MATH 20100 section 4.2

Theorem 3 - Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Theorem 4 - The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\left.\frac{df}{dx}\right|_{x=c} = f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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2) $f(x) = x^{\frac{2}{3}} = \left(\sqrt[3]{x}\right)^2$ $[0, 1]$ $f(0) = \left(\sqrt[3]{(0)}\right)^2 = (0)^2 = 0$

$$\frac{df}{dx} = \frac{2}{3} x^{\frac{-1}{3}} = \frac{2}{3\left(\sqrt[3]{x}\right)}$$

$$f(1) = \left(\sqrt[3]{(1)}\right)^2 = (1)^2 = 1$$

$$\left.\frac{df}{dx}\right|_{x=c} = \frac{2}{3\left(\sqrt[3]{c}\right)} = \frac{f(1) - f(0)}{(1) - (0)} = \frac{(1) - (0)}{1} = 1$$

$$\frac{2}{3\left(\sqrt[3]{c}\right)} = \frac{1}{1} \Rightarrow \frac{3\left(\sqrt[3]{c}\right)}{2} = 1 \Rightarrow \sqrt[3]{c} = \frac{2}{3}$$

$$c = \left(\frac{2}{3}\right)^3 = \underline{\underline{\frac{8}{27}}}$$

4) $f(x) = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$ $[1, 3]$ $f(1) = \sqrt{(1)-1} = \sqrt{0} = 0$

$f(3) = \sqrt{(3)-1} = \sqrt{2}$

$\frac{df}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}(1) = \frac{1}{2\sqrt{x-1}}$

$\left.\frac{df}{dx}\right|_{x=c} = \frac{1}{2\sqrt{c-1}} = \frac{f(3)-f(1)}{(3)-(1)} = \frac{(\sqrt{2})-(0)}{2} = \frac{\sqrt{2}}{2}$

$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2}$

$\frac{1}{\sqrt{c-1}} = \frac{\sqrt{2}}{1}$ $\Rightarrow$ $\frac{\sqrt{c-1}}{1} = \frac{1}{\sqrt{2}}$

$c-1 = \left(\frac{1}{\sqrt{2}}\right)^2$ $\Rightarrow$ $c-1 = \frac{1}{2}$

$\underline{\underline{c = \frac{3}{2}}}$

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6) $f(x) = \ln(x-1)$ $[2, 4]$ $f(2) = \ln((2)-1) = \ln(1) = 0$

$f(4) = \ln((4)-1) = \ln(3)$

$\frac{df}{dx} = \frac{1}{x-1}(1) = \frac{1}{x-1}$

$\left.\frac{df}{dx}\right|_{x=c} = \frac{1}{c-1} = \frac{f(4)-f(2)}{(4)-(2)} = \frac{(\ln 3)-(0)}{2} = \frac{\ln 3}{2}$

$\frac{1}{c-1} = \frac{\ln 3}{2}$ $\Rightarrow$ $\frac{c-1}{1} = \frac{2}{\ln 3}$ $\Rightarrow$ $\underline{\underline{c = 1 + \frac{2}{\ln 3}}}$

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8) $g(x) = \begin{cases} x^3 & -2 \le x \le 0 \\ x^2 & 0 < x \le 2 \end{cases}$ $\frac{dg}{dx} = \begin{cases} 3x^2 & -2 < x < 0 \\ 2x & 0 < x < 2 \end{cases}$

$g(-2) = (-2)^3 = -8$ $g(2) = (2)^2 = 4$

$\left.\frac{dg}{dx}\right|_{x=c} = g'(c) = \frac{g(2)-g(-2)}{(2)-(-2)} = \frac{(4)-(-8)}{2+2} = \frac{12}{4} = 3$

interval: $-2 \le x \le 0$

$\left.\frac{dg}{dx}\right|_{x=c} = 3c^2 = g'(c) = 3$

$3c^2 = 3$

$c^2 = 1$

$c = \pm 1$

$\underline{\underline{c = -1}}$ $c = 1$ discard

interval $0 < x \le 2$

$\left.\frac{dg}{dx}\right|_{x=c} = 2c = g'(c) = 3$

$2c = 3$

$\underline{\underline{c = \frac{3}{2}}}$

10) $f(x) = x^{\frac{4}{5}} = (\sqrt[5]{x})^4 \quad [0,1]$

$$\frac{df}{dx} = \frac{4}{5}x^{-\frac{1}{5}} = \frac{4}{5(\sqrt[5]{x})}$$

Since $f(x)$ is continuous on $[0,1]$ and $\frac{df}{dx}$ exist for $(0,1)$, $f(x)$ satisfies the hypothesis of the Mean Value Theorem on $[0,1]$

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12)
$$f(x) = \begin{cases} \frac{\sin x}{x} & -\pi \le x < 0 \\ 0 & x = 0 \end{cases}$$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(\frac{\sin x}{x}\right) = 1 \neq 0 \, f(0)$, $f(x)$ is not continuous at $x=0$. Therefore, $f(x)$ does not satisfy the hypothesis of the Mean Value Theorem on $[-\pi, 0]$.

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14)
$$f(x) = \begin{cases} 2x-3 & 0 \le x \le 2 \\ 6x - x^2 - 7 & 2 < x \le 3 \end{cases} \qquad \frac{df}{dx} = \begin{cases} 2 & 0 < x < 2 \\ 6-2x & 2 < x < 3 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x-3) = 2(2^-)-3 = 4-3 = 1 = f(2)$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(6x - x^2 - 7) = 6(2^+) - (2^+)^2 - 7 = 12-4-7 = 1$$

Since $\lim_{x \to 2^-} f(x) = f(2) = 1 = \lim_{x \to 2^+} f(x)$, $f(x)$ is continuous at $x=2$ and on the interval $(0,3)$.

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2(2+h)-3) - (2(2)-3)}{h} = \lim_{h \to 0^-} \frac{(4+2h-3)-(4-3)}{h} = \lim_{h \to 0^-} \frac{2h}{h}$$

$$= \lim_{h \to 0^-} 2 = 2$$

14) continued

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{(6(2+h)-(2+h)^2-7)-(6(2)-(2)^2-7)}{h}$$

$$= \lim_{h\to 0^+} \frac{(12+6h-(4+4h+h^2)-7)-(12-4-7)}{h}$$

$$= \lim_{h\to 0^+} \frac{(12+6h-4-4h-h^2-7)-(12-4-7)}{h} = \lim_{h\to 0^+} \frac{(6h-4h-h^2)}{h}$$

$$= \lim_{h\to 0^+} \frac{(2h-h^2)}{h} = \lim_{h\to 0^+} \frac{h(2-h)}{h} = \lim_{h\to 0^+} (2-h) = 2-(0^+) = 2$$

Since $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$, $f(x)$ is differentiable at $x=2$ and on interval $(0,3)$.

$f(x)$ is continuous on $[0,3]$ and differentiable on $(0,3)$, $f(x)$ does satisfy the hypothesis of the Mean Value Theorem on $[0,3]$.

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16)

$$f(x) = \begin{cases} 3 & x=0 \\ -x^2+3x+a & 0<x<1 \\ mx+b & 1\le x\le 2 \end{cases} \qquad \frac{df}{dx} = \begin{cases} 0 & x=0 \\ -2x+3 & 0<x<1 \\ m & 1<x<2 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-x^2+3x+a) = -(0^+)^2+3(0^+)+a = a$$

$$f(0)=3$$

16) continued

to be continuous at $x=0$ $\quad a = \lim_{x\to 0^+} f(x) = f(0) = 3$

so $a=3$.

to be continuous at $x=1$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x^2+3x+a) = \lim_{x\to 1^-} (-x^2+3x+3)$

$= -(1^-)^2 + 3(1^-) + 3 = -(1) + 3 + 3 = 5$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (mx+b) = m(1^+)+b = m+b = f(1)$

$5 = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = m+b$

so $m+b=5$

now for differentiability at $x=1$

$\lim_{x\to 1^-} \frac{df}{dx} = \lim_{x\to 1^-} (-2x+3) = -2(1^-)+3 = -2+3 = 1$

$\lim_{x\to 1^+} \frac{df}{dx} = \lim_{x\to 1^+} (m) = m$ $\qquad$ so $m=1$

$(1)+b=5$

$b=4$

Therefore $a=3$, $b=4$, $m=1$ to satisfy the hypothesis of the Mean Value Theorem on $[0,2]$.

18) the function f is both differentiable and continuous on $[a, b]$ and we have $f(r_1) = f(r_2) = f(r_3) = 0$ where $a < r_1 < r_2 < r_3 < b$, then by Rolle's Theorem there exists a value c_1 ($r_1 < c_1 < r_2$) such that $f'(c_1) = 0$ and c_2 ($r_2 < c_2 < r_3$) such that $f'(c_2) = 0$. Again, since f' is both differentiable and continuous on $[a, b]$, by applying Rolle's Theorem again we have a value α between c_1 and c_2 ($c_1 < \alpha < c_2$) such that $f''(\alpha) = 0$.

To generalize this result, if a function f has $(n+1)$ zeros in $[a, b]$ and $f^{(n)}$ exist and is continuous on $[a, b]$, then $f^{(n)}$ has at least one zero between a and b.

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20) Assume that $f(x)$ is a cubic polynomial with four or more zeros, by Rolle's Theorem $f'(x)$ has three or more zeros and $f''(x)$ has two or more zeros also $f'''(x)$ has at least one zero.

For this to happen $f'''(x)$ is at least a linear function. This is a contradiction because $f'''(x)$ is a non-zero constant when $f(x)$ is a cubic polynomial.

22) $f(x) = x^3 + \frac{4}{x^2} + 7 = x^3 + 4x^{-2} + 7 \qquad (-\infty, 0)$

$\frac{df}{dx} = [3x^2] + 4[-2x^{-3}] + [0] = 3x^2 - \frac{8}{x^3}$

$\frac{df}{dx} > 0$ on $(-\infty, 0) \Rightarrow f(x)$ is increasing on $(-\infty, 0)$

$0 = f(x) = x^3 + \frac{4}{x^2} + 7 \Rightarrow x = -2$

if $x < -2$, $f(x) = x^3 + \frac{4}{x^2} + 7 < 0$ (negative)

if $-2 < x < 0$, $f(x) = x^3 + \frac{4}{x^2} + 7 > 0$ (positive)

Therefore $f(x)$ has exactly one zero on $(-\infty, 0)$.

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24) $g(t) = \frac{1}{1-t} + \sqrt{1+t} - 3.1 = (1-t)^{-1} + (1+t)^{\frac{1}{2}} - 3.1 \qquad (-1, 1)$

$\frac{dg}{dt} = [-1(1-t)^{-2}(-1)] + [\frac{1}{2}(1+t)^{-\frac{1}{2}}(1)] - [0] = \frac{1}{(1-t)^2} + \frac{1}{2\sqrt{1+t}}$

$\frac{dg}{dt} > 0$ on $(-1, 1) \Rightarrow g(t)$ is increasing on $(-1, 1)$

$g\left(\frac{-9}{10}\right) = \frac{1}{1-\left(\frac{-9}{10}\right)} + \sqrt{1+\left(\frac{-9}{10}\right)} - 3.1 = \frac{1}{\frac{19}{10}} + \sqrt{\frac{1}{10}} - 3.1 = \frac{10}{19} + \frac{1}{\sqrt{10}} - 3.1 < 0$

$g\left(\frac{9}{10}\right) = \frac{1}{1-\left(\frac{9}{10}\right)} + \sqrt{1+\left(\frac{9}{10}\right)} - 3.1 = \frac{1}{\frac{1}{10}} + \sqrt{\frac{19}{10}} - 3.1 = 10 + \frac{\sqrt{19}}{\sqrt{10}} - 3.1 > 0$

Therefore $g(t)$ has exactly one zero on $(-1, 1)$.

26) $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}$ $\qquad (-\infty, \infty)$

$$\frac{dr}{d\theta} = 2[1] - [2\cos\theta \sin\theta (1)] + [0] = 2 - 2\sin\theta\cos\theta = 2 - \sin(2\theta)$$

$\frac{dr}{d\theta} > 0$ on $(-\infty, \infty) \Rightarrow r(\theta)$ is increasing on $(-\infty, \infty)$

$$r(-2\pi) = 2(-2\pi) - \cos^2(-2\pi) + \sqrt{2} = -4\pi - (1)^2 + \sqrt{2} = -4\pi - 1 + \sqrt{2} < 0$$

$$r(2\pi) = 2(2\pi) - \cos^2(2\pi) + \sqrt{2} = 4\pi - (1)^2 + \sqrt{2} = 4\pi - 1 + \sqrt{2} > 0$$

Therefore $r(\theta)$ has exactly one zero in $(-\infty, \infty)$.

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28) $r(\theta) = \tan\theta - \cot\theta - \theta$ $\qquad (0, \frac{\pi}{2})$

$$\frac{dr}{d\theta} = [\sec^2\theta(1)] - [-\csc^2\theta(1)] - [1] = \sec^2\theta + \csc^2\theta - 1 = \sec^2\theta + \cot^2\theta$$

$\frac{dr}{d\theta} > 0$ on $(0, \frac{\pi}{2}) \Rightarrow r(\theta)$ is increasing on $(0, \frac{\pi}{2})$

$$r\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{6}\right) - \left(\frac{\pi}{6}\right) = \left(\frac{1}{\sqrt{3}}\right) - \left(\frac{\sqrt{3}}{1}\right) - \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \sqrt{3} - \frac{\pi}{6} < 0$$

$$r\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) - \cot\left(\frac{\pi}{3}\right) - \left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{1}\right) - \left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{3} = \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{3} > 0$$

Therefore $r(\theta)$ has exactly one zero in $(0, \frac{\pi}{2})$.

54)

start	$t=0$	$t=2.2$ hrs
distance	0 miles	26.2 miles

$$\text{average speed} = \frac{(26.2)-(0)}{(2.2)-(0)} = \frac{26.2}{2.2} = \frac{262}{22} = \frac{131}{11} > \frac{121}{11} = \frac{(11)^2}{11} = 11$$

By Mean Value theorem, the runner must have gone at least once the average speed of $\frac{131}{11}$ mph. Since the initial speed and final speed are both 0 mph and the runner's speed is continuous, by the Intermediate Value theorem, the runner's speed must have been 11 mph at least twice (at least once during speed increase and at least once during speed decrease).

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64) $0 < f'(x) < \frac{1}{2}$ for all x implies $f'(x)$ exists for all x, thus f is differentiable on $(-1,1)$ and $f(x)$ is continuous on $[-1,1]$. So $f(x)$ satisfies the conditions of the Mean Value Theorem.

$$\frac{f(1)-f(-1)}{(1)-(-1)} = f'(c) \text{ for some } c \text{ in } [-1,1]$$

$$0 < \frac{f(1)-f(-1)}{2} = f'(c) < \frac{1}{2} \quad \text{"multiply all terms by 2"}$$

$$0 < f(1)-f(-1) < 1$$

64) continued

left side

$$0 < f(1) - f(-1) \Rightarrow f(-1) < f(1)$$

right side

$$f(1) - f(-1) < 1$$

$$f(1) < 1 + f(-1) < 2 + f(-1)$$

By combining the results above,

$$f(-1) < f(1) < 2 + f(-1)$$