = 1 - (x+1)^3$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = [0] - [3(x+1)^2(1)] = -3(x+1)^2 \qquad \frac{d^2y}{dx^2} = -3[2(x+1)'(1)] = -6(x+1)$$

critical point

$0 = \frac{dy}{dx} = -3(x+1)^2$

$0 = -3(x+1)^2$

$0 = (x+1)^2$

$0 = x+1$

$x = -1$

inflection point

$0 = \frac{d^2y}{dx^2} - 6(x+1)$

$0 = -6(x+1)$

$0 = x+1$

$x = -1$

$\boxed{x=-2}$ "full test" $\boxed{x=0}$

$\frac{dy}{dx}$: dec. — -1 — dec.

$\frac{d^2y}{dx^2}$: C.U. — -1 — C.D.

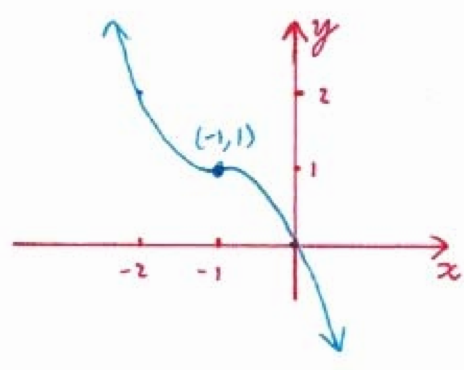
$x = -1$ is an inflection point with slope 0

$$y\Big|_{x=-1} = 1 - ((-1)+1)^3 = 1 - (0)^3 = 1 \quad (-1, 1)$$

at $x = -2$:

$$\frac{dy}{dx}\Big|_{x=-2} = -3((-2)+1)^2 = -3(-1)^2 < 0 \text{ dec.}$$

$$\frac{d^2y}{dx^2}\Big|_{x=-2} = -6((-2)+1) = -6(-1) > 0 \text{ C.U.}$$



at $x = 0$:

$$\frac{dy}{dx}\Big|_{x=0} = -3((0)+1)^2 = -3(1)^2 < 0 \text{ dec.}$$

$$\frac{d^2y}{dx^2}\Big|_{x=0} = -6((0)+1) = -6(1) < 0 \text{ C.D.}$$

18) $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = -[4x^3] + 6[2x] - [0] = -4x^3 + 12x = -4x(x^2 - 3)$$

$$\frac{d^2y}{dx^2} = -4[3x^2] + 12[1] = -12x^2 + 12 = -12(x^2 - 1)$$

critical point

$$0 = \frac{dy}{dx} = -4x(x^2-3)$$

$$0 = -4x(x^2-3)$$

$$0 = (x+\sqrt{3})(-4x)(x-\sqrt{3})$$

$x+\sqrt{3}=0$	$-4x=0$	$x-\sqrt{3}=0$
$x=-\sqrt{3}$	$x=0$	$x=\sqrt{3}$

inflection point

$$0 = \frac{d^2y}{dx^2} = -12(x^2-1)$$

$$0 = -12(x^2-1)$$

$$0 = -12(x+1)(x-1)$$

$x+1=0$	$x-1=0$
$x=-1$	$x=1$

$\frac{dy}{dx}$: INC. | $-\sqrt{3}$ Local Max | dec. | 0 local min | INC. | $\sqrt{3}$ Local Max | dec.

$\frac{d^2y}{dx^2}$: C.D. | -1 | C.U. | 1 | C.D.

at $x=-\sqrt{3}$: $\left.\frac{d^2y}{dx^2}\right|_{x=-\sqrt{3}} = -12((-\sqrt{3})^2-1) = -12(3-1) < 0$ C.D. Local Max

$\left.y\right|_{x=-\sqrt{3}} = (-\sqrt{3})^2(6-(-\sqrt{3})^2) - 4 = (3)(6-3) - 4 = (3)(3) - 4 = 5$ $(-\sqrt{3}, 5)$

at $x=0$: $\left.\frac{d^2y}{dx^2}\right|_{x=0} = -12((0)^2-1) = -12(-1) > 0$ C.U. local min

$\left.y\right|_{x=0} = (0)^2(6-(0)^2) - 4 = 0 - 4 = -4$ $(0, -4)$

18) continued

at $x=\sqrt{3}$: $\left.\dfrac{d^2y}{dx^2}\right|_{x=\sqrt{3}} = -12\left((\sqrt{3})^2-1\right) = -12(3-1) < 0$ C.D. Local Max

$y|_{x=\sqrt{3}} = (\sqrt{3})^2(6-(\sqrt{3})^2)-4 = (3)(6-3)-4 = (3)(3)-4 = 5$ $(\sqrt{3}, 5)$

$y|_{x=-1} = (-1)^2(6-(-1)^2)-4$

$= (1)(6-1)-4$

$= 5-4 = 1$

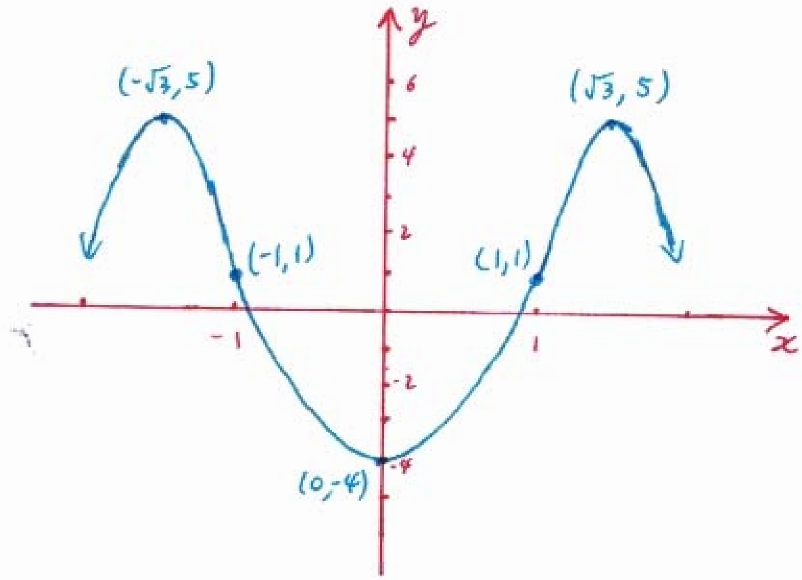
$(-1, 1)$

$y|_{x=1} = (1)^2(6-(1)^2)-4$

$= (1)(6-1)-4$

$= 5-4 = 1$

$(1, 1)$



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20) $y = x^4 + 2x^3 = x^3(x+2)$ domain: $(-\infty, \infty)$

$\dfrac{dy}{dx} = [4x^3] + 2[3x^2] = 4x^3 + 6x^2 = 2x^2(2x+3)$

$\dfrac{d^2y}{dx^2} = 4[3x^2] + 6[2x] = 12x^2 + 12x = 12x(x+1)$

critical points

$0 = \dfrac{dy}{dx} = 2x^2(2x+3)$

$0 = 2x^2(2x+3)$

$2x^2 = 0$ | $2x+3 = 0$

$x^2 = 0$ | $2x = -3$

$x = 0$ | $x = \frac{-3}{2}$

inflection points

$0 = \dfrac{d^2y}{dx^2} = 12x(x+1)$

$0 = 12x(x+1)$

$12x = 0$ | $x+1 = 0$

$x = 0$ | $x = -1$

20) continued

"full test" $\boxed{x=1}$

$\frac{dy}{dx}$: dec. | local min at $\frac{-3}{2}$ | INC. | 0 | INC

$\frac{d^2y}{dx^2}$: C.U. | -1 | C.D. | 0 | C.U.

$y\big|_{x=-1} = (-1)^3((-1)+2) = (-1)(1) = -1 \quad (-1,-1)$

$x = -\frac{3}{2}$:

$\frac{d^2y}{dx^2}\Big|_{x=-\frac{3}{2}} = 12\left(\frac{-3}{2}\right)\left(\left(\frac{-3}{2}\right)+1\right)$

$= 12\left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right) > 0$

C.U.

local min $\left(\frac{-3}{2}, \frac{-27}{16}\right)$

$y\big|_{x=-\frac{3}{2}} = \left(\frac{-3}{2}\right)^3\left(\left(\frac{-3}{2}\right)+2\right)$

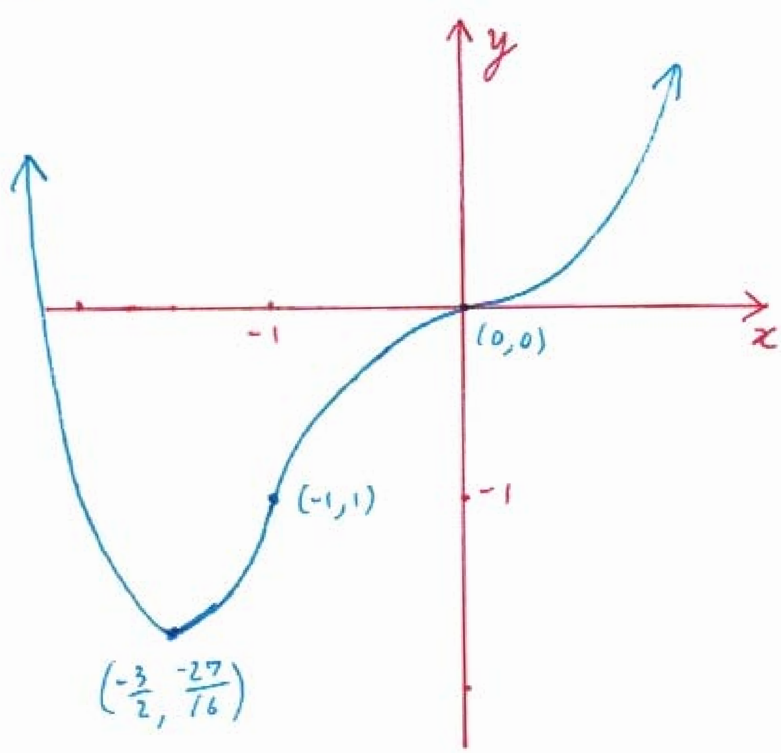
$= \left(\frac{-27}{8}\right)\left(\frac{1}{2}\right) = \frac{-27}{16}$

at $x=1$:

$\frac{dy}{dx}\Big|_{x=1} = 2(1)^2(2(1)+3) > 0$ INC $\quad \frac{d^2y}{dx^2}\Big|_{x=1} = 12(1)((1)+1) > 0$ C.U.

$x=0$ is an inflection point with slope 0.

$y\big|_{x=0} = (0)^3((0)+2) = 0 \quad (0,0)$



22) $y = x\left(\frac{x}{2}-5\right)^4$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = (x)\left[4\left(\frac{x}{2}-5\right)^3\left(\frac{1}{2}\right)\right] + \left(\left(\frac{x}{2}-5\right)^4\right)[1] = \left(\frac{x}{2}-5\right)^3\left\{(x)[2] + \left(\frac{x}{2}-5\right)[1]\right\}$$

$$= \left(\frac{x}{2}-5\right)^3\left\{\frac{5x}{2}-5\right\} = \left(\frac{5x}{2}-5\right)\left(\frac{x}{2}-5\right)^3 = 5\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-5\right)^3$$

$$\frac{d^2y}{dx^2} = \left(\frac{5x}{2}-5\right)\left[3\left(\frac{x}{2}-5\right)^2\left(\frac{1}{2}\right)\right] + \left(\left(\frac{x}{2}-5\right)^3\right)\left[\frac{5}{2}\right]$$

$$= \left(\frac{x}{2}-5\right)^2\left\{\left(\frac{5x}{2}-5\right)\left[\frac{3}{2}\right] + \left(\frac{x}{2}-5\right)\left[\frac{5}{2}\right]\right\} = \left(\frac{x}{2}-5\right)^2\left\{\frac{15x}{4}-\frac{15}{2}+\frac{5x}{4}-\frac{25}{2}\right\}$$

$$= \left(\frac{x}{2}-5\right)^2\left\{\frac{20x}{4}-\frac{40}{2}\right\} = \{5x-20\}\left(\frac{x}{2}-5\right)^2 = 5(x-4)\left(\frac{x}{2}-5\right)^2$$

critical points

$$0 = \frac{dy}{dx} = 5\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-5\right)^3$$

$$0 = 5\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-5\right)^3$$

$\frac{x}{2}-1=0$	$\left(\frac{x}{2}-5\right)^3=0$
$\frac{x}{2}=1$	$\left(\frac{x}{2}-5\right)=0$
$x=2$	$\frac{x}{2}=5$
	$x=10$

inflection point

$$0 = \frac{d^2y}{dx^2} = 5(x-4)\left(\frac{x}{2}-5\right)^2$$

$$0 = 5(x-4)\left(\frac{x}{2}-5\right)^2$$

$x-4=0$	$\left(\frac{x}{2}-5\right)^2=0$
$x=4$	$\left(\frac{x}{2}-5\right)=0$
	$\frac{x}{2}=5$
	$x=10$

$\frac{dy}{dx}$: INC. — Local Max at 2 — dec. — "full test" local min at 10 — INC $\boxed{x=12}$

$\frac{d^2y}{dx^2}$: C.D. — 4 — C.U. — 10 — C.U.

22) continued

at $x=2$: $\left.\frac{d^2y}{dx^2}\right|_{x=2} = 5((2)-4)\left(\frac{(2)}{2}-5\right)^2 = 5(-2)(-4)^2 < 0$ C.D.

Local Max $\left.y\right|_{x=2} = (2)\left(\frac{(2)}{2}-5\right)^4 = (2)(-4)^4 = 512$ $(2, 512)$

$\left.y\right|_{x=4} = (4)\left(\frac{(4)}{2}-5\right)^4 = (4)(-3)^4 = 324$ $(4, 324)$

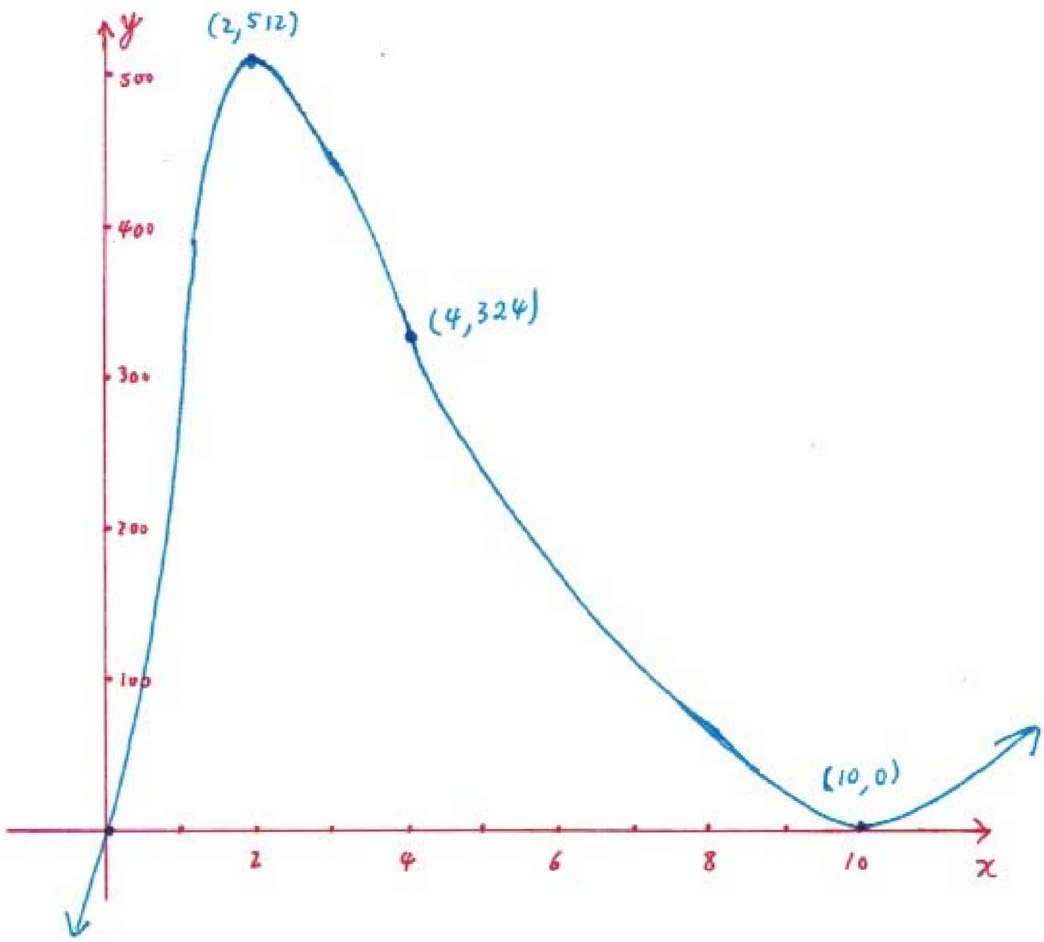
at $x=12$:

$\left.\frac{dy}{dx}\right|_{x=12} = 5\left(\frac{(12)}{2}-1\right)\left(\frac{(12)}{2}-5\right)^3 = 5(5)(1)^3 > 0$ INC.

$\left.\frac{d^2y}{dx^2}\right|_{x=12} = 5((12)-4)\left(\frac{(12)}{2}-5\right)^2 = 5(8)(1)^2 > 0$ C.U.

$x=10$ is a local minimum

$\left.y\right|_{x=10} = (10)\left(\frac{(10)}{2}-5\right)^4 = (10)(0)^4 = 0$ $(0, 0)$



24) $y = x - \sin x \qquad 0 \le x \le 2\pi$

$\frac{dy}{dx} = [1] - [\cos x(1)] = 1 - \cos x \qquad \frac{d^2y}{dx^2} = [0] - [-\sin x(1)] = \sin x$

critical point — inflection point — $\boxed{x = \frac{\pi}{2}}$ "full test"

$0 = \frac{dy}{dx} = 1 - \cos x \qquad 0 = \frac{d^2y}{dx^2} = \sin x$

$\frac{dy}{dx}$ $0[$ ——— INC. ——— $]2\pi$

$0 = 1 - \cos x \qquad 0 = \sin x$

$\frac{d^2y}{dx^2}$ $0[$ — C.U. — π — C.D. — $]2\pi$

$\cos x = 1 \qquad x = 0 \mid x = \pi \mid x = 2\pi$

$x = 0 \mid x = 2\pi$

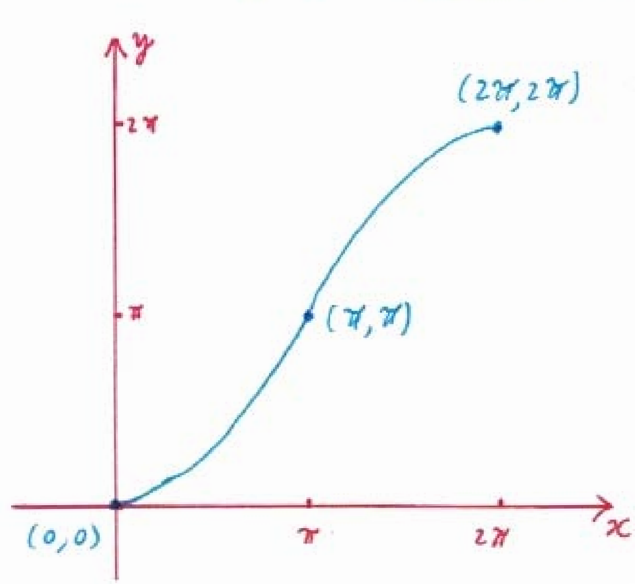
at $x = \frac{\pi}{2}$: $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = 1 - \cos\left(\frac{\pi}{2}\right) = 1 - (0) = 1 > 0$ INC.

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) = 1 > 0$ C.U.

$y|_{x=0} = (0) - \sin(0) = (0) - (0) = 0 \quad (0,0)$ local/abs min

$y|_{x=\pi} = (\pi) - \sin(\pi) = \pi - (0) = \pi \quad (\pi, \pi)$

$y|_{x=2\pi} = (2\pi) - \sin(2\pi) = 2\pi - (0) = \quad (2\pi, 2\pi)$ Local/Abs Max



26) $y = \frac{4}{3}x - \tan x \qquad \frac{-\pi}{2} < x < \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{4}{3}[1] - [\sec^2 x(1)] = \frac{4}{3} - \sec^2 x \qquad \frac{d^2y}{dx^2} = [0] - [2\sec x(\sec x \tan x(1))] = -2\sec^2 x \tan x$$

critical point

$$0 = \frac{dy}{dx} = \frac{4}{3} - \sec^2 x$$

$$0 = \frac{4}{3} - \sec^2 x$$

$$0 = \left(\frac{2}{\sqrt{3}} + \sec x\right)\left(\frac{2}{\sqrt{3}} - \sec x\right)$$

$\frac{2}{\sqrt{3}} + \sec x = 0$ | $\frac{2}{\sqrt{3}} - \sec x$

$\sec x = \frac{-2}{\sqrt{3}}$ | $\sec x = \frac{2}{\sqrt{3}}$

$\Downarrow$ | $\Downarrow$

$\cos x = \frac{-\sqrt{3}}{2}$ | $\cos x = \frac{\sqrt{3}}{2}$

discard not in $(\frac{-\pi}{2}, \frac{\pi}{2})$ | $x = \frac{-\pi}{6}$ | $x = \frac{\pi}{6}$

inflection point

$$0 = \frac{d^2y}{dx^2} = -2\sec^2 x \tan x$$

$$0 = -2\sec^2 x \tan x$$

$-2\sec^2 x = 0$ | $\tan x = 0$

$\sec^2 x = 0$ | $x = 0$

$\sec x = 0$

no solution

$\frac{dy}{dx}$: $\frac{-\pi}{2}$ (—— $\frac{-\pi}{6}$ —— $\frac{\pi}{6}$ ——) $\frac{\pi}{2}$, local min at $\frac{-\pi}{6}$

$\frac{d^2y}{dx^2}$: $\frac{-\pi}{2}$ (—— C.U. —— 0 ——) $\frac{\pi}{2}$

$$y\Big|_{x=0} = \frac{4}{3}(0) - \tan(0) = 0 - 0 = 0$$

$(0,0)$

---

at $x = \frac{-\pi}{6}$: $\frac{d^2y}{dx^2}\Big|_{x=\frac{-\pi}{6}} = -2\sec^2\left(\frac{-\pi}{6}\right)\tan\left(\frac{-\pi}{6}\right) = -2\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{-1}{\sqrt{3}}\right) > 0$

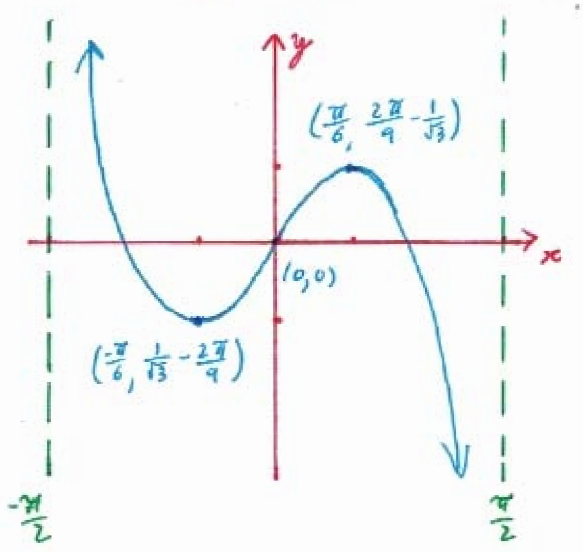
C.U. local min $\left(\frac{-\pi}{6}, \frac{1}{\sqrt{3}} - \frac{2\pi}{9}\right)$

$$y\Big|_{x=\frac{-\pi}{6}} = \frac{4}{3}\left(\frac{-\pi}{6}\right) - \tan\left(\frac{-\pi}{6}\right) = \frac{-2\pi}{9} - \left(\frac{-1}{\sqrt{3}}\right) = \frac{-2\pi}{9} + \frac{1}{\sqrt{3}}$$

at $x = \frac{\pi}{6}$: $\frac{d^2y}{dx^2}\Big|_{x=\frac{\pi}{6}} = -2\sec^2\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right) = -2\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) < 0$

C.D. Local Max $\left(\frac{\pi}{6}, \frac{2\pi}{9} - \frac{1}{\sqrt{3}}\right)$

$$y\Big|_{x=\frac{\pi}{6}} = \frac{4}{3}\left(\frac{\pi}{6}\right) - \tan\left(\frac{\pi}{6}\right) = \frac{2\pi}{9} - \left(\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{9} - \frac{1}{\sqrt{3}}$$



28) $y = \cos x + \sqrt{3} \sin x \qquad 0 \le x \le 2\pi$

$$\frac{dy}{dx} = [-\sin x(1)] + \sqrt{3}\,[\cos x(1)] = -\sin x + \sqrt{3}\cos x$$

$$\frac{d^2y}{dx^2} = -[\cos x(1)] + \sqrt{3}\,[-\sin x(1)] = -\cos x - \sqrt{3}\sin x$$

critical points

$$0 = \frac{dy}{dx} = -\sin x + \sqrt{3}\cos x$$

$$0 = -\sin x + \sqrt{3}\cos x$$

$$\sin x = \sqrt{3}\cos x$$

$$\frac{\sin x}{\cos x} = \sqrt{3}$$

$$\tan x = \sqrt{3} = \frac{\sqrt{3}}{1}$$

$$x = \frac{\pi}{3} \;\Big|\; x = \frac{4\pi}{3}$$

inflection point

$$0 = \frac{d^2y}{dx^2} = -\cos x - \sqrt{3}\sin x$$

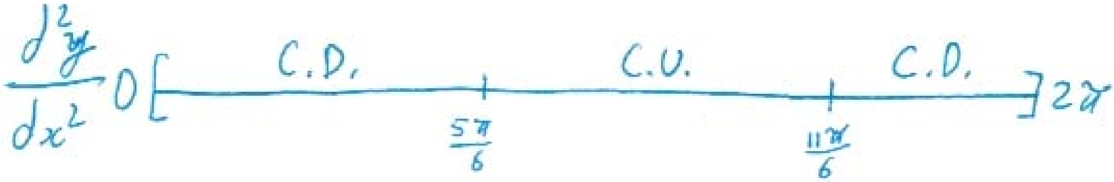
$$0 = -\cos x - \sqrt{3}\sin x$$

$$\sqrt{3}\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = \frac{-1}{\sqrt{3}}$$

$$\tan x = \frac{-1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \;\Big|\; x = \frac{11\pi}{6}$$



at $x = \frac{\pi}{3}$; $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{3}} = -\cos\left(\frac{\pi}{3}\right) - \sqrt{3}\sin\left(\frac{\pi}{3}\right) = -\left(\frac{1}{2}\right) - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) < 0$

C.D. Local Max $\left(\frac{\pi}{3}, 2\right)$

$$y\Big|_{x=\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + \sqrt{3}\sin\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right) + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$$

28) continued

at $x = \frac{4\pi}{3}$: $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{4\pi}{3}} = -\cos\left(\frac{4\pi}{3}\right) - \sqrt{3}\sin\left(\frac{4\pi}{3}\right) = -\left(\frac{-1}{2}\right) - \sqrt{3}\left(\frac{-\sqrt{3}}{2}\right) > 0$

C.U. local min $\left(\frac{4\pi}{3}, -2\right)$

$y|_{x=\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) + \sqrt{3}\sin\left(\frac{4\pi}{3}\right) = \left(\frac{-1}{2}\right) + \sqrt{3}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-1}{2} - \frac{3}{2} = \frac{-4}{2} = -2$

at $x = \frac{5\pi}{6}$: $y|_{x=\frac{5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}\sin\left(\frac{5\pi}{6}\right) = \left(\frac{-\sqrt{3}}{2}\right) + \sqrt{3}\left(\frac{1}{2}\right) = 0 \quad \left(\frac{5\pi}{6}, 0\right)$

at $x = \frac{11\pi}{6}$: $y|_{x=\frac{11\pi}{6}} = \cos\left(\frac{11\pi}{6}\right) + \sqrt{3}\sin\left(\frac{11\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right) + \sqrt{3}\left(\frac{-1}{2}\right) = 0 \quad \left(\frac{11\pi}{6}, 0\right)$

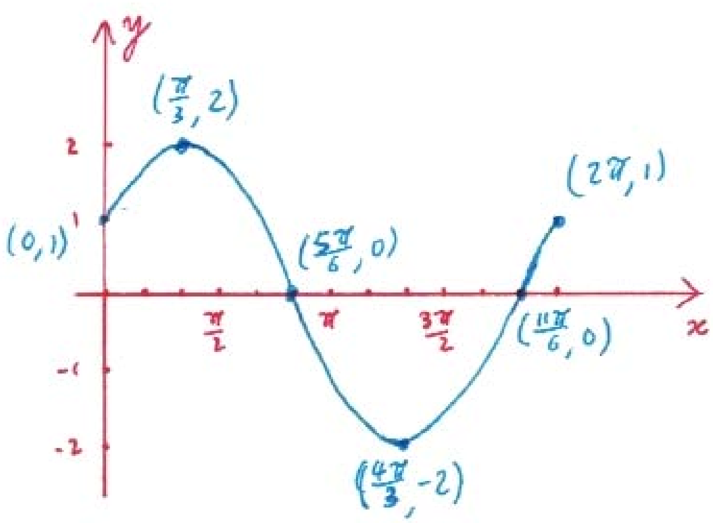
endpoints:

$x = 0$ local min

$y|_{x=0} = \cos(0) + \sqrt{3}\sin(0) = (1) + \sqrt{3}(0) = 1 \quad (0, 1)$

$x = 2\pi$ Local Max

$y|_{x=2\pi} = \cos(2\pi) + \sqrt{3}\sin(2\pi) = (1) + \sqrt{3}(0) = 1 \quad (2\pi, 1)$



86) $y = \frac{x^2-49}{x^2+5x-14} = \frac{(x+7)(x-7)}{(x+7)(x-2)} = \frac{x-7}{x-2}$

V.A.: $x^2+5x-14=0$

$(x+7)(x-2)=0$

$x+7=0$	$x-2=0$
$x=-7$	$x=2$
not V.A. because numerator is 0 when $x=-7$ "missing point"	V.A.

domain: $(-\infty,-7)\cup(-7,2)\cup(2,\infty)$

H.A.: $y = \lim_{x\to\infty} \frac{x^2-49}{x^2+5x-14} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}-\frac{49}{x^2}}{\frac{x^2}{x^2}+\frac{5x}{x^2}-\frac{14}{x^2}}$

$= \lim_{x\to\infty} \frac{1-\frac{49}{x^2}}{1+\frac{5}{x}-\frac{14}{x^2}} = \frac{1-0}{1+0-0} = \frac{1}{1} = 1$

$$\frac{dy}{dx} = \frac{(x-2)[1]-(x-7)[1]}{(x-2)^2} = \frac{x-2-x+7}{(x-2)^2} = \frac{5}{(x-2)^2} = 5(x-2)^{-2}$$

$$\frac{d^2y}{dx^2} = 5\left[-2(x-2)^{-3}(1)\right] = \frac{-10}{(x-2)^3}$$

$\Big\}\ x \neq -7$

"full test"

	$\boxed{x=-8}$		$\boxed{x=0}$		$\boxed{x=3}$
$\frac{dy}{dx}$	INC.	)(-7)(	INC.	) 2 (	INC.
$\frac{d^2y}{dx^2}$	C.U.	)(-7)(	C.U.	) 2 (	C.D.

critical point: $0 = \frac{dy}{dx} = \frac{5}{(x-2)^2}$ no solution

inflection point: $0 = \frac{d^2y}{dx^2} = \frac{-10}{(x-2)^3}$ no solution

at $x=-8$:

$\left.\frac{dy}{dx}\right|_{x=-8} = \frac{5}{((-8)-2)^2} > 0$ INC. $\quad \left.\frac{d^2y}{dx^2}\right|_{x=-8} = \frac{-10}{((-8)-2)^3} > 0$ C.U.

at $x=0$:

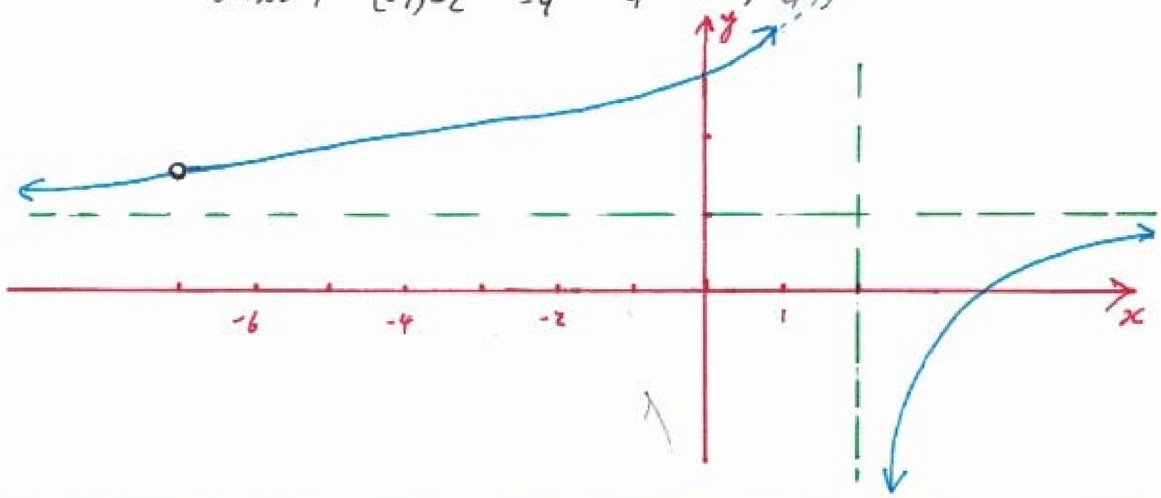
$\left.\frac{dy}{dx}\right|_{x=0} = \frac{5}{((0)-2)^2} > 0$ INC. $\quad \left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{-10}{((0)-2)^3} > 0$ C.U.

86) continued

at $x=3$:

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{5}{((3)-2)^2} > 0 \text{ INC.} \qquad \left.\frac{d^2y}{dx^2}\right|_{x=3} = \frac{-10}{((3)-2)^3} < 0 \quad \text{C.D.}$$

at $x=-7$: $\left.y\right|_{x=-7} = \frac{(-7)-7}{(-7)-2} = \frac{-14}{-9} = \frac{14}{9}$ $\left(-7, \frac{14}{9}\right)$



88) $y = \dfrac{x^2-4}{2x}$ V.A.: $2x=0$, $x=0$ domain: $(-\infty,0)\cup(0,\infty)$

Oblique asymptote:

$$\begin{array}{r} \tfrac{1}{2}x \\ 2x\,\overline{)\,x^2+0x-4} \\ \underline{(x^2)} \\ 0+0x-4 \end{array}$$

$$y = \frac{x^2-4x}{2x} = \frac{1}{2}x + \frac{(-4)}{2x} = \frac{1}{2}x - \frac{2}{x}$$

$$\underline{y = \frac{1}{2}x}$$

$$\frac{dy}{dx} = \frac{(2x)[2x]-(x^2-4)[2]}{(2x)^2} = \frac{4x^2-2x^2+8}{4x^2} = \frac{2x^2+8}{4x^2}$$

$$= \frac{2(x^2+4)}{4x^2} = \frac{x^2+4}{2x^2}$$

88) continued

$$\frac{d^2y}{dx^2} = \frac{(2x^2)[2x] - (x^2+4)[4x]}{(2x^2)^2} = \frac{4x\{(x^2)[1] - (x^2+4)[1]\}}{4x^4}$$

$$= \frac{x^2 - x^2 - 4}{x^3} = \frac{-4}{x^3}$$

"full test"

$\boxed{x=-1}$ $\boxed{x=1}$

$\frac{dy}{dx}$ INC) 0 (

$\frac{d^2y}{dx^2}$ C.U.) 0 (

critical point

$0 = \frac{dy}{dx} = \frac{x^2+4}{2x^2}$ no solution

inflection point

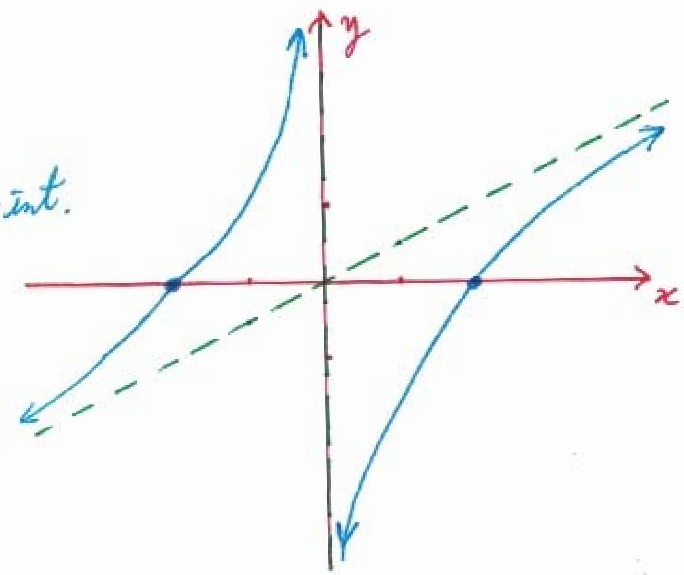
$0 = \frac{d^2y}{dx^2} = \frac{-4}{x^3}$ no solution

at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = \frac{(-1)^2+4}{2(-1)^2} > 0$ INC. $\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{-4}{(-1)^3} > 0$ C.U.

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \frac{(1)^2-4}{2(1)^2} > 0$ INC. $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{-4}{(1)^3} < 0$ C.D.

y-int: none

$$\left.\begin{array}{l}\text{at } x=-2 \\ y|_{x=-2} = \frac{(-2)^2-4}{2(-2)} = \frac{4-4}{-4} = 0 \\ (-2,0) \\ \text{at } x=2 \\ y|_{x=2} = \frac{(2)^2-4}{2(2)} = \frac{4-4}{4} = 0 \\ (2,0)\end{array}\right\} x\text{-int.}$$



90) $y = \dfrac{x^2}{x^2-1}$

V.A.: $x^2-1=0$

$(x+1)(x-1)=0$ domain: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

$x+1=0 \;|\; x-1=0$

$x=-1 \;|\; x=1$

H.A.: $y = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2-1} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{1-\frac{1}{x^2}} = \dfrac{1}{1-0} = \dfrac{1}{1} = 1$

$$\frac{dy}{dx} = \frac{(x^2-1)[2x]-(x^2)[2x]}{(x^2-1)^2} = \frac{2x^3-2x-2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^2-1)^2)[-2]-(-2x)[2(x^2-1)(2x)]}{((x^2-1)^2)^2} = \frac{2(x^2-1)\{(x^2-1)[-1]-(-2x)[2x]\}}{(x^2-1)^4}$$

$$= \frac{2\{-x^2+1+4x^2\}}{(x^2-1)^3} = \frac{2\{3x^2+1\}}{(x^2-1)^3}$$

critical point

$0 = \dfrac{dy}{dx} = \dfrac{-2x}{(x^2-1)^2}$

$0 = \dfrac{-2x}{(x^2-1)^2}$

$0 = -2x$

$x = 0$

inflection point

$0 = \dfrac{d^2y}{dx^2} = \dfrac{2\{3x^2+1\}}{(x^2-1)^3}$

$0 = \dfrac{2\{3x^2+1\}}{(x^2-1)^3}$

$0 = 3x^2+1$

no solution

at $x=0$:

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = \dfrac{2\{3(0)^2+1\}}{((0)^2-1)^3} < 0$ C.D.

Local Max $(0,0)$

$\left.y\right|_{x=0} = \dfrac{(0)^2}{(0)^2+1} = \dfrac{0}{1} = 0$

"full test" $\boxed{x=-2}$ — "full test" $\boxed{x=2}$

$\dfrac{dy}{dx}$: INC $)\,-1\,($ INC — Local Max at 0 — dec $)\,1\,($

$\dfrac{d^2y}{dx^2}$: C.U. $)\,-1\,($ C.D. $)\,1\,($

90) continued

at $x=-2$:

$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{-2(-2)}{((-2)^2-1)^2} > 0$ INC $\quad \left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{2\{3(-2)^2+1\}}{((-2)^2-1)^3} > 0$ C.U.

$\left.y\right|_{x=-2} = \frac{(-2)^2}{(-2)^2-1} = \frac{4}{4-1} = \frac{4}{3} \quad (-2, \frac{4}{3})$

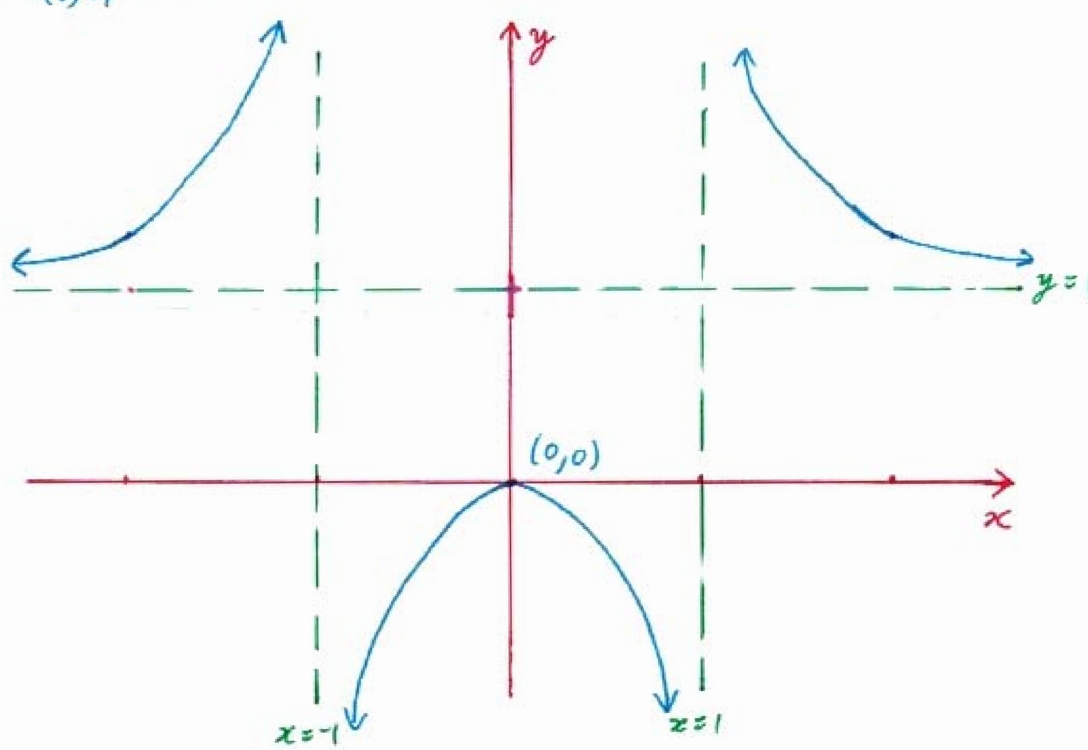
at $x=2$:

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{-2(2)}{((2)^2-1)^2} < 0$ dec. $\quad \left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{2\{3(2)^2+1\}}{((2)^2-1)^3} > 0$ C.U.

$\left.y\right|_{x=2} = \frac{(2)^2}{(2)^2-1} = \frac{4}{4-1} = \frac{4}{3} \quad (2, \frac{4}{3})$

y-int: $y = \frac{(0)^2}{(0)^2-1} = 0$

x-int: $0 = \frac{x^2}{x^2-1} \Rightarrow 0 = x^2 \Rightarrow 0 = x$



92) $y = \dfrac{x^2-4}{x^2-2}$

V.A.: $x^2-2=0$

$(x+\sqrt{2})(x-\sqrt{2})=0$

$x+\sqrt{2}=0 \mid x-\sqrt{2}=0$

$x=-\sqrt{2} \mid x=\sqrt{2}$

domain: $(-\infty,-\sqrt{2})\cup(-\sqrt{2},\sqrt{2})\cup(\sqrt{2},\infty)$

H.A.: $y = \lim_{x\to\infty} \dfrac{x^2-4}{x^2-2} = \lim_{x\to\infty} \dfrac{\frac{x^2}{x^2}-\frac{4}{x^2}}{\frac{x^2}{x^2}-\frac{2}{x^2}} = \lim_{x\to\infty} \dfrac{1-\frac{4}{x^2}}{1-\frac{2}{x^2}} = \dfrac{1-0}{1-0} = \dfrac{1}{1} = 1$

$$\frac{dy}{dx} = \frac{(x^2-2)[2x]-(x^2-4)[2x]}{(x^2-2)^2} = \frac{2x\{(x^2-2)[1]-(x^2-4)[1]\}}{(x^2-2)^2}$$

$$= \frac{2x\{x^2-2-x^2+4\}}{(x^2-2)^2} = \frac{2x\{2\}}{(x^2-2)^2} = \frac{4x}{(x^2-2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^2-2)^2)[4]-(4x)[2(x^2-2)'(2x)]}{((x^2-2)^2)^2} = \frac{4(x^2-2)\{(x^2-2)[1]-(x)[4x]\}}{(x^2-2)^4}$$

$$= \frac{4\{x^2-2-4x^2\}}{(x^2-2)^3} = \frac{4\{-3x^2-2\}}{(x^2-2)^3} = \frac{-4\{3x^2+2\}}{(x^2-2)^3}$$

critical point

$0 = \dfrac{dy}{dx} = \dfrac{4x}{(x^2-2)^2}$

inflection point

$0 = \dfrac{d^2y}{dx^2} = \dfrac{-4\{3x^2+2\}}{(x^2-2)^3}$ none no solution

---

$0 = \dfrac{4x}{(x^2-2)^2}$

$0 = 4x$

$x = 0$

at $x=0$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = \dfrac{-4\{3(0)^2+2\}}{((0)^2-2)^3} > 0$ C.U. local min

$y|_{x=0} = \dfrac{(0)^2-4}{(0)^2-2} = \dfrac{-4}{-2} = 2$ $(0,2)$ "also y-int"

$\dfrac{dy}{dx}$: "full test" $\boxed{x=-2}$ dec. $)-\sqrt{2}($ dec | local min at 0 | inc $)\sqrt{2}($ "full test" $\boxed{x=2}$ INC.

$\dfrac{d^2y}{dx^2}$: C.D. $)-\sqrt{2}($ C.U. $)\sqrt{2}($ C.D.

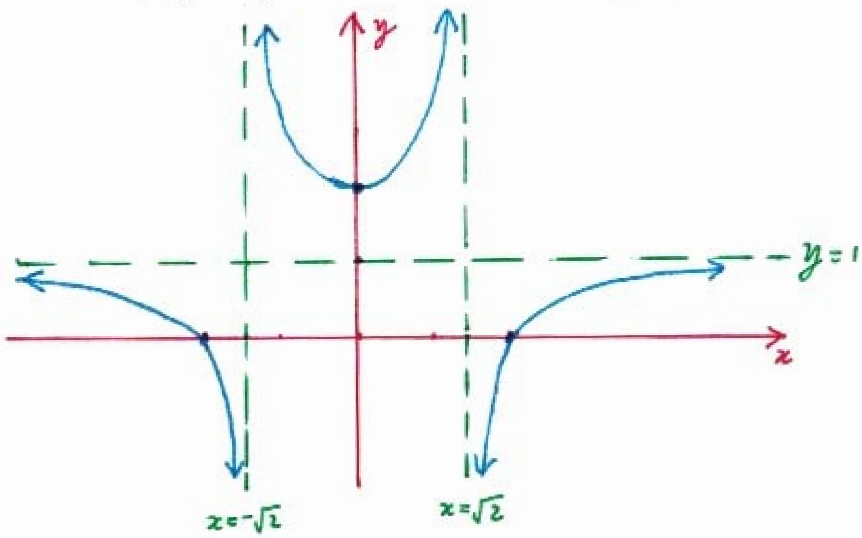
92) continued

at $x=-2$: $\left.\frac{dy}{dx}\right|_{x=-2} = \frac{4(-2)}{((-2)^2-2)^2} < 0$ dec. $\left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{-4\{3(-2)^2+2\}}{((-2)^2-2)^3} < 0$ C.D.

$y|_{x=-2} = \frac{(-2)^2-4}{(-2)^2-2} = \frac{4-4}{4-2} = \frac{0}{2} = 0$ $(-2,0)$ "also x-int."

at $x=2$: $\left.\frac{dy}{dx}\right|_{x=2} = \frac{4(2)}{((2)^2-2)^2} > 0$ INC $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{-4\{3(2)^2+2\}}{((2)^2-2)^3} < 0$ C.D.

$y|_{x=2} = \frac{(2)^2-4}{(2)^2-2} = \frac{4-4}{4-2} = \frac{0}{2} = 0$ $(2,0)$ "also x-int"



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94) $y = -\frac{x^2-4}{x+1} = \frac{-(x^2-4)}{x+1} = \frac{-x^2+4}{x+1}$

V.A.: $x+1=0$
$x=-1$

domain: $(-\infty,-1) \cup (-1,\infty)$

Oblique asymptote

$$\begin{array}{r|l} & -x+1 \\ \hline x+1 & -x^2+0x+4 \\ & \underline{-(-x^2-x)} \\ & +x+4 \\ & \underline{-(x+1)} \\ & +3 \end{array}$$

$y = \frac{-x^2+4}{x+1} = -x+1 + \frac{(+3)}{x+1}$

$\underline{y=-x+1}$

94) continued

$$\frac{dy}{dx} = \frac{(x+1)[-2x] - (-x^2+4)[1]}{(x+1)^2} = \frac{-2x^2 - 2x + x^2 - 4}{(x+1)^2} = \frac{-x^2 - 2x - 4}{(x+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x+1)^2)[-2x-2] - (-x^2-2x-4)[2(x+1)'(1)]}{((x+1)^2)^2}$$

$$= \frac{-2(x+1)\{(x+1)[x+1] - (x^2+2x+4)[1]\}}{(x+1)^4} = \frac{-2\{x^2+2x+1-x^2-2x-4\}}{(x+1)^3}$$

$$= \frac{-2\{-3\}}{(x+1)^3} = \frac{6}{(x+1)^3}$$

critical point

$$0 = \frac{dy}{dx} = \frac{-x^2-2x-4}{(x+1)^2}$$

$$0 = \frac{-x^2-2x-4}{(x+1)^2}$$

$$0 = -x^2-2x-4$$

$$0 = x^2+2x+4$$

no solution

none

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{6}{(x+1)^3}$$

no solution

none

y-int:

$$y = \frac{-(0)^2+4}{(0)+1} = \frac{4}{1} = 4$$

$(0,4)$

x-int:

$$0 = \frac{-x^2+4}{x+1}$$

$$0 = -x^2+4$$

$$x^2-4=0$$

$$(x+2)(x-2)=0$$

$x+2=0$	$x-2=0$
$x=-2$	$x=2$
$(-2,0)$	$(2,0)$

"full test"

$\boxed{x=-2}$ $\boxed{x=0}$

$\frac{dy}{dx}$ dec. $)-1($ dec.

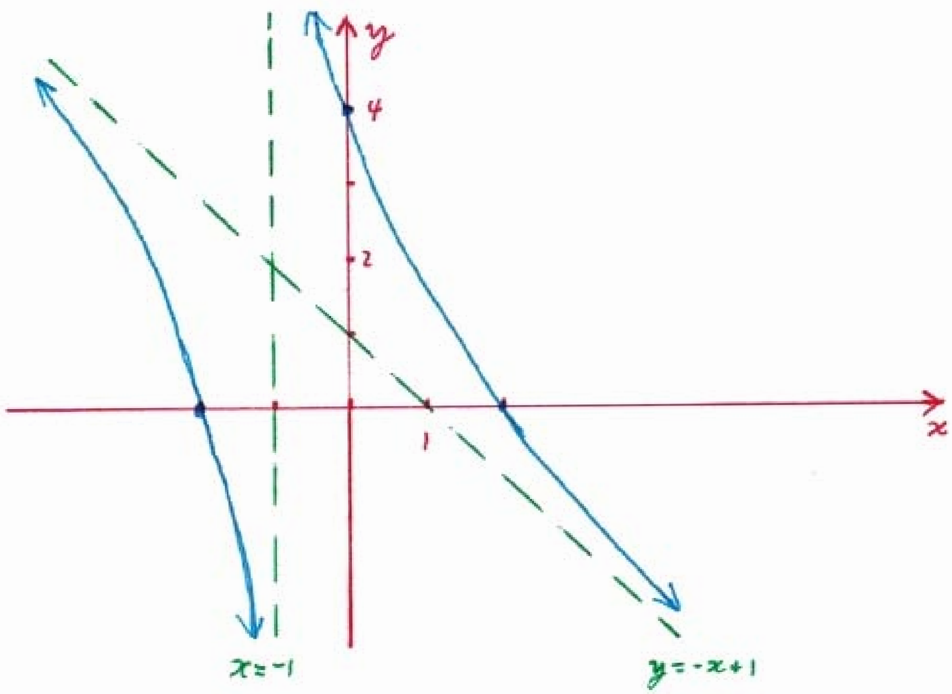
$\frac{d^2y}{dx^2}$ C.D. $)-1($ C.U.

94) continued

at $x=-2$: $\left.\frac{dy}{dx}\right|_{x=-2} = \frac{-(-2)^2-2(-2)-4}{((-2)+1)^2} = \frac{-4+4-4}{(-1)^2} < 0$ dec.

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{6}{((-2)+1)^3} = \frac{6}{(-1)^3} < 0$ C.D.

at $x=2$: $\left.\frac{dy}{dx}\right|_{x=2} = \frac{-(2)^2-2(2)-4}{((2)+1)^2} = \frac{-4-4-4}{(3)^2} < 0$ dec $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{6}{((2)+1)^3} > 0$ C.U.



96) $y = -\frac{x^2-x+1}{x-1} = \frac{-x^2+x-1}{x-1}$ V.A.: $x-1=0$, $x=1$ domain: $(-\infty,1)\cup(1,\infty)$

Oblique asymptote $y = \frac{-x^2+x-1}{x-1} = -x + \frac{(-1)}{x-1}$

$$\begin{array}{r|l} & -x \\ x-1 & -x^2+x-1 \\ & -(-x^2+x) \\ & \quad 0 - 1 \end{array}$$

$\underline{y=-x}$

$\frac{dy}{dx} = \frac{(x-1)[-2x+1]-(-x^2+x-1)[1]}{(x-1)^2} = \frac{-2x^2+3x-1+x^2-x+1}{(x-1)^2} = \frac{-x^2+2x}{(x-1)^2}$

$= \frac{x(2-x)}{(x-1)^2}$

96) continued

$$\frac{d^2y}{dx^2} = \frac{((x-1)^2)[-2x+2] - (-x^2+2x)[2(x-1)'(1)]}{((x-1)^2)^2}$$

$$= \frac{2(x-1)\{(x-1)[-x+1] - (x^2+2x)[1]\}}{(x-1)^4} = \frac{2\{-x^2+2x-1+x^2-2x\}}{(x-1)^3}$$

$$= \frac{2\{-1\}}{(x-1)^3} = \frac{-2}{(x-1)^3}$$

critical points

$$0 = \frac{dy}{dx} = \frac{x(2-x)}{(x-1)^2}$$

$$0 = \frac{x(2-x)}{(x-1)^2}$$

$$0 = x(2-x)$$

$x = 0$ | $2 - x = 0$, $x = 2$

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{-2}{(x-1)^3}$$ none

no solution

x-int:

$$0 = \frac{-x^2+x-1}{x-1} \Rightarrow$$ $x^2 - x + 1 = 0$ no solution

$0 = -x^2 + x - 1$ none

$\frac{dy}{dx}$: dec — local max at 0 — INC) | (INC — local Max at 2 — dec

$\frac{d^2y}{dx^2}$: C.U.) | (C.D

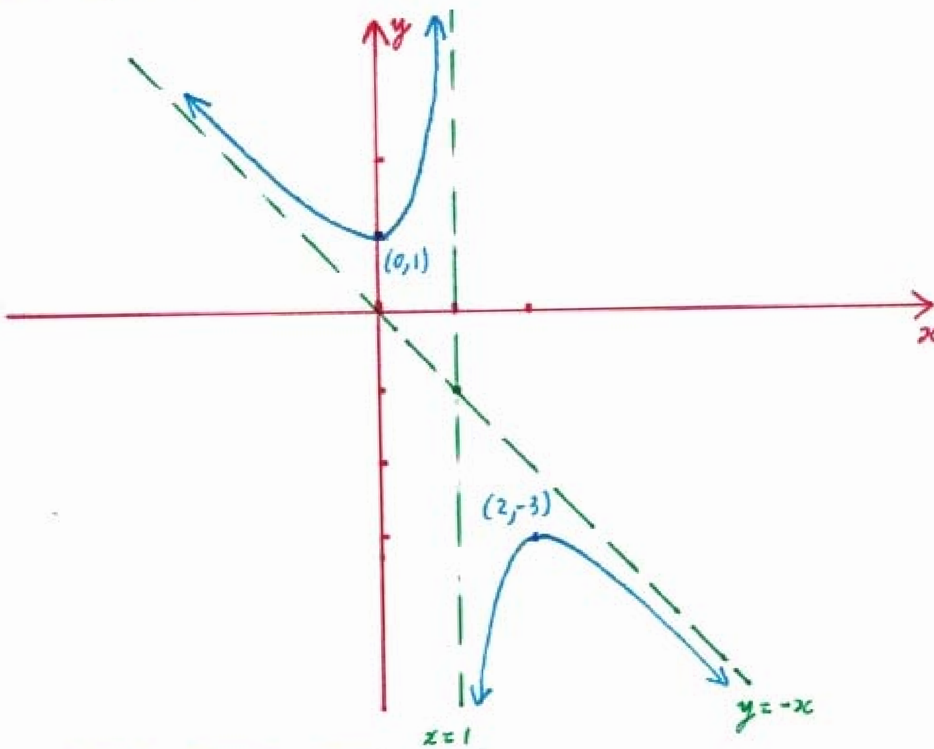
at $x=0$: $\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{-2}{((0)-1)^3} > 0$ C.U. local min

$y|_{x=0} = \frac{-(0)^2+(0)-1}{(0)-1} = \frac{-1}{-1} = 1$ $(0,1)$ "also y-int"

96) continued

at $x=2$: $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{-2}{((2)-1)^3} < 0$ C.D. Local Max

$\left. y \right|_{x=2} = \frac{-(2)^2+(2)-1}{(2)-1} = \frac{-4+2-1}{1} = \frac{-3}{1} = -3$ $(2,-3)$



98) $y = \frac{x^3+x-2}{x-x^2}$

when $x=1$ $x^3+x-2=0$

so $(x-1)$ is a factor of x^3+x-2

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{)\, x^3+0x^2+x-2} \\ -(x^3-x^2) \\ \hline +x^2+x \\ -(x^2-x) \\ \hline +2x-2 \\ -(2x-2) \\ \hline 0 \end{array}$$

$y = \frac{(x-1)(x^2+x+2)}{x(1-x)} = \frac{(x-1)(x^2+x+2)}{-x(x-1)}$

$y = \frac{x^2+x+2}{-x} = \frac{-x^2-x-2}{x}$ $x \neq 1$

$= \frac{-x^2}{x} - \frac{x}{x} - \frac{2}{x} = -x-1-\frac{2}{x} = -x-1-2x^{-1}$

V.A.: $x-x^2=0$

$x(1-x)=0$

$x=0$ | $1-x=0$

$x=1$

missing point

domain: $(-\infty,0)\cup(0,1)\cup(1,\infty)$

$\left. y \right|_{x=1} = \frac{-(1)^2-(1)-2}{(1)} = \frac{-1-1-2}{1} = \frac{-4}{1} = -4$

$(1,-4)$

98) continued

$$\left.\begin{aligned}\frac{dy}{dx} &= -[1]-[0]-2[-1x^{-2}] = -1+2x^{-2} = -1+\frac{2}{x^2} = \frac{2}{x^2}-1 \\ \frac{d^2y}{dx^2} &= [0]+2[-2x^{-3}] = -4x^{-3} = \frac{-4}{x^3}\end{aligned}\right\} x \neq 1$$

critical point

$0 = \frac{dy}{dx} = \frac{2}{x^2} - 1$

$0 = \frac{2}{x^2} - 1$

$1 = \frac{2}{x^2}$

$x^2 = 2$

$x^2 - 2 = 0$

$(x+\sqrt{2})(x-\sqrt{2}) = 0$

$x+\sqrt{2} = 0 \mid x-\sqrt{2} = 0$

$x = -\sqrt{2} \mid x = \sqrt{2}$

inflection point

$0 = \frac{d^2y}{dx^2} = \frac{-4}{x^3}$

no solution

none

Oblique asymptote

$$\begin{array}{r|l} & -x-1 \\ -x^2+x & x^3+0x^2+x-2 \\ & -(x^3-x^2) \\ & \quad +x^2+x \\ & \quad -(x^2-x) \\ & \qquad +2x-2 \end{array}$$

$y = \frac{x^3+x-2}{x-x^2} = -x-1+\frac{(+2x-2)}{x-x^2}$

$\underline{y = -x-1}$

"full test"

$\boxed{x = \frac{1}{2}}$

$\frac{dy}{dx}$ INC, Local Max $-\sqrt{2}$, dec) 0 (INC) 1 (dec. local min $\sqrt{2}$ INC

$\frac{d^2y}{dx^2}$ C.U.) 0 (C.D.) 1 (C.D.

98) continued

at $x = -\sqrt{2}$; $\left.\dfrac{d^2y}{dx^2}\right|_{x=-\sqrt{2}} = \dfrac{-4}{(-\sqrt{2})^3} > 0$ C.U. Local Max

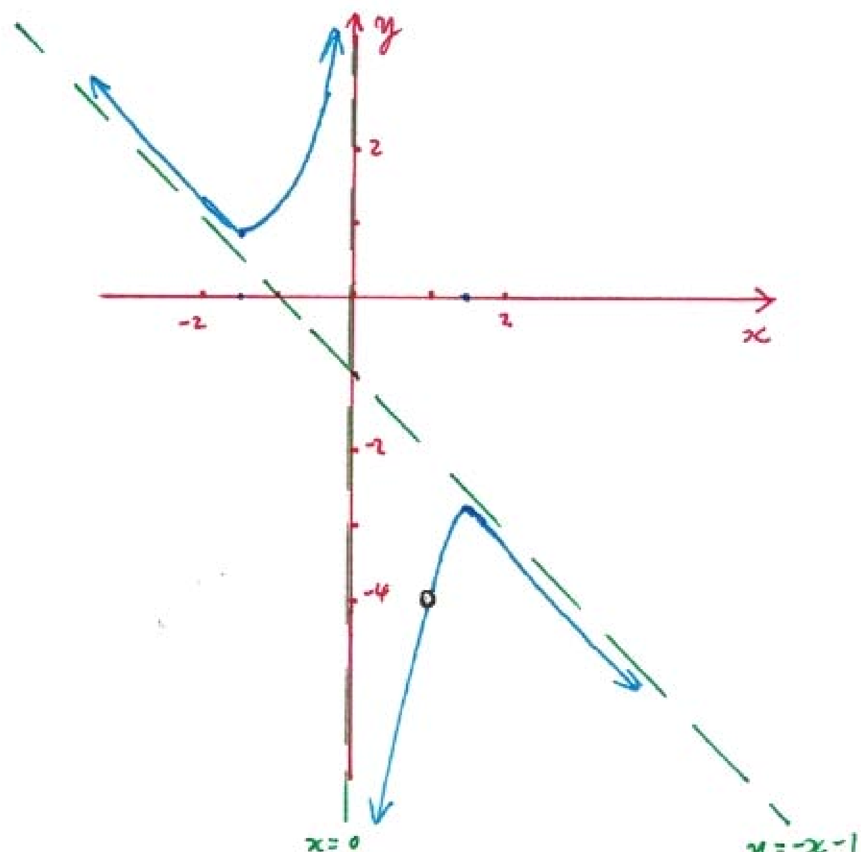
$\left. y \right|_{x=-\sqrt{2}} = \dfrac{-(-\sqrt{2})^2 - (-\sqrt{2}) - 2}{(-\sqrt{2})} = \dfrac{-2 + \sqrt{2} - 2}{-\sqrt{2}} = \dfrac{-4 + \sqrt{2}}{-\sqrt{2}} = 2\sqrt{2} - 1$ $\quad (-\sqrt{2},\ 2\sqrt{2} - 1)$

at $x = \sqrt{2}$; $\left.\dfrac{d^2y}{dx^2}\right|_{x=\sqrt{2}} = \dfrac{-4}{(\sqrt{2})^3} < 0$ C.D. local min

$\left. y \right|_{x=\sqrt{2}} = \dfrac{-(\sqrt{2})^2 - (\sqrt{2}) - 2}{(\sqrt{2})} = \dfrac{-2 - \sqrt{2} - 2}{\sqrt{2}} = \dfrac{-4 - \sqrt{2}}{\sqrt{2}} = -2\sqrt{2} - 1$ $\quad (\sqrt{2},\ -2\sqrt{2} - 1)$

at $x = \frac{1}{2}$: $\left.\dfrac{dy}{dx}\right|_{x=\frac{1}{2}} = \dfrac{2}{(\frac{1}{2})^2} - 1 = \dfrac{2}{(\frac{1}{4})} - 1 = 8 - 1 > 0$ INC.

$\left.\dfrac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} = \dfrac{-4}{(\frac{1}{2})^3} = \dfrac{-4}{(\frac{1}{8})} < 0$ C.D.



100) $y = \dfrac{x-1}{x^2(x-2)} = \dfrac{x-1}{x^3-2x^2}$ VA: $x^2(x-2)=0$

$x^2=0 \quad | \quad x-2=0$

$x=0 \quad | \quad x=2$

domain: $(-\infty,0)\cup(0,2)\cup(2,\infty)$

H.A.: $y = \lim\limits_{x\to\infty} \dfrac{x-1}{x^3-2x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{x}{x^3}-\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{2x^2}{x^3}} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^2}-\frac{1}{x^3}}{1-\frac{2}{x}} = \dfrac{0-0}{1-0} = 0$

$$\frac{dy}{dx} = \frac{(x^3-2x^2)[1]-(x-1)[3x^2-4x]}{(x^3-2x^2)^2} = \frac{(x^3-2x^2)-(3x^3-7x^2+4x)}{(x^3-2x^2)^2}$$

$$= \frac{-2x^3+5x^2-4x}{(x^3-2x^2)^2} = \frac{x(-2x^2+5x-4)}{(x^2(x-2))^2} = \frac{x(-2x^2+5x-4)}{x^4(x-2)^2} = \frac{-2x^2+5x-4}{x^3(x-2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^3-2x^2)^2)[-6x^2+10x-4]-(-2x^3+5x^2-4x)[2(x^3-2x^2)'(3x^2-4x)]}{((x^3-2x^2)^2)^2}$$

$$= \frac{2(x^3-2x^2)\{(x^3-2x^2)[-3x^2+5x-2]-(-2x^3+5x^2-4x)[3x^2-4x]\}}{(x^3-2x^2)^4}$$

$$= \frac{2\{(-3x^5+5x^4-2x^3+6x^4-10x^3+4x^2)-(-6x^5+15x^4-12x^3+8x^4-20x^3+16x^2)\}}{(x^3-2x^2)^3}$$

$$= \frac{2\{(-3x^5+11x^4-12x^3+4x^2)-(-6x^5+23x^4-32x^3+16x^2)\}}{(x^2(x-2))^3}$$

$$= \frac{2\{3x^5-12x^4+20x^3-12x^2\}}{x^6(x-2)^3} = \frac{2x^2\{3x^3-12x^2+20x-12\}}{x^6(x-2)^3}$$

$$= \frac{2\{3x^3-12x^2+20x-12\}}{x^4(x-2)^3}$$

100) continued

critical point

$$0 = \frac{dy}{dx} = \frac{-2x^2+5x-4}{x^3(x-2)^2}$$

$$0 = \frac{-2x^2+5x-4}{x^3(x-2)^2}$$

$$0 = -2x^2+5x-4$$

$$2x^2-5x+4=0$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2-4(2)(4)}}{2(2)} = \frac{5 \pm \sqrt{25-32}}{4}$$

no solution

none

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{2\{3x^3-12x^2+20x-12\}}{x^4(x-2)^3}$$

$$0 = \frac{2\{3x^3-12x^2+20x-12\}}{x^4(x-2)^3}$$

$$0 = 2\{3x^3-12x^2+20x-12\}$$

$$0 = 3x^3-12x^2+20x-12$$

the solution is not a rational number. So see below:

at $x=1$: $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2\{3(1)^3-12(1)^2+20(1)-12\}}{(1)^4((1)-2)^3} = \frac{2\{3-12+20-12\}}{(1)(-1)^3} = \frac{2\{-1\}}{-1} = 2 > 0$ C.U.

at $x=\frac{3}{2}$: $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{3}{2}} = \frac{2\{3(\frac{3}{2})^3-12(\frac{3}{2})^2+20(\frac{3}{2})-12\}}{(\frac{3}{2})^4((\frac{3}{2})-2)^3} = \frac{2\{\frac{81}{8}-27+30-12\}}{(\frac{81}{16})(\frac{-1}{2})^3}$

$$= \frac{2\{\frac{81}{9}-9\}}{(\frac{81}{16})(\frac{-1}{8})} = \frac{2\{\frac{9}{8}\}}{(\frac{81}{16})(\frac{-1}{8})} < 0 \quad \text{C.D.}$$

By Intermediate Value Theorem for Continuous Functions on pg 97, we know that there is a value c in $[1, \frac{3}{2}]$ such that $\left.\frac{d^2y}{dx^2}\right|_{x=c} = 0$.

There is an inflection point.

100) continued

to show the actual real number solution of $0 = 3x^3 - 12x^2 + 20x - 12$,
I used MAPLE software.

there are 2 additional roots that are Complex Number.

$$x = \frac{4}{3} - \frac{\sqrt[3]{2 + 2\sqrt{17}}}{3} + \frac{4}{3\left(\sqrt[3]{2+2\sqrt{17}}\right)} \approx 1.223223466$$

$\boxed{x=-1}$ "fall test" $\boxed{x=1}$ $\boxed{x=3}$

$\frac{dy}{dx}$ INC)0(dec.)2(dec.

$\frac{d^2y}{dx^2}$ C.U)0(C.U. | C.D.)2(C.U.

≈ 1.22

$$\text{at } x=1: \left.\frac{dy}{dx}\right|_{x=1} = \frac{-2(1)^2 + 5(1) - 4}{(1)^3((1)-2)^2} = \frac{-1}{(1)(-1)^2} < 0 \text{ dec.}$$

$$\text{at } x=3: \left.\frac{dy}{dx}\right|_{x=3} = \frac{-2(3)^2 + 5(3) - 4}{(3)^3((3)-2)^2} = \frac{-18+15-4}{(27)(1)^2} < 0 \text{ dec.}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = \frac{2\{3(3)^3 - 12(3)^2 + 20(3) - 12\}}{(3)^4((3)-2)^3} = \frac{2\{81 - 108 + 60 - 12\}}{(81)(1)^3} = \frac{2\{21\}}{81} > 0 \text{ C.U.}$$

$$\text{at } x=-1: \left.\frac{dy}{dx}\right|_{x=-1} = \frac{-2(-1)^2 + 5(-1) - 4}{(-1)^3((-1)-2)^2} = \frac{-2-5-4}{(-1)(-1)^2} > 0 \text{ INC}$$

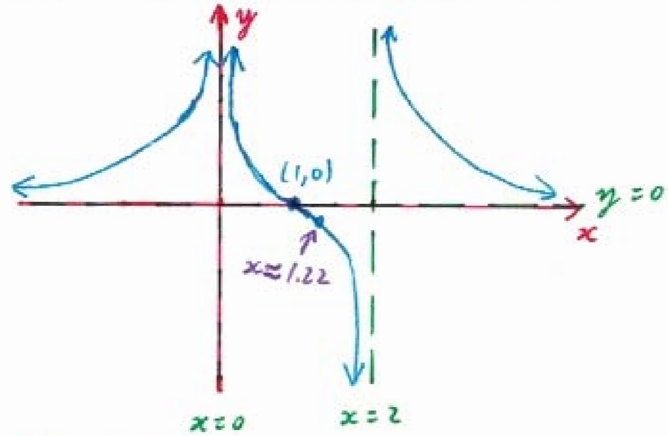
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{2\{3(-1)^3 - 12(-1)^2 + 20(-1) - 12\}}{(-1)^4((-1)-2)^3} = \frac{2\{-3-12-20-12\}}{(1)(-3)^3} > 0 \text{ C.U.}$$

100) continued y-int: none because $x=0$ is V.A.

x-int: $0 = \frac{x-1}{x^2(x-2)}$

(1, 0) $0 = x-1$

$x = 1$



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102) $y = \frac{4x}{x^2+4}$ V.A.: $x^2+4=0$
no solution domain: $(-\infty, \infty)$
none

H.A.: $y = \lim_{x\to\infty} \frac{4x}{x^2+4} = \lim_{x\to\infty} \frac{\frac{4x}{x^2}}{\frac{x^2}{x^2}+\frac{4}{x^2}} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{4}{x^2}} = \frac{0}{1+0} = 0$

y-int: $y = \frac{4(0)}{(0)^2+4} = \frac{0}{4} = 0$ x-int: $0 = \frac{4x}{x^2+4} \Rightarrow 0 = 4x \Rightarrow x = 0$

(0,0) both x-int, y-int

$$\frac{dy}{dx} = \frac{(x^2+4)[4]-(4x)[2x]}{(x^2+4)^2} = \frac{4x^2+16-8x^2}{(x^2+4)^2} = \frac{-4x^2+16}{(x^2+4)^2} = \frac{-4(x^2-4)}{(x^2+4)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^2+4)^2)[-8x]-(-4x^2+16)[2(x^2+4)^1(2x)]}{((x^2+4)^2)^2}$$

$$= \frac{-8x(x^2+4)\{(x^2+4)[1]-(x^2-4)[2]\}}{(x^2+4)^4} = \frac{-8x\{x^2+4-2x^2+8\}}{(x^2+4)^3}$$

$$= \frac{-8x\{-x^2+12\}}{(x^2+4)^3} = \frac{8x\{x^2-12\}}{(x^2+4)^3}$$

102) continued

critical point

$$0 = \frac{dy}{dx} = \frac{-4(x^2-4)}{(x^2+4)^2}$$

$$0 = -4(x^2-4)$$

$$0 = x^2-4$$

$$0 = (x+2)(x-2)$$

$x+2=0$ | $x-2=0$
$x=-2$ | $x=2$

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{8x\{x^2-12\}}{(x^2+4)^3}$$

$$0 = \frac{8x\{x^2-12\}}{(x^2+4)^3}$$

$$0 = 8x\{x^2-12\}$$

$$0 = (x+\sqrt{12})(8x)(x-\sqrt{12})$$

$x+\sqrt{12}=0$ | $8x=0$ | $x-\sqrt{12}=0$
$x=-\sqrt{12}$ | $x=0$ | $x=\sqrt{12}$
$x=-2\sqrt{3}$ | | $x=2\sqrt{3}$

$\frac{dy}{dx}$: dec, -2 (local min), INC, 2 (Local Max), dec

$\frac{d^2y}{dx^2}$: $-2\sqrt{3}$, C.U., 0, C.D., $2\sqrt{3}$

at $x=-2$, $\left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{8(-2)\{(-2)^2-12\}}{((-2)^2+4)^3} = \frac{8(-2)\{-8\}}{(8)^3} > 0$ C.U. local min

$$y\big|_{x=-2} = \frac{4(-2)}{(-2)^2+4} = \frac{-8}{4+4} = \frac{-8}{8} = -1 \quad (-2,-1)$$

at $x=2$: $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{8(2)\{(2)^2-12\}}{((2)^2+4)^3} = \frac{8(2)\{-8\}}{(8)^3} < 0$ C.D. Local Max

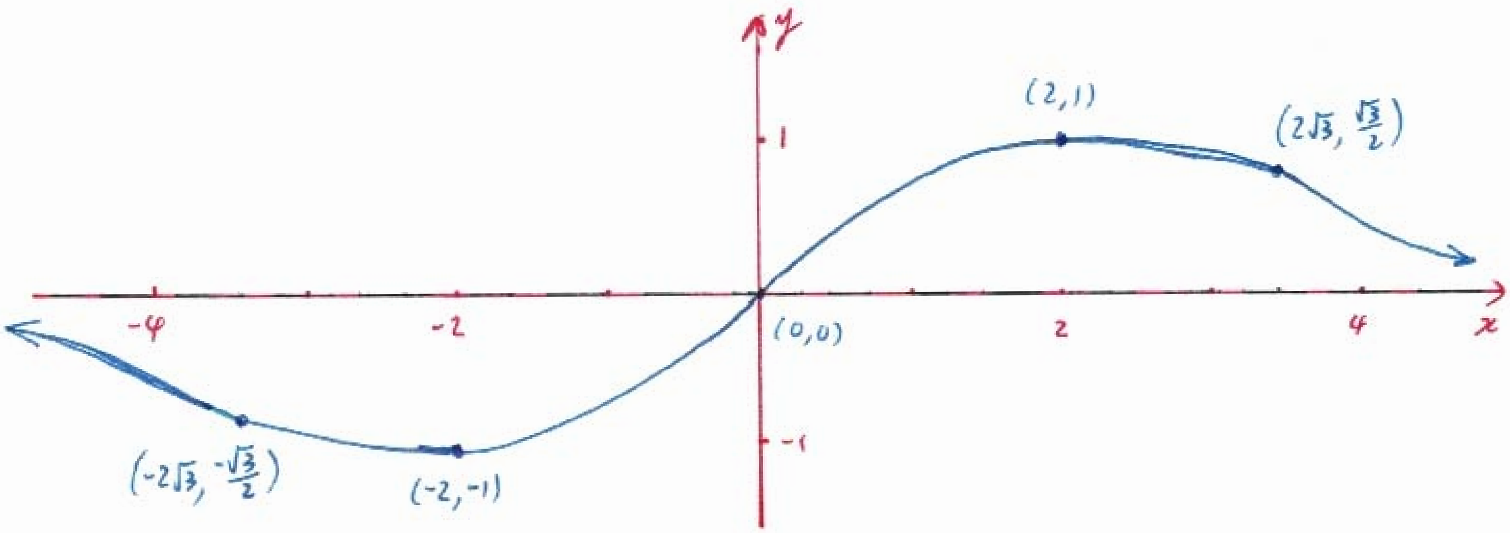
$$y\big|_{x=2} = \frac{4(2)}{(2)^2+4} = \frac{8}{4+4} = \frac{8}{8} = 1 \quad (2,1)$$

102) continued

at $x=0$: $y=0$ (from intercept calculation) $(0,0)$

at $x=-2\sqrt{3}$: $y\big|_{x=-2\sqrt{3}} = \dfrac{4(-2\sqrt{3})}{(-2\sqrt{3})^2+4} = \dfrac{-8\sqrt{3}}{4(3)+4} = \dfrac{-8\sqrt{3}}{12+4} = \dfrac{-8\sqrt{3}}{16} = \dfrac{-\sqrt{3}}{2}$ $\left(-2\sqrt{3}, \dfrac{-\sqrt{3}}{2}\right)$

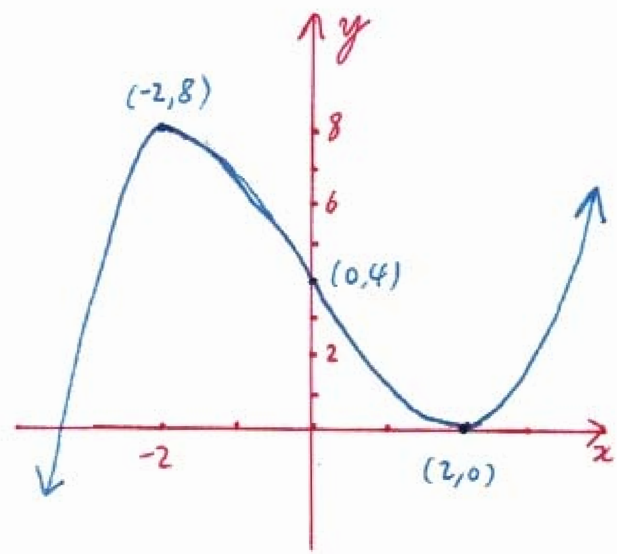
at $x=2\sqrt{3}$: $y\big|_{x=2\sqrt{3}} = \dfrac{4(2\sqrt{3})}{(2\sqrt{3})^2+4} = \dfrac{8\sqrt{3}}{4(3)+4} = \dfrac{8\sqrt{3}}{12+4} = \dfrac{8\sqrt{3}}{16} = \dfrac{\sqrt{3}}{2}$ $\left(2\sqrt{3}, \dfrac{\sqrt{3}}{2}\right)$



104)

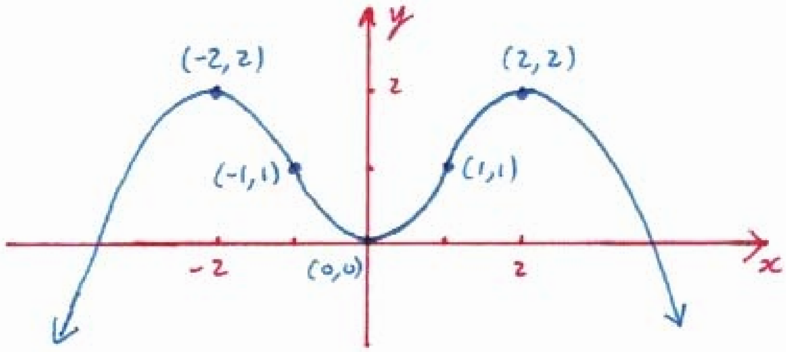
$\dfrac{dy}{dx}$: INC | -2 $(-2,8)$ | dec. | 2 $(2,0)$ | INC

$\dfrac{d^2y}{dx^2}$: C.D. | 0 $(0,4)$ | C.U.



106) $\frac{dy}{dx}$: + INC | −2 $(-2, 2)$ | − dec | 0 $(0,0)$ | + INC | 2 $(2,2)$ | − dec

$\frac{d^2y}{dx^2}$: − C.D. | −1 $(-1,1)$ | + C.U. | 1 $(1,1)$ | − C.D.



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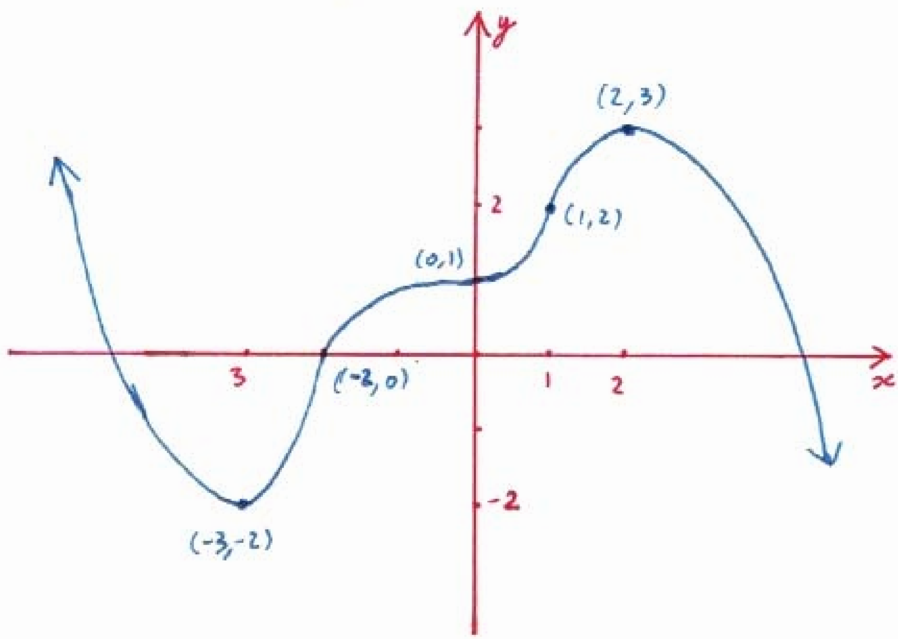
108) $\frac{dy}{dx}$: − dec. | −3 $(-3,-2)$ | + INC | 0 $(0,1)$ | + INC | 2 $(2,3)$ | − dec

$\frac{d^2y}{dx^2}$: + C.U. | −2 $(-2,0)$ | − C.D. | 0 | + C.U. | 1 $(1,2)$ | − C.D.



30) $y = x^{2/5} = (\sqrt[5]{x})^2$ domain: $(-\infty, \infty)$

y-int: $y = (\sqrt[5]{(0)})^2 = 0$ x-int: $0 = (\sqrt[5]{x})^2 \Rightarrow 0 = \sqrt[5]{x} \Rightarrow 0 = x$ $(0,0)$

$$\frac{dy}{dx} = \left[\frac{2}{5}x^{-3/5}\right] = \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{5(\sqrt[5]{x})^3} \qquad \frac{d^2y}{dx^2} = \frac{2}{5}\left[\frac{-3}{5}x^{-8/5}\right] = \frac{-6}{25(\sqrt[5]{x})^8}$$

critical point

$0 = \frac{dy}{dx} = \frac{2}{5(\sqrt[5]{x})^3}$

$0 = \frac{2}{5(\sqrt[5]{x})^3}$

no solution
but denominato 0
when $x = 0$

inflection point

$0 = \frac{d^2y}{dx^2} = \frac{-6}{25(\sqrt[5]{x})^8}$

$0 = \frac{-6}{25(\sqrt[5]{x})^8}$

no solution
but denominator 0
when $x = 0$

"full test"

$\boxed{x=-1}$ $\boxed{x=1}$

$\frac{dy}{dx}$: dec. | 0 | INC.

$\frac{d^2y}{dx^2}$: C.D. | 0 | C.D.

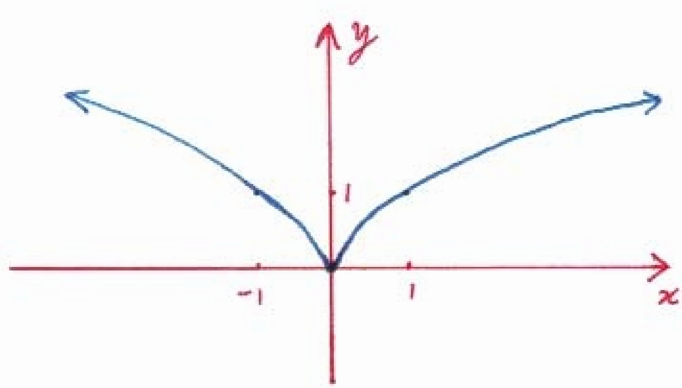
at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = \frac{2}{5(\sqrt[5]{(-1)})^3} = \frac{2}{5(-1)^3} < 0$ dec.

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{-6}{25(\sqrt[5]{(-1)})^8} = \frac{-6}{25(-1)^8} < 0$ C.D.

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \frac{2}{5(\sqrt[5]{(1)})^3} > 0$ INC

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{-6}{25(\sqrt[5]{(1)})^8} < 0$ C.D.

at $x=0$
this is a cusp.
local min



32) $y = \dfrac{\sqrt{1-x^2}}{2x+1} = \dfrac{(1-x^2)^{\frac{1}{2}}}{2x+1}$ from numerator: $1-x^2 \geq 0$ and $[-1,1]$

V.A.: $2x+1=0$
$2x=-1$
$x=\frac{-1}{2}$

since $x=\frac{-1}{2}$ is in $[-1,1]$, our domain: $[-1,\frac{-1}{2}) \cup (\frac{-1}{2},1]$

$$\frac{dy}{dx} = \frac{(2x+1)\left[\frac{1}{2}(1-x^2)^{\frac{-1}{2}}(-2x)\right] - \left((1-x^2)^{\frac{1}{2}}\right)[2]}{(2x+1)^2}$$

$$= \frac{\frac{-x(2x+1)}{\sqrt{1-x^2}} - 2\sqrt{1-x^2}}{(2x+1)^2} = \frac{\frac{-x(2x+1)}{\sqrt{1-x^2}} - \left(\frac{2\sqrt{1-x^2}}{1}\right)\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)}{(2x+1)^2}$$

$$= \frac{\frac{-2x^2-x-2(1-x^2)}{\sqrt{1-x^2}}}{(2x+1)^2} = \frac{-2x^2-x-2+2x^2}{\sqrt{1-x^2}\,(2x+1)^2} = \frac{-x-2}{\sqrt{1-x^2}\,(2x+1)^2}$$

$b = \sqrt{1-x^2}\,(2x+1)^2 = (1-x^2)^{\frac{1}{2}}(2x+1)^2$

$$\frac{db}{dx} = \left((1-x^2)^{\frac{1}{2}}\right)\left[2(2x+1)^1(2)\right] + \left((2x+1)^2\right)\left[\frac{1}{2}(1-x^2)^{\frac{-1}{2}}(-2x)\right]$$

$$= 4(2x+1)\sqrt{1-x^2} - \frac{x(2x+1)^2}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\sqrt{1-x^2}\,(2x+1)^2\right)[-1] - (-x-2)\left[4(2x+1)\sqrt{1-x^2} - \frac{x(2x+1)^2}{\sqrt{1-x^2}}\right]}{\left(\sqrt{1-x^2}\,(2x+1)^2\right)^2}$$

$$= \frac{(2x+1)\left\{-(2x+1)\sqrt{1-x^2} + 4(x+2)\sqrt{1-x^2} - \frac{x(x+2)(2x+1)}{\sqrt{1-x^2}}\right\}}{\left(\sqrt{1-x^2}\right)^2(2x+1)^4}$$

32) continued

$$\frac{d^2y}{dx^2} = \frac{\frac{-(2x+1)\sqrt{1-x^2}}{1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right) + \frac{4(x+2)\sqrt{1-x^2}}{1}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right) - \frac{x(x+2)(2x+1)}{\sqrt{1-x^2}}}{\left(\sqrt{1-x^2}\right)^2(2x+1)^3}$$

$$= \frac{-(2x+1)(1-x^2) + 4(x+2)(1-x^2) - x(x+2)(2x+1)}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

$$= \frac{-(-2x^3-x^2+2x+1) + 4(-x^3-2x^2+x+2) - x(2x^2+5x+2)}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

$$= \frac{2x^3+x^2-2x-1-4x^3-8x^2+4x+8-2x^3-5x^2-2x}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

$$= \frac{-4x^3-12x^2+7}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

critical point

$$0 = \frac{dy}{dx} = \frac{-x-2}{\sqrt{1-x^2}\,(2x+1)^2}$$

$$0 = \frac{-x-2}{\sqrt{1-x^2}\,(2x+1)^2}$$

$$0 = -x-2$$

$$x = -2$$

discard

not in domain

none

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{-4x^3-12x^2+7}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

$$0 = \frac{-4x^3-12x^2+7}{(2x+1)^3\left(\sqrt{1-x^2}\right)^3}$$

$$0 = -4x^3-12x^2+7$$

using MAPLE software

$$\left.\frac{d^2y}{dx^2}\right|_{x=-0.95} \approx 18.05 > 0 \quad \text{C.U.}$$

continued next page

32) continued $\left.\dfrac{d^2y}{dx^2}\right|_{x=-0.90} \approx -4.62 < 0$ C.D.

so there is an inflection point between $(-0.95, -0.90)$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0.65} \approx 0.16 > 0$ C.U. $\left.\dfrac{d^2y}{dx^2}\right|_{x=0.70} \approx -0.05 < 0$ C.D.

and another inflection point between $(0.65, 0.70)$

"full test"

$\boxed{x = \frac{-3}{4}}$ $\boxed{x=0}$

$\dfrac{dy}{dx}$ -1 [dec.) $\frac{-1}{2}$ (dec.] 1

$\dfrac{d^2y}{dx^2}$ -1 [C.U. -0.95 -0.90 C.D.) $\frac{-1}{2}$ (C.U. 0.65 0.70 C.D] 1

at $x = \frac{-3}{4}$: $\left.\dfrac{dy}{dx}\right|_{x=\frac{-3}{4}} = \dfrac{-\left(\frac{-3}{4}\right)-2}{\sqrt{1-\left(\frac{-3}{4}\right)^2}\,\left(2\left(\frac{-3}{4}\right)+1\right)^2} = \dfrac{\frac{-1}{4}}{\sqrt{\frac{7}{16}}\,\left(\frac{-1}{2}\right)^2} < 0$ dec.

at $x=0$: $\left.\dfrac{dy}{dx}\right|_{x=0} = \dfrac{-(0)-2}{\sqrt{1-(0)^2}\,(2(0)+1)^2} = \dfrac{-2}{\sqrt{1}\,(1)^2} < 0$ dec.

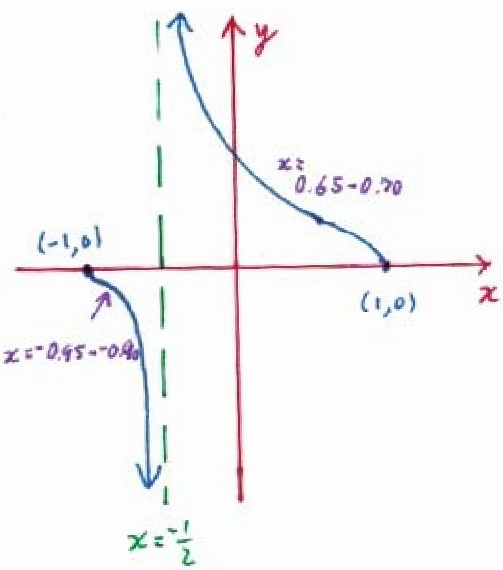
endpoints:

$x=-1$: $\left.y\right|_{x=-1} = \dfrac{\sqrt{1-(-1)^2}}{2(-1)+1} = 0$

Local Max: $(-1,0)$

$x=1$: $\left.y\right|_{x=1} = \dfrac{\sqrt{1-(1)^2}}{2(1)+1} = 0$

local min: $(1,0)$



34) $y = 5x^{\frac{2}{5}} - 2x = 5(\sqrt[5]{x})^2 - 2x$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = 5\left[\frac{2}{5}x^{-\frac{3}{5}}\right] - 2[1] = 2x^{-\frac{3}{5}} - 2 = \frac{2}{(\sqrt[5]{x})^3} - 2 = 2\left(\frac{1}{(\sqrt[5]{x})^3} - 1\right)$$

$$\frac{d^2y}{dx^2} = 2\left[\frac{-3}{5}x^{-\frac{8}{5}}\right] = \frac{-6}{5(\sqrt[5]{x})^8}$$

critical point

$$0 = \frac{dy}{dx} = 2\left(\frac{1}{(\sqrt[5]{x})^3} - 1\right)$$

$$0 = 2\left(\frac{1}{(\sqrt[5]{x})^3} - 1\right)$$

$$0 = \frac{1}{(\sqrt[5]{x})^3} - 1$$

$$1 = \frac{1}{(\sqrt[5]{x})^3}$$

$$(\sqrt[5]{x})^3 = 1$$

$$\sqrt[5]{x} = 1$$

$$x = 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{-6}{5(\sqrt[5]{(1)})^8} < 0$$

C.D.

Local Max

$$y|_{x=1} = 5(\sqrt[5]{(1)})^2 - 2(1) = 3$$

also denominator 0, when $x=0$

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{-6}{5(\sqrt[5]{x})^8}$$

no solution

also denominator 0, when $x = 0$

$$y|_{x=0} = 5(\sqrt[5]{(0)})^2 - 2(0) = 0$$

$(0,0)$

"full test" $\boxed{x=-1}$

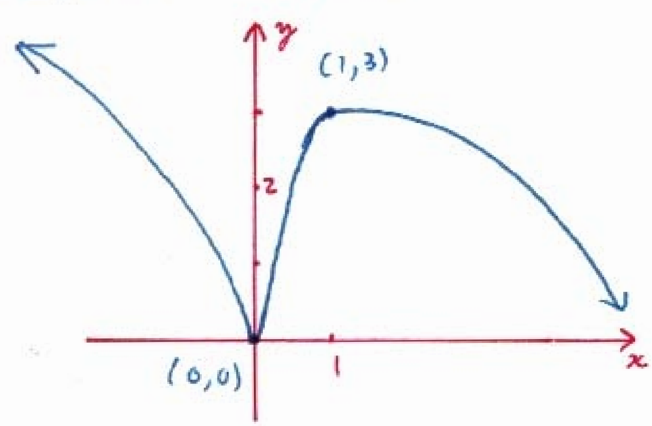
$\frac{dy}{dx}$: dec. | 0 | INC. | 1 (Local Max) | dec.

$\frac{d^2y}{dx^2}$: C.D. | 0 | C.D.

at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = 2\left(\frac{1}{(\sqrt[5]{(-1)})^3} - 1\right) < 0$ dec.

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{-6}{5(\sqrt[5]{(-1)})^8} < 0$ C.D.

we have a cusp at $x=0$ [local min]



36) $y = x^{2/3}(x-5) = x^{5/3} - 5x^{2/3}$ domain: $(-\infty, \infty)$

$= (\sqrt[3]{x})^2(x-5)$

$$\frac{dy}{dx} = \left[\frac{5}{3}x^{2/3}\right] - 5\left[\frac{2}{3}x^{-\frac{1}{3}}\right] = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{5}{3}\left\{(\sqrt[3]{x})^2 - \frac{2}{\sqrt[3]{x}}\right\}$$

$$= \frac{5}{3}\left\{\frac{(\sqrt[3]{x})^2}{1}\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}}\right) - \frac{2}{\sqrt[3]{x}}\right\} = \frac{5}{3}\left\{\frac{x-2}{\sqrt[3]{x}}\right\}$$

$$\frac{d^2y}{dx^2} = \frac{5}{3}\left[\frac{2}{3}x^{-\frac{1}{3}}\right] - \frac{10}{3}\left[\frac{-1}{3}x^{-4/3}\right] = \frac{10}{9}\left\{\frac{1}{\sqrt[3]{x}} + \frac{1}{(\sqrt[3]{x})^4}\right\}$$

$$= \frac{10}{9}\left\{\frac{1}{\sqrt[3]{x}}\left(\frac{(\sqrt[3]{x})^3}{(\sqrt[3]{x})^3}\right) + \frac{1}{(\sqrt[3]{x})^4}\right\} = \frac{10}{9}\left\{\frac{x+1}{(\sqrt[3]{x})^4}\right\}$$

critical points

$$0 = \frac{dy}{dx} = \frac{5}{3}\left\{\frac{x-2}{\sqrt[3]{x}}\right\}$$

$$0 = \frac{5}{3}\left\{\frac{x-2}{\sqrt[3]{x}}\right\}$$

$0 = x - 2$

$x = 2$

also denominator 0 when $x = 0$

inflection points

$$0 = \frac{d^2y}{dx^2} = \frac{10}{9}\left\{\frac{x+1}{(\sqrt[3]{x})^4}\right\}$$

$$0 = \frac{10}{9}\left\{\frac{x+1}{(\sqrt[3]{x})^4}\right\}$$

$0 = x + 1$

$x = -1$

also denominator 0 when $x = 0$

"full test" $\boxed{x=-2}$

$\frac{dy}{dx}$: INC. | $x = 0$ (X) | dec. | $x = 2$ local min | INC.

$\frac{d^2y}{dx^2}$: C.D. | $x = -1$ | C.U. | $x = 0$ (X) | C.U.

36) continued

at $x=2$, $\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{10}{9}\left\{\frac{(2)+1}{(\sqrt[3]{(2)})^4}\right\} > 0$ C.U. local min

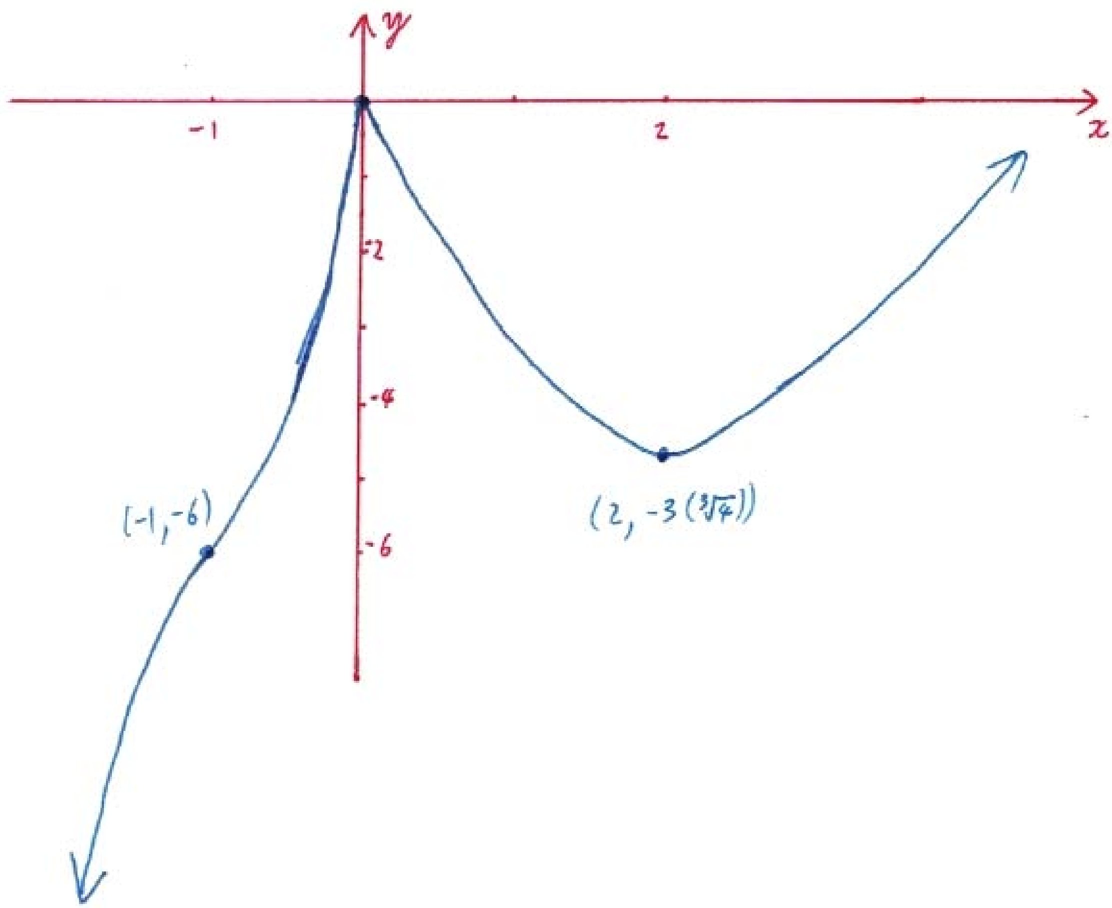
$\left.y\right|_{x=2} = (\sqrt[3]{(2)})^2((2)-5) = (\sqrt[3]{4})(-3) = -3(\sqrt[3]{4})$ $(2, -3(\sqrt[3]{4}))$

at $x=-2$: $\left.\frac{dy}{dx}\right|_{x=-2} = \frac{5}{3}\left\{\frac{(-2)-2}{\sqrt[3]{(-2)}}\right\} = \frac{5}{3}\left\{\frac{-4}{-\sqrt[3]{2}}\right\} = \frac{5}{3}\left\{\frac{4}{\sqrt[3]{2}}\right\} > 0$ INC.

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{10}{9}\left\{\frac{(-2)+1}{(\sqrt[3]{(-2)})^4}\right\} = \frac{10}{9}\left\{\frac{-1}{(\sqrt[3]{2})^4}\right\} < 0$ C.D.

at $x=-1$: $\left.y\right|_{x=-1} = (\sqrt[3]{(-1)})^2((-1)-5) = (1)(-6) = -6$ $(-1,-6)$

at $x=0$: $\left.y\right|_{x=0} = (\sqrt[3]{(0)})^2((0)-5) = (0)(-5) = 0$ $(0,0)$ cusp at $x=0$ [Local Max]



38) $y = (2-x^2)^{3/2} = \left(\sqrt{2-x^2}\right)^3$ domain: $[-\sqrt{2}, \sqrt{2}]$

$$\frac{dy}{dx} = \frac{3}{2}(2-x^2)^{\frac{1}{2}}(-2x) = -3x(2-x^2)^{\frac{1}{2}} = -3x\sqrt{2-x^2}$$

$$\frac{d^2y}{dx^2} = (-3x)\left[\frac{1}{2}(2-x^2)^{-\frac{1}{2}}(-2x)\right] + \left((2-x^2)^{\frac{1}{2}}\right)[-3] = \frac{3x^2}{\sqrt{2-x^2}} - 3\sqrt{2-x^2}$$

$$= \frac{3x^2}{\sqrt{2-x^2}} - \frac{3\sqrt{2-x^2}}{1}\left(\frac{\sqrt{2-x^2}}{\sqrt{2-x^2}}\right) = \frac{3x^2-3(2-x^2)}{\sqrt{2-x^2}} = \frac{3x^2-6+3x^2}{\sqrt{2-x^2}} = \frac{6x^2-6}{\sqrt{2-x^2}} = \frac{6(x^2-1)}{\sqrt{2-x^2}}$$

critical point

$0 = \frac{dy}{dx} = -3x\sqrt{2-x^2}$

$0 = -3x\sqrt{2-x^2}$

$-3x = 0$	$\sqrt{2-x^2} = 0$
$x = 0$	$2-x^2 = 0$
	$(\sqrt{2}+x)(\sqrt{2}-x) = 0$
	$\sqrt{2}+x = 0$; $\sqrt{2}-x = 0$
	$x = -\sqrt{2}$; $x = \sqrt{2}$
	also endpoints

inflection points

$0 = \frac{d^2y}{dx^2} = \frac{6(x^2-1)}{\sqrt{2-x^2}}$

$0 = \frac{6(x^2-1)}{\sqrt{2-x^2}}$

$0 = 6(x^2-1)$

$0 = x^2-1$

$0 = (x+1)(x-1)$

$x+1 = 0$	$x-1 = 0$
$x = -1$	$x = 1$

also denominator 0

$\sqrt{2-x^2} = 0$

$x = -\sqrt{2}$, $x = \sqrt{2}$

endpoints

$\frac{dy}{dx}$ $-\sqrt{2}$ [INC. — Local Max (0) — dec.] $\sqrt{2}$

$\frac{d^2y}{dx^2}$ $-\sqrt{2}$ [C.U. — -1 — C.D. — 1 — C.U.] $\sqrt{2}$

38) continued

at $x=0$: $\left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{6((0)^2-1)}{\sqrt{2-(0)^2}} = \frac{6(-1)}{\sqrt{2}} < 0$ C.P. Local Max

$y|_{x=0} = \left(\sqrt{2-(0)^2}\right)^3 = (\sqrt{2})^3 = 2\sqrt{2}$ $(0, 2\sqrt{2})$

at $x=-1$: $y|_{x=-1} = \left(\sqrt{2-(-1)^2}\right)^3 = \left(\sqrt{2-1}\right)^3 = (\sqrt{1})^3 = 1$ $(-1, 1)$

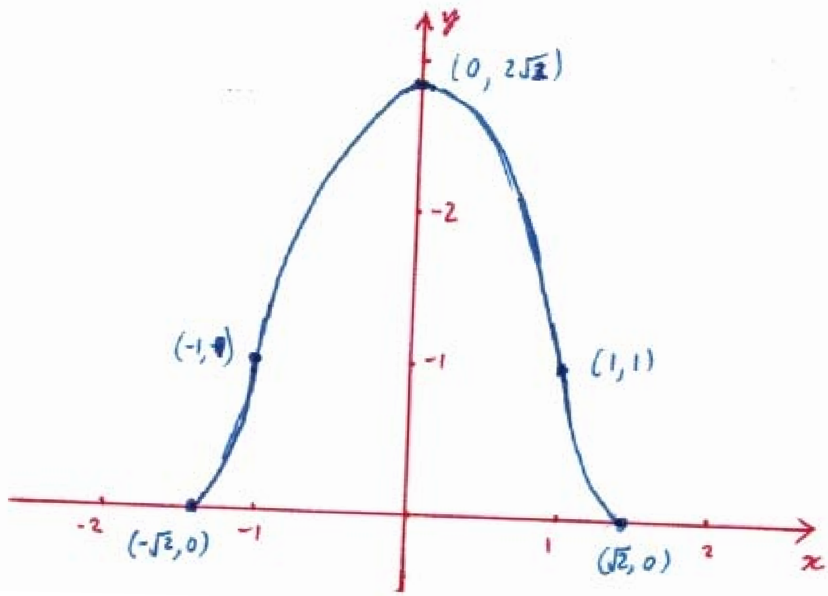
at $x=1$: $y|_{x=1} = \left(\sqrt{2-(1)^2}\right)^3 = \left(\sqrt{2-1}\right)^3 = (\sqrt{1})^3 = 1$ $(1, 1)$

endpoints:

$x=-\sqrt{2}$: $y|_{x=-\sqrt{2}} = \left(\sqrt{2-(-\sqrt{2})^2}\right)^3 = \left(\sqrt{2-2}\right)^3 = (\sqrt{0})^3 = 0$ $(-\sqrt{2}, 0)$

$x=\sqrt{2}$: $y|_{x=\sqrt{2}} = \left(\sqrt{2-(\sqrt{2})^2}\right)^3 = \left(\sqrt{2-2}\right)^3 = (\sqrt{0})^3 = 0$ $(\sqrt{2}, 0)$

these endpoints are also local min.



40) $y = x^2 + \frac{2}{x} = x^2 + 2x^{-1}$ V.A.: $x=0$ domain: $(-\infty, 0) \cup (0, \infty)$

$= \frac{x^3+2}{x}$

this function has an Oblique Asymptote $y = x^2$ [polynomial]

$$\frac{dy}{dx} = [2x] + 2[-1x^{-2}] = 2x - 2x^{-2} = 2x - \frac{2}{x^2} = \frac{2x}{1}\left(\frac{x^2}{x^2}\right) - \frac{2}{x^2} = \frac{2x^3-2}{x^2} = \frac{2(x^3-1)}{x^2}$$

$$\frac{d^2y}{dx^2} = 2[1] - 2[-2x^{-3}] = 2 + \frac{4}{x^3} = \frac{2x^3+4}{x^3}$$

critical point

$0 = \frac{dy}{dx} = \frac{2(x^3-1)}{x^2}$

$0 = \frac{2(x^3-1)}{x^2}$

$0 = 2(x^3-1)$

$0 = x^3 - 1$

$0 = (x-1)(x^2+x+1)$

$x - 1 = 0$ | $x^2+x+1=0$

$x = 1$ | no real solution

inflection point

$0 = \frac{d^2y}{dx^2} = \frac{2x^3+4}{x^3}$

$0 = \frac{2x^3+4}{x^3}$

$0 = 2x^3+4$

$0 = 2(x^3+2)$

$0 = x^3 + (\sqrt[3]{2})^3$

$0 = (x+\sqrt[3]{2})(x^2 - \sqrt[3]{2}x + (\sqrt[3]{2})^2)$

$x + \sqrt[3]{2} = 0$ | $x^2 - \sqrt[3]{2}x + (\sqrt[3]{2})^2 = 0$

$x = -\sqrt[3]{2}$ | no real solution

"full test" $\boxed{x=-1}$

$\frac{dy}{dx}$: dec.) 0 (dec. | local min at 1 | INC.

$\frac{d^2y}{dx^2}$: C.U. | $-\sqrt[3]{2}$ | C.D.) 0 (C.U.

40) continued

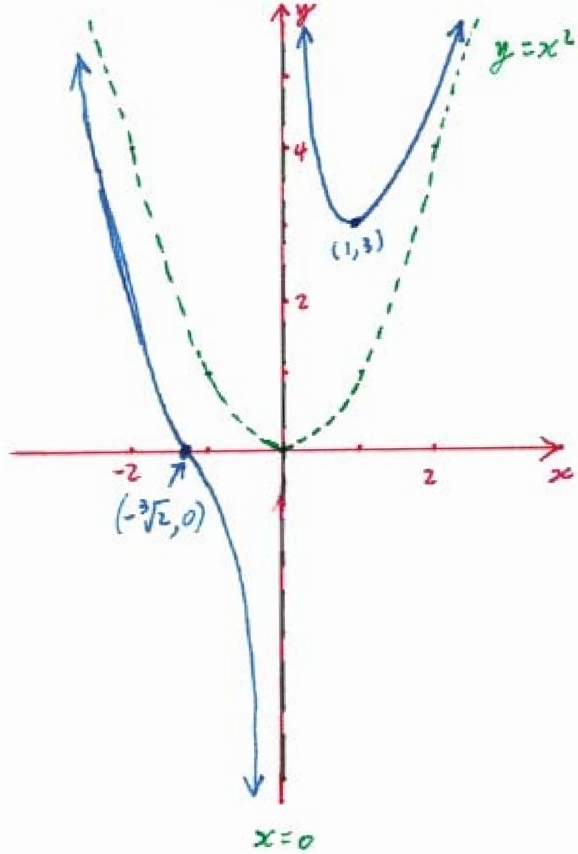
at $x=1$: $\left.\frac{d^2y}{dx^2}\right|_{x=1} = 2+\frac{4}{(1)^3} > 0$ C.U. local min

$\left.y\right|_{x=1} = (1)^2 + \frac{2}{(1)} = 1+2 = 3$ $(1,3)$

at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) - \frac{2}{(-1)^2} = -2-\frac{2}{1} = -4 < 0$ dec.

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2+\frac{4}{(-1)^3} = 2+\frac{4}{-1} = 2-4 = -2 < 0$ C.D.

at $x=-\sqrt[3]{2}$: $\left.y\right|_{x=-\sqrt[3]{2}} = \frac{(-\sqrt[3]{2})^3+2}{(-\sqrt[3]{2})} = \frac{-2+2}{-\sqrt[3]{2}} = \frac{0}{-\sqrt[3]{2}} = 0$ $(-\sqrt[3]{2}, 0)$



42) $y = \sqrt[3]{x^3+1} = (x^3+1)^{\frac{1}{3}}$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = \frac{1}{3}(x^3+1)^{-\frac{2}{3}}(3x^2) = \frac{x^2}{(x^3+1)^{2/3}} = \frac{x^2}{(\sqrt[3]{x^3+1})^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^3+1)^{2/3})[2x] - (x^2)[\frac{2}{3}(x^3+1)^{-\frac{1}{3}}(3x^2)]}{((x^3+1)^{2/3})^2} = \frac{2x(\sqrt[3]{x^3+1})^2 - \frac{2x^4}{(\sqrt[3]{x^3+1})}}{(\sqrt[3]{x^3+1})^4}$$

$$= \frac{\frac{2x(\sqrt[3]{x^3+1})^2}{1}\left(\frac{(\sqrt[3]{x^3+1})}{(\sqrt[3]{x^3+1})}\right) - \frac{2x^4}{(\sqrt[3]{x^3+1})}}{(\sqrt[3]{x^3+1})^4} = \frac{\frac{2x(x^3+1) - 2x^4}{\sqrt[3]{x^3+1}}}{(\sqrt[3]{x^3+1})^4}$$

$$= \frac{2x^4 + 2x - 2x^4}{(\sqrt[3]{x^3+1})^5} = \frac{2x}{(\sqrt[3]{x^3+1})^5}$$

critical points

$$0 = \frac{dy}{dx} = \frac{x^2}{(\sqrt[3]{x^3+1})^2}$$

$$0 = \frac{x^2}{(\sqrt[3]{x^3+1})^2}$$

$0 = x^2$

$x = 0$

also denominator 0

$(\sqrt[3]{x^3+1})^2 = 0$

$\sqrt[3]{x^3+1} = 0$

$\Rightarrow$ $x^3+1=0$

$(x+1)(x^2-x+1)=0$

$x+1=0$, $x=-1$ | $x^2-x+1=0$ no real solution

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{2x}{(\sqrt[3]{x^3+1})^5}$$

$$0 = \frac{2x}{(\sqrt[3]{x^3+1})^5}$$

$0 = 2x$

$x = 0$

also denominator 0

{see left}

$x = -1$

42) continued

"full test"

	$\boxed{x=-2}$		$\boxed{x=\frac{-1}{2}}$		$\boxed{x=1}$
$\frac{dy}{dx}$	INC	-1	INC.	0	INC
$\frac{d^2y}{dx^2}$	C.U.	-1	C.D.	0	C.U.

at $x=-2$: $\left.\frac{dy}{dx}\right|_{x=-2} = \frac{(-2)^2}{\left(\sqrt[3]{(-2)^3+1}\right)^2} = \frac{4}{(\sqrt[3]{-7})^2} > 0$ INC.

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = \frac{2(-2)}{\left(\sqrt[3]{(-2)^3+1}\right)^5} = \frac{-4}{(\sqrt[3]{-7})^5} > 0 \quad \text{C.U.}$$

at $x=\frac{-1}{2}$: $\left.\frac{dy}{dx}\right|_{x=\frac{-1}{2}} = \frac{\left(\frac{-1}{2}\right)^2}{\left(\sqrt[3]{\left(\frac{-1}{2}\right)^3+1}\right)^2} > 0$ INC

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{-1}{2}} = \frac{2\left(\frac{-1}{2}\right)}{\left(\sqrt[3]{\left(\frac{-1}{2}\right)^3+1}\right)^5} = \frac{-1}{\left(\sqrt[3]{\frac{-1}{8}+1}\right)^5} < 0 \quad \text{C.D.}$$

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \frac{(1)^2}{\left(\sqrt[3]{(1)^3+1}\right)^2} > 0$ INC

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2(1)}{\left(\sqrt[3]{(1)^3+1}\right)^5} > 0 \quad \text{C.U.}$$

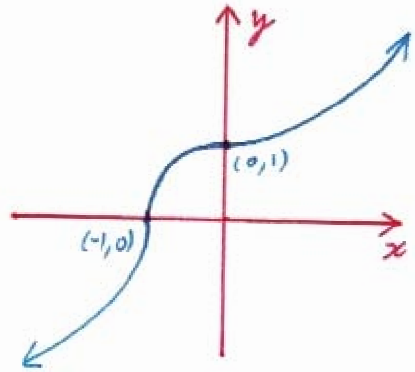
at $x=-1$; this is an inflection point with slope undefined

$$y|_{x=-1} = \sqrt[3]{(-1)^3+1} = \sqrt[3]{-1+1} = \sqrt[3]{0} = 0 \qquad (-1,0)$$

42) continued

at $x=0$: this is an inflection point with slope 0

$$y\big|_{x=0} = \sqrt[3]{(0)^3+1} = \sqrt[3]{1} = 1 \quad (0,1)$$



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44) $y = \dfrac{5}{x^4+5} = 5(x^4+5)^{-1}$ V.A.: $x^4+5=0$ no solution none domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = 5\left[-1\,(x^4+5)^{-2}(4x^3)\right] = -20x^3(x^4+5)^{-2} = \frac{-20x^3}{(x^4+5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((x^4+5)^2)[-60x^2] - (-20x^3)[2(x^4+5)^1(4x^3)]}{((x^4+5)^2)^2}$$

$$= \frac{20x^2(x^4+5)\{(x^4+5)[-3] - (-x)[8x^3]\}}{(x^4+5)^4}$$

$$= \frac{20x^2\{-5x^4-15+8x^4\}}{(x^4+5)^3} = \frac{20x^2\{5x^4-15\}}{(x^4+5)^3} = \frac{100x^2\{x^4-3\}}{(x^4+5)^3}$$

critical points

$$0 = \frac{dy}{dx} = \frac{-20x^3}{(x^4+5)^2}$$

$$0 = \frac{-20x^3}{(x^4+5)^2}$$

$$0 = -20x^3$$

$$x = 0$$

inflection points

$$0 = \frac{d^2y}{dx^2} = \frac{100x^2\{x^4-3\}}{(x^4+5)^3}$$

$$0 = \frac{100x^2\{x^4-3\}}{(x^4+5)^3}$$

$$0 = 100x^2\{x^4-3\} \Rightarrow$$

$0 = 100x^2$; $x^4-3=0$

$x = 0$

$(x^2+\sqrt{3})(x^2-\sqrt{3})=0$

$(x^2+\sqrt{3})(x+\sqrt{\sqrt{3}})(x-\sqrt{\sqrt{3}})=0$

no solution

$x+\sqrt{\sqrt{3}}=0$	$x-\sqrt{\sqrt{3}}=0$
$x=-\sqrt{\sqrt{3}}$	$x=\sqrt{\sqrt{3}}$
$x=-\sqrt[4]{3}$	$x=\sqrt[4]{3}$

44) continued

"full test"

$\frac{dy}{dx}$: $\boxed{x=-1}$ INC. | Local Max at 0 | $\boxed{x=1}$ dec.

$\frac{d^2y}{dx^2}$: C.U. | $-\sqrt[4]{3}$ | C.D. | 0 | C.D. | $\sqrt[4]{3}$ | C.U.

at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = \frac{-20(-1)^3}{((-1)^4+5)^2} > 0$ INC. $\left.\frac{d^2y}{dx^2}\right|_{x=-1} = \frac{100(-1)^2\{(-1)^4-3\}}{((-1)^4+5)^3} < 0$ C.D.

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \frac{-20(1)^3}{((1)^4+5)^2} < 0$ dec. $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{100(1)^2\{(1)^4-3\}}{((1)^4+5)^3} < 0$ C.D.

so at $x=0$, we have a Local Max.

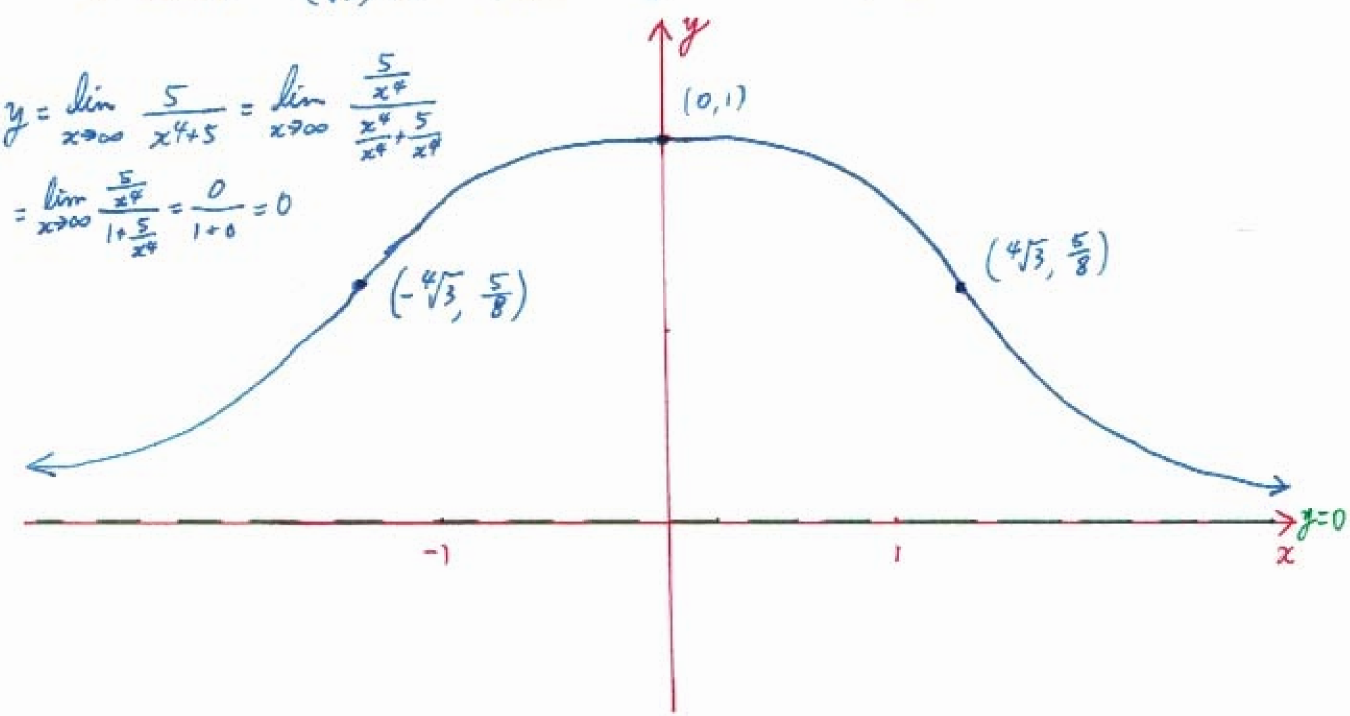
$x=0$: $y|_{x=0} = \frac{5}{(0)^4+5} = \frac{5}{5} = 1$ $\quad (0,1)$

$x=-\sqrt[4]{3}$: $y|_{x=-\sqrt[4]{3}} = \frac{5}{(-\sqrt[4]{3})^4+5} = \frac{5}{3+5} = \frac{5}{8}$ $\quad \left(-\sqrt[4]{3}, \frac{5}{8}\right)$

$x=\sqrt[4]{3}$: $y|_{x=\sqrt[4]{3}} = \frac{5}{(\sqrt[4]{3})^4+5} = \frac{5}{3+5} = \frac{5}{8}$ $\quad \left(\sqrt[4]{3}, \frac{5}{8}\right)$

H.A.: $y = \lim_{x\to\infty} \frac{5}{x^4+5} = \lim_{x\to\infty} \frac{\frac{5}{x^4}}{\frac{x^4}{x^4}+\frac{5}{x^4}}$

$= \lim_{x\to\infty} \frac{\frac{5}{x^4}}{1+\frac{5}{x^4}} = \frac{0}{1+0} = 0$



46) $y = |x^2 - 2x|$ domain: $(-\infty, \infty)$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \mid x - 2 = 0$, $x = 2$

	$(-\infty, 0)$	$(0,2)$	$(2,\infty)$
x	neg	POS	POS
$(x-2)$	neg	neg	POS
$x(x-2)$	POS	neg	POS

$$y = \begin{cases} (x^2 - 2x) & x < 0 \\ -(x^2 - 2x) = 2x - x^2 & 0 \le x \le 2 \\ (x^2 - 2x) & 2 < x \text{ or } x > 2 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} 2x - 2 & x < 0 \\ 2 - 2x & 0 < x < 2 \\ 2x - 2 & 2 < x \end{cases} \qquad \frac{d^2y}{dx^2} = \begin{cases} 2 & x < 0 \\ -2 & 0 < x < 2 \\ 2 & 2 < x \end{cases}$$

critical point

$$0 = \frac{dy}{dx} = \begin{cases} 2x - 2 & x < 0 \\ 2 - 2x & 0 < x < 2 \\ 2x - 2 & 2 < x \end{cases}$$

$0 = 2x - 2$ $0 = 2 - 2x$

$2 = 2x$ $2x = 2$

$1 = x$ $x = 1$

only 1 critical point in $0 < x < 2$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = -2$ C.D. Local Max

$y|_{x=1} = |(1)^2 - 2(1)| = |1-2| = |-1| = 1$ $(1,1)$

at $x=-1$: $\left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) - 2 < 0$ dec.

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2$ C.U.

at $x=3$: $\left.\frac{dy}{dx}\right|_{x=3} = 2(3) - 2 > 0$ INC.

$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 2$ C.U.

inflection point

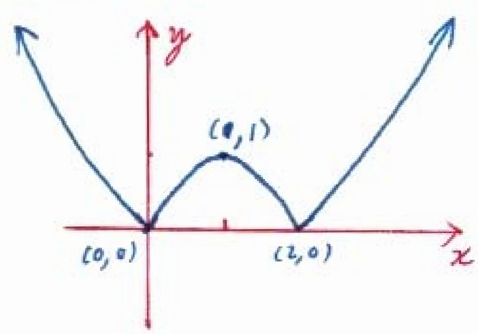
$$0 = \frac{d^2y}{dx^2} = \begin{cases} 2 & x < 0 \\ -2 & 0 < x < 2 \\ 2 & 2 < x \end{cases}$$

none (no solution)

"full test" $[x=-1]$ Local Max   "full test" $\boxed{x=3}$

$\frac{dy}{dx}$: dec. | 0 | INC. | 1 | dec. | 2 | INC.

$\frac{d^2y}{dx^2}$: C.U. | 0 | C.D. | 2 | C.U.



$x=0$: $y|_{x=0} = |(0)^2 - 2(0)| = 0$ $(0,0)$

$x=2$: $y|_{x=2} = |(2)^2 - 2(2)| = 0$ $(2,0)$

these points are cusp and local min.

48) $y=\sqrt{|x-4|}$

Since we have an absolute value inside an even root, the domain is $(-\infty, \infty)$

$$y=\begin{cases}\sqrt{-(x-4)}=\sqrt{4-x}=(4-x)^{\frac{1}{2}}, & x<4\\ \sqrt{+(x-4)}=\sqrt{x-4}=(x-4)^{\frac{1}{2}}, & 4\le x\end{cases}$$

$x-4=0 \Rightarrow x=4$

$$\frac{dy}{dx}=\begin{cases}\frac{1}{2}(4-x)^{-\frac{1}{2}}(-1)=\frac{-1}{2}(4-x)^{-\frac{1}{2}}=\frac{-1}{2\sqrt{4-x}}, & x<4\\ \frac{1}{2}(x-4)^{-\frac{1}{2}}(1)=\frac{1}{2}(x-4)^{-\frac{1}{2}}=\frac{1}{2\sqrt{x-4}}, & 4<x\end{cases}$$

$$\frac{d^2y}{dx^2}=\begin{cases}\frac{-1}{2}\left[\frac{-1}{2}(4-x)^{-\frac{3}{2}}(-1)\right]=\frac{-1}{4(\sqrt{4-x})^3}, & x<4\\ \frac{1}{2}\left[\frac{-1}{2}(x-4)^{-3/2}(1)\right]=\frac{-1}{4(\sqrt{x-4})^3}, & 4<x\end{cases}$$

critical point

$$0=\frac{dy}{dx}=\begin{cases}\frac{-1}{2\sqrt{4-x}} & x<4\\ \frac{1}{2\sqrt{x-4}} & 4<x\end{cases}$$

no solution

denominator 0, $x=4$

inflection point

$$0=\frac{d^2y}{dx^2}=\begin{cases}\frac{-1}{4(\sqrt{4-x})^3} & x<4\\ \frac{-1}{4(\sqrt{x-4})^3} & 4<x\end{cases}$$

no solution

denominator 0, $x=4$

"full test"

$\boxed{x=0}$ $\boxed{x=5}$

$\frac{dy}{dx}$: dec. | 4 | INC.

$\frac{d^2y}{dx^2}$: C.D. | 4 | C.D.

at $x=0$: $\left.\frac{dy}{dx}\right|_{x=0}=\frac{-1}{2\sqrt{4-(0)}}<0$ dec.

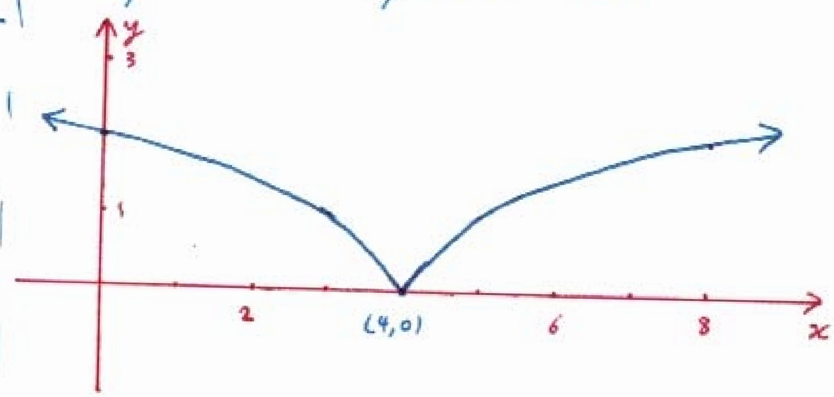
$\left.\frac{d^2y}{dx^2}\right|_{x=0}=\frac{-1}{4(\sqrt{4-(0)})^3}<0$ C.D.

at $x=5$: $\left.\frac{dy}{dx}\right|_{x=5}=\frac{1}{2\sqrt{(5)-4}}>0$ INC.

$\left.\frac{d^2y}{dx^2}\right|_{x=5}=\frac{-1}{4\sqrt{(5)-4}}<0$ C.D.

at $x=4$: $y|_{x=4}=\sqrt{|(4)-4|}=\sqrt{|0|}=0$ $(4,0)$

this point is a cusp and local min



50) $y = \dfrac{x^2}{1-x}$ V.A.: $1-x=0$, $1=x$ domain: $(-\infty,1)\cup(1,\infty)$

Oblique asymptote

$$y = \frac{x^2}{1-x} = -x-1+\frac{(+1)}{1-x} \qquad y=-x-1$$

$$\begin{array}{r} -x-1 \\ -x+1\,\overline{)\,x^2+0x+0} \\ -(x^2-x) \\ \hline +x+0 \\ -(x-1) \\ \hline +1 \end{array}$$

$\dfrac{dy}{dx}$: dec. — local min (0) — INC.) | (INC. — Local Max (2) — dec.

$\dfrac{d^2y}{dx^2}$: C.U.) | (C.D.

$$\frac{dy}{dx} = \frac{(1-x)[2x]-(x^2)[-1]}{(1-x)^2} = \frac{2x-2x^2+x^2}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((1-x)^2)[2-2x]-(2x-x^2)[2(1-x)^1(-1)]}{((1-x)^2)^2} = \frac{2(1-x)\{(1-x)[1-x]-(2x-x^2)[-1]\}}{(1-x)^4}$$

$$= \frac{2\{(1-2x+x^2)+(2x-x^2)\}}{(1-x)^3} = \frac{2\{1\}}{(1-x)^3} = \frac{2}{(1-x)^3}$$

critical points

$0 = \dfrac{dy}{dx} = \dfrac{2x-x^2}{(1-x)^2}$

$0 = \dfrac{2x-x^2}{(1-x)^2}$

$0 = 2x-x^2$

$0 = x(2-x)$

$x=0$ | $2-x=0$, $x=2$

inflection point

$0 = \dfrac{d^2y}{dx^2} = \dfrac{2}{(1-x)^3}$

no solution

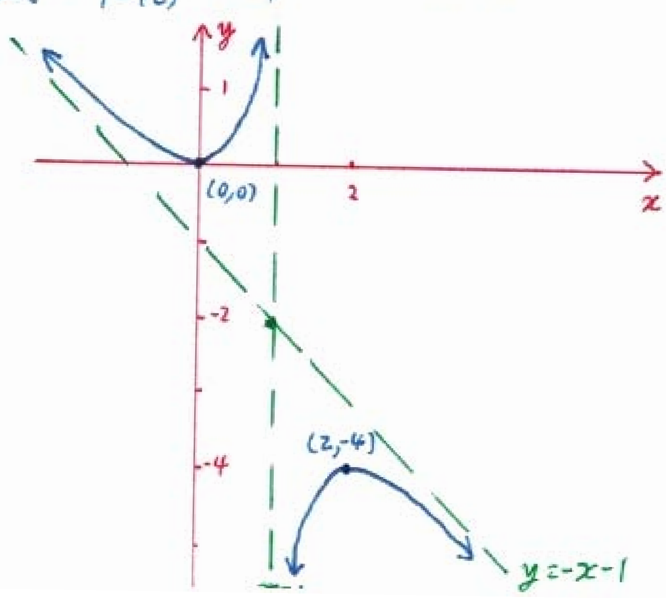
none

at $x=0$: $\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = \dfrac{2}{(1-(0))^3} > 0$ C.U. local min

$\left.y\right|_{x=0} = \dfrac{(0)^2}{1-(0)} = 0 \quad (0,0)$

at $x=2$: $\left.\dfrac{d^2y}{dx^2}\right|_{x=2} = \dfrac{2}{(1-(2))^3} < 0$ C.D. Local Max

$\left.y\right|_{x=2} = \dfrac{(2)^2}{1-(2)} = \dfrac{4}{-1} = -4 \quad (2,-4)$



52) $y = (\ln x)^2$ domain: $(0, \infty)$

$$\frac{dy}{dx} = 2(\ln x)'\left(\frac{1}{x}(1)\right) = \frac{2\ln x}{x}$$

$$\frac{d^2y}{dx^2} = \frac{(x)\left[2\left(\frac{1}{x}(1)\right)\right] - (2\ln x)[1]}{(x)^2} = \frac{2 - 2\ln x}{x^2}$$

critical point

$0 = \frac{dy}{dx} = \frac{2\ln x}{x}$

$0 = \frac{2\ln x}{x}$

$0 = 2\ln x$

$0 = \ln x$

$\Downarrow$

$x = e^0 = 1$

inflection point

$0 = \frac{d^2y}{dx^2} = \frac{2 - 2\ln x}{x^2}$

$0 = \frac{2 - 2\ln x}{x^2}$

$0 = 2 - 2\ln x$

$2\ln x = 2$

$\ln x = 1$

$\Downarrow$

$x = e^1 = e$

$\frac{dy}{dx}$: 0 (dec. | 1 (local min) | INC.

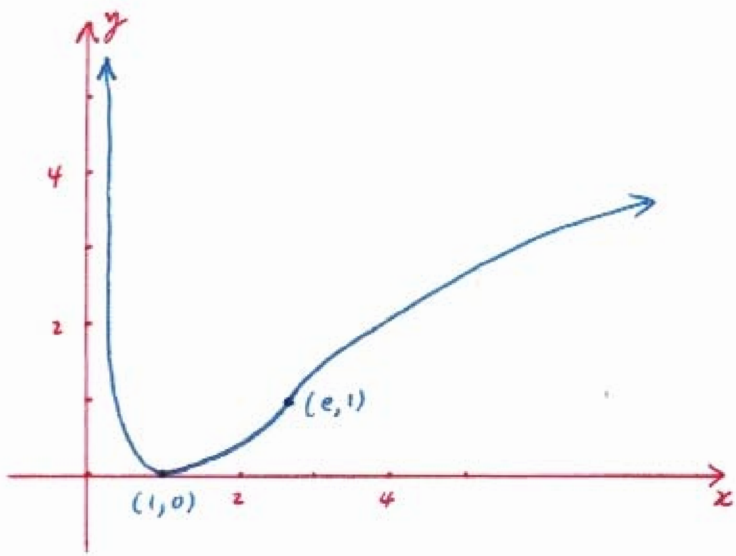
$\frac{d^2y}{dx^2}$: 0 (C.U. | e | C.D.

at $x = 1$: $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2 - 2\ln(1)}{(1)^2} > 0$ C.U. local min

$y|_{x=1} = (\ln(1))^2 = (0)^2 = 0$ $(1, 0)$

at $x = e$: $y|_{x=e} = (\ln(e))^2 = (1)^2 = 1$

$(e, 1)$



54) $y = xe^{-x} = \frac{x}{e^x}$ domain: $(-\infty, \infty)$

H.A. $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x} \overset{L}{=} \lim\limits_{x\to\infty} \frac{1}{e^x} = 0 \Rightarrow y = 0$ as $x \to \infty$

we will learn how to evaluate in section 4.5.

$\lim\limits_{x\to-\infty} xe^{-x} = (-\infty)e^{-(-\infty)} = (-\infty)e^{\infty} = (-\infty)(\infty) = -\infty$ none

$$\frac{dy}{dx} = (x)\left[e^{-x}(-1)\right] + (e^{-x})[1] = -xe^{-x} + e^{-x} = e^{-x} - xe^{-x} = e^{-x}(1-x) = \frac{1-x}{e^{x}}$$

$$\frac{d^2y}{dx^2} = \left\{(-x)\left[e^{-x}(-1)\right] + (e^{-x})[-1]\right\} + \left[e^{-x}(-1)\right] = \left\{xe^{-x} - e^{-x}\right\} - e^{-x} = xe^{-x} - 2e^{-x}$$

$$= e^{-x}(x-2) = \frac{x-2}{e^x}$$

critical point

$0 = \frac{dy}{dx} = \frac{1-x}{e^x}$

$0 = \frac{1-x}{e^x}$

$0 = 1 - x$

$x = 1$

inflection point

$0 = \frac{d^2y}{dx^2} = \frac{x-2}{e^x}$

$0 = \frac{x-2}{e^x}$

$0 = x - 2$

$x = 2$

$\frac{dy}{dx}$: INC. | Local Max at 1 | dec.

$\frac{d^2y}{dx^2}$: C.D. | 2 | C.U.

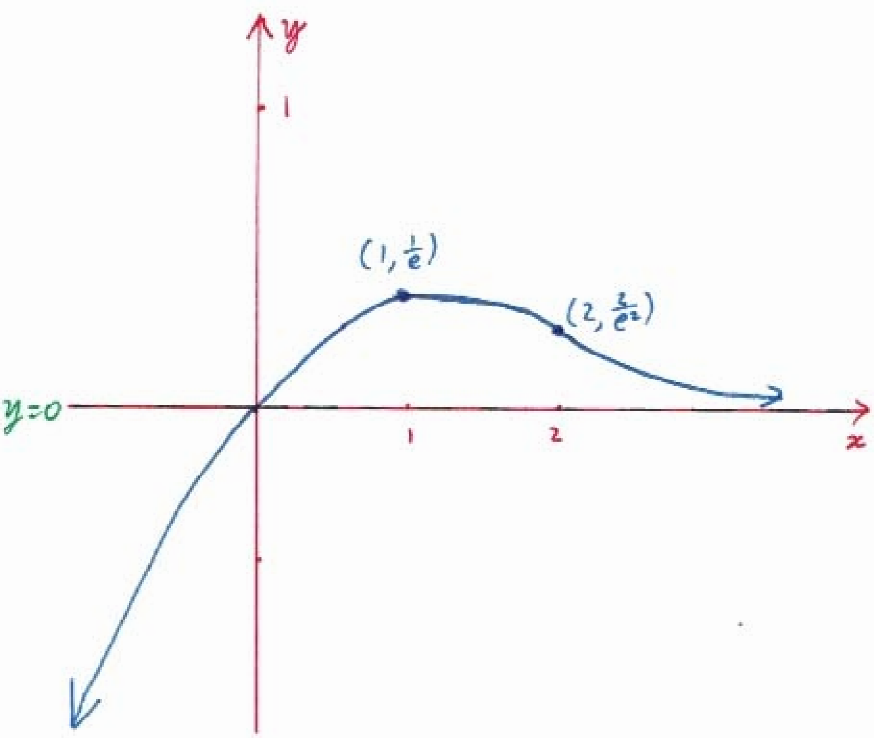
at $x=1$: $\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{(1)-2}{e^{(1)}} < 0$ C.D.

Local Max

$\left.y\right|_{x=1} = \frac{(1)}{e^{(1)}} = \frac{1}{e}$ $\left(1, \frac{1}{e}\right)$

at $x=2$, $\left.y\right|_{x=2} = \frac{(2)}{e^{(2)}} = \frac{2}{e^2}$

$\left(2, \frac{2}{e^2}\right)$



56) $y = \dfrac{\ln x}{\sqrt{x}} = \dfrac{\ln x}{x^{\frac{1}{2}}}$ domain: $(0, \infty)$

H.A.: $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0 \Rightarrow y = 0 \text{ as } x \to \infty$

we will learn how to evaluate in section 4.5

$$\frac{dy}{dx} = \frac{(x^{\frac{1}{2}})\left[\frac{1}{x}(1)\right] - (\ln x)\left[\frac{1}{2}x^{-\frac{1}{2}}\right]}{(x^{\frac{1}{2}})^2} = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{\frac{2-\ln x}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{2 - \ln x}{2(\sqrt{x})^3} = \frac{2 - \ln x}{2x^{3/2}}$$

$$\frac{d^2y}{dx^2} = \frac{(2x^{3/2})\left[\frac{-1}{x}(1)\right] - (2 - \ln x)\left[2\left[\frac{3}{2}x^{\frac{1}{2}}\right]\right]}{(2x^{3/2})^2} = \frac{\frac{-2(\sqrt{x})^3}{x} - 3\sqrt{x}(2 - \ln x)}{4(\sqrt{x})^6}$$

$$= \frac{-2\sqrt{x} - 6\sqrt{x} + 3\sqrt{x}\ln x}{4(\sqrt{x})^6} = \frac{3\sqrt{x}\ln x - 8\sqrt{x}}{4(\sqrt{x})^6} = \frac{\sqrt{x}(3\ln x - 8)}{4(\sqrt{x})^6} = \frac{3\ln x - 8}{4(\sqrt{x})^5}$$

critical point

$0 = \dfrac{dy}{dx} = \dfrac{2 - \ln x}{2(\sqrt{x})^3}$

$0 = \dfrac{2 - \ln x}{2(\sqrt{x})^3}$

$0 = 2 - \ln x$

$\ln x = 2$

$\Downarrow$

$x = e^2$

inflection point

$0 = \dfrac{d^2y}{dx^2} = \dfrac{3\ln x - 8}{4(\sqrt{x})^5}$

$0 = \dfrac{3\ln x - 8}{4(\sqrt{x})^5}$

$0 = 3\ln x - 8$

$8 = 3\ln x$

$\frac{8}{3} = \ln x$

$\Downarrow$

$x = e^{\frac{8}{3}}$

$\frac{dy}{dx}$: 0 (INC. | Local Max at e^2 | dec.

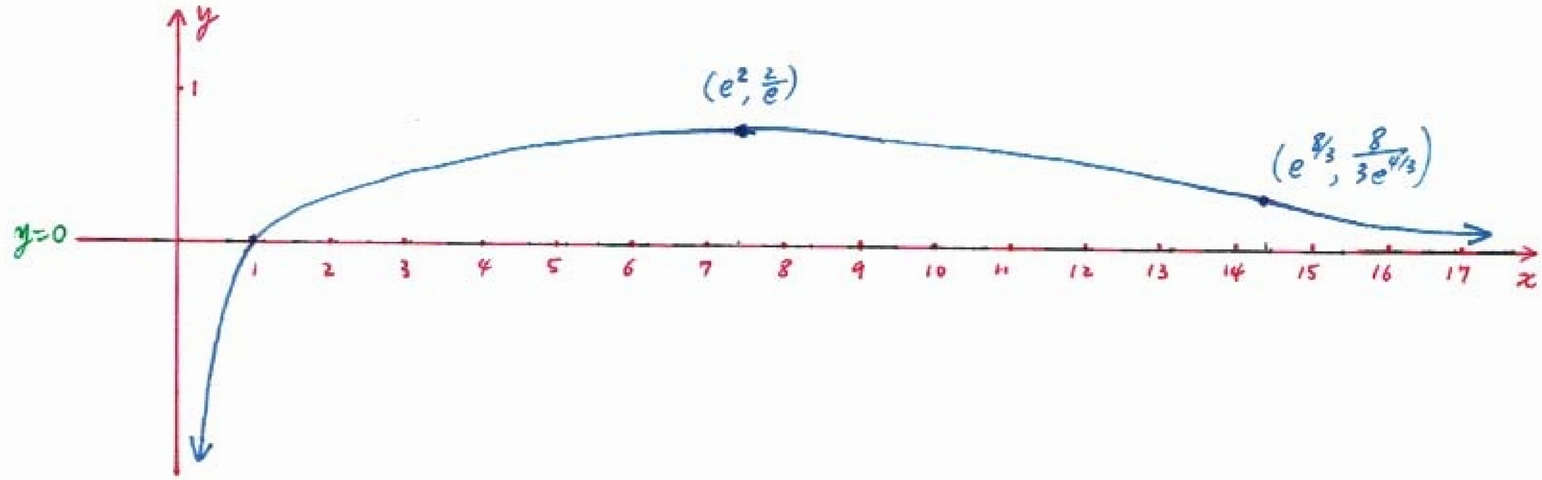
$\frac{d^2y}{dx^2}$: 0 (C.D. | $e^{8/3}$ | C.U.

at $x = e^2$: $\left.\dfrac{d^2y}{dx^2}\right|_{x=e^2} = \dfrac{3\ln(e^2) - 8}{4(\sqrt{e^2})^5} = \dfrac{3(2) - 8}{4e^5} < 0$ C.D. Local Max

$\left. y\right|_{x=e^2} = \dfrac{\ln(e^2)}{\sqrt{(e^2)}} = \dfrac{2}{e}$ $\left(e^2, \frac{2}{e}\right)$

56) continued

at $x = e^{\frac{8}{3}}$: $y\Big|_{x=e^{\frac{8}{3}}} = \dfrac{\ln\left(e^{\frac{8}{3}}\right)}{\sqrt{\left(e^{\frac{8}{3}}\right)}} = \dfrac{\frac{8}{3}}{e^{4/3}} = \dfrac{8}{3e^{4/3}}$ $\quad \left(e^{8/3}, \dfrac{8}{3e^{4/3}}\right)$



58) $y = \dfrac{e^x}{1+e^x}$ $\quad$ V.A.: $1+e^x = 0$ no solution $\quad$ none $\quad$ domain: $(-\infty, \infty)$

H.A.: $\displaystyle\lim_{x\to\infty} \frac{e^x}{1+e^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{1}{e^x}+\frac{e^x}{e^x}} = \lim_{x\to\infty} \frac{1}{\frac{1}{e^x}+1} = \frac{1}{0+1} = \frac{1}{1} = 1$ $\quad y=1$ as $x \to \infty$

$\displaystyle\lim_{x\to-\infty} \frac{e^x}{1+e^x} = \frac{0}{1+0} = 0$ $\quad y = 0$ as $x \to -\infty$

$$\frac{dy}{dx} = \frac{(1+e^x)[e^x(1)] - (e^x)[e^x(1)]}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{((1+e^x)^2)[e^x(1)] - (e^x)[2(1+e^x)^1(e^x(1))]}{((1+e^x)^2)^2}$$

$$= \frac{e^x(1+e^x)\{(1+e^x)[1] - (e^x)[2]\}}{(1+e^x)^4} = \frac{e^x\{1+e^x-2e^x\}}{(1+e^x)^3} = \frac{e^x\{1-e^x\}}{(1+e^x)^3}$$

58) continued

critical point

$$0 = \frac{dy}{dx} = \frac{e^x}{(1+e^x)^2}$$

$$0 = \frac{e^x}{(1+e^x)^2}$$

$$0 = e^x$$

no solution

none

inflection point

$$0 = \frac{d^2y}{dx^2} = \frac{e^x\{1-e^x\}}{(1+e^x)^3}$$

$$0 = \frac{e^x\{1-e^x\}}{(1+e^x)^3}$$

$$0 = e^x\{1-e^x\}$$

$e^x = 0$ no solution | $1-e^x=0$, $e^x=1$ $\Downarrow$ $x=\ln(1)$, $x=0$

"full test" $\boxed{x=1}$

at $x=1$: $\left.\frac{dy}{dx}\right|_{x=1} = \frac{e^{(1)}}{(1+e^{(1)})^2} > 0$ INC.

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{e^{(1)}\{1-e^{(1)}\}}{(1+e^{(1)})^3} < 0 \text{ C.D.}$$

at $x=0$: $y|_{x=0} = \frac{e^{(0)}}{1+e^{(0)}} = \frac{1}{1+1} = \frac{1}{2}$

$(0, \frac{1}{2})$

$\frac{dy}{dx}$: INC.

$\frac{d^2y}{dx^2}$: C.U. | 0 | C.D.