MATH 20100 section 4.1

Definitions Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } D$$

and an absolute minimum value on D at c if

$$f(x) \geq f(c) \quad \text{for all } x \text{ in } D.$$

Theorem 1 - The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Definitions

A function f has a local maximum value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a local minimum value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Theorem 2 - The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain and if f' is defined at c, then

$$f'(c) = 0.$$

Definition

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1) Find all critical points of f on the interval.
2) Evaluate f at all critical points and endpoints.
3) Take the largest and smallest of these values.

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2) Abs. Max at $x=c$ abs. min. at $x=b$
Thm 1 guarantees the existence of such extreme values because f is continuous on $[a,b]$.

4) Abs. Max. none abs. min. none
function is not defined on a closed interval, so Thm 1's conclusion does not apply.

6) Abs. Max. at $x=a$ abs. min. at $x=c$
when the hypothesis of Thm 1 is satisfied then extrema are guaranteed.
when the hypothesis of Thm 1 is not satisfied, absolute extrema may or may not occur.

8) Abs. Max. at $(0,2)$ abs. min. at $(-2,0), (2,0)$

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10) Abs. Max. at $(1,2)$ abs. min. at $(0,-1)$

Local Max. at $(-3,0)$ local min. at $(2,0)$

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12) (a) slope of tangent line $= 0$ at points a and b with negative slope at point c

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14) (a) only graph that the function is not differentiable at a and b

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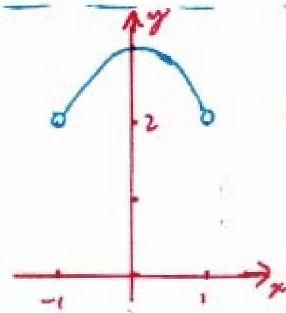
16)



$y = \frac{6}{x^2+1}$, $-1 < x < 1$

Abs Max at $x=0$ $(0,3)$ abs. min. none

since function is defined on open interval, conclusion of Thm 1 does not hold.

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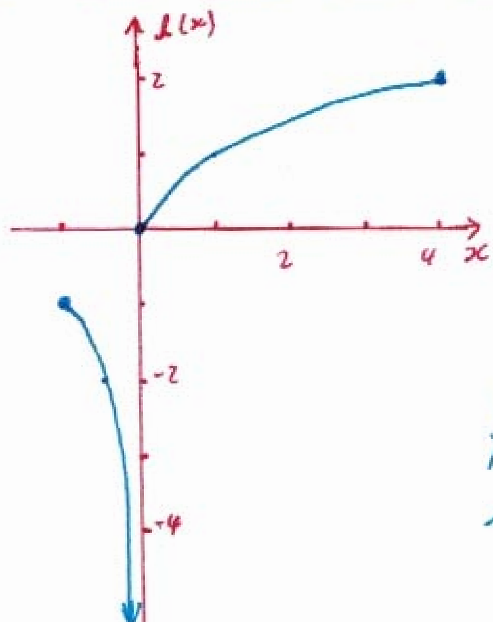
18)



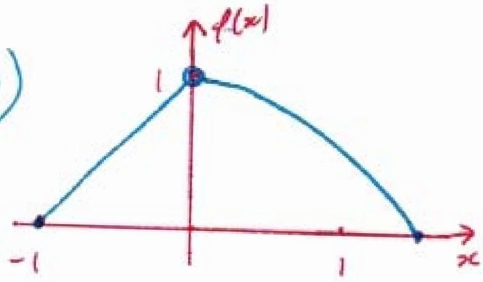
$$h(x) = \begin{cases} \frac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$$

Abs. Max. at $x=4$ $(4,2)$

abs min none.

function is not continuous at $x=0$, so conclusion of Thm 1 not satisfied.

20)



$$f(x) = \begin{cases} x+1\ , & -1 \le x\ 0 \\ \cos x\ , & 0 < x \le \frac{\pi}{2} \end{cases}$$

Abs. Max. none   abs. min. at $x=-1$ $(-1,0)$ and $x=\frac{\pi}{2}$ $(\frac{\pi}{2},0)$

Since $f(x)$ is defined on a union of half-open intervals and not continuous at $x=0$, $f(x)$ does not meet the conclusions of Theorem 1.

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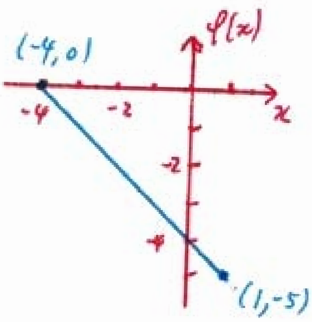
22)



$f(x) = -x-4\ ,\quad -4 \le x \le 1$

$f(0) = -(0)-4 = -4$

$f(1) = -(1)-4 = -5$

$\frac{df}{dx} = -[1] = -1$ no critical points

Abs Max is 0 at $x=-4$ abs. min is -5 at $x=1$

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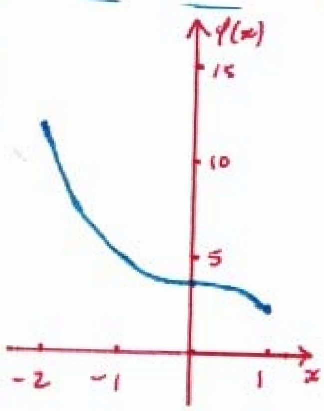
24)



$f(x) = 4 - x^3\ ,\quad -2 \le x \le 1$

$\frac{df}{dx} = -[3x^2] = -3x^2$

critical point :
$0 = \frac{df}{dx} = -3x^2$
$0 = -3x^2$
$0 = x^2$
$0 = x$

$f(0) = 4-(0)^3 = 4$

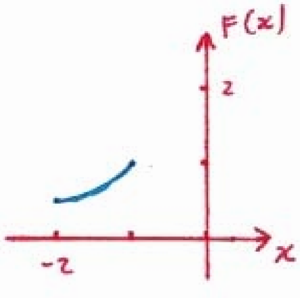
$f(-2) = 4-(-2)^3 = 4+8 = 12$

$f(1) = 4-(1)^3 = 3$

Abs. Max is 12 at $x=-2$

abs. min is 3 at $x=1$

26)



$F(x) = -\frac{1}{x} = \frac{-1}{x} \quad -2 \le x \le -1$

$F(x) = -x^{-1}$

$\frac{dF}{dx} = -\left[-1 x^{-2}\right] = \frac{1}{x^2}$

critical point none because the value $x=0$ is critical value for $\frac{dF}{dx} = \frac{1}{x^2}$ but $x=0$ is not in $-2 \le x \le -1$.

$F(-2) = \frac{-1}{(-2)} = \frac{1}{2}$ $\qquad$ $F(-1) = \frac{-1}{(-1)} = 1$

Abs Max is 1 at $x=-1$ $\quad$ abs min is $\frac{1}{2}$ at $x = \frac{1}{2}$

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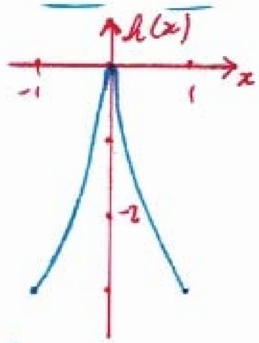
28)



$h(x) = -3x^{2/3} = -3(\sqrt[3]{x})^2 \quad -1 \le x \le 1$

$\frac{dh}{dx} = -3\left[\frac{2}{3} x^{-\frac{1}{3}}\right] = \frac{-2}{\sqrt[3]{x}}$

critical point $\quad$ but at $x=0$ $h(x)$ is undefined.

$0 = \frac{dh}{dx} = \frac{-2}{\sqrt[3]{x}}$ $\qquad$ $h(0) = -3\left(\sqrt[3]{(0)}\right)^2 = 0$

no solution $\qquad$ $h(-1) = -3\left(\sqrt[3]{(-1)}\right)^2 = -3(-1)^2 = -3$

$h(1) = -3\left(\sqrt[3]{(1)}\right)^2 = -3(1)^2 = -3$

Abs Max is 0 at $x=0$ $\quad$ abs min is -3 at $x=-1$ and $x=1$

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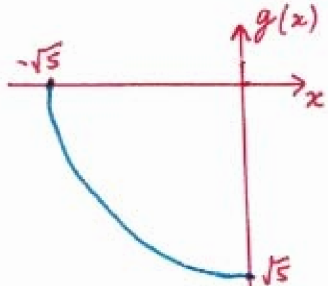
30)



$g(x) = -\sqrt{5-x^2} = -\sqrt{(\sqrt{5})^2 - x^2}$

$= -(5-x^2)^{\frac{1}{2}} \qquad -\sqrt{5} \le x \le 0$

$\frac{dg}{dx} = -\left[\frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x)\right] = \frac{x}{\sqrt{5-x^2}}$

30) continued

critical points:

$0 = \frac{dg}{dx} = \frac{x}{\sqrt{5-x^2}}$ also when $x = -\sqrt{5}$ and $x = \sqrt{5}$ $g(x)$ is undefined

$0 = x$

discard $x = \sqrt{5}$, $\sqrt{5}$ not in $-\sqrt{5} \le x \le 0$

$g(-\sqrt{5}) = -\sqrt{5-(-\sqrt{5})^2} = -\sqrt{5-5} = -\sqrt{0} = 0$

$g(0) = -\sqrt{5-(0)^2} = -\sqrt{5}$

Abs Max is 0 at $x = -\sqrt{5}$ abs min is $-\sqrt{5}$ at $x = 0$

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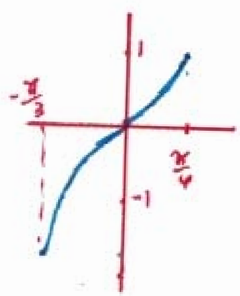
32)



$f(\theta) = \tan\theta$ $\quad \frac{-\pi}{3} \le \theta \le \frac{\pi}{4}$

$\frac{df}{d\theta} = [\sec^2\theta(1)] = \sec^2\theta$

$f(\theta)$ has no critical points in $(\frac{-\pi}{3}, \frac{\pi}{4})$

$f(\frac{-\pi}{3}) = \tan(\frac{-\pi}{3}) = -\sqrt{3}$ $\quad f(\frac{\pi}{4}) = \tan(\frac{\pi}{4}) = 1$

Abs Max is 1 at $\theta = \frac{\pi}{4}$ abs min is $-\sqrt{3}$ at $\theta = \frac{-\pi}{3}$

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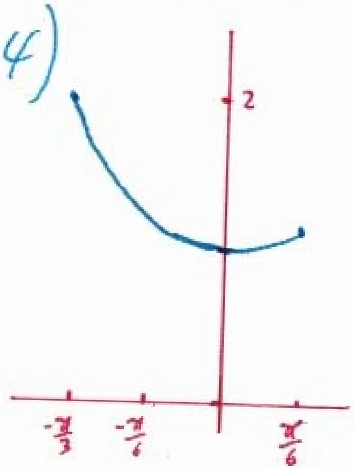
34)



$g(x) = \sec x$ $\quad \frac{-\pi}{3} \le x \le \frac{\pi}{6}$

$\frac{dg}{dx} = [\sec x \tan x (1)] = \sec x \tan x$

critical point

$0 = \frac{dg}{dx} = \sec x \tan x$

$\sec(0) = 1$, $\tan(0) = 0 \Rightarrow x = 0$

34) continued

$g(0) = \sec(0) = 1 \quad g\left(\frac{-\pi}{3}\right) = \sec\left(\frac{-\pi}{3}\right) = 2 \quad g\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

Abs Max is 2 at $x = \frac{-\pi}{3}$ abs min is 1 at $x = 0$

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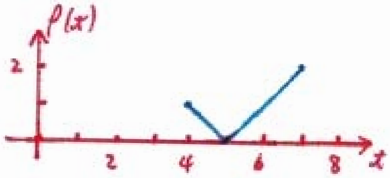
36)



$f(t) = |t-5| \qquad 4 \le t \le 7$

$f(t) = \sqrt{(t-5)^2} = ((t-5)^2)^{\frac{1}{2}}$

$$\frac{df}{dt} = \left[\frac{1}{2}\left((t-5)^2\right)^{-\frac{1}{2}}(2(t-5)(1))\right] = \frac{t-5}{\sqrt{(t-5)^2}} = \frac{t-5}{|t-5|}$$

critical point $x = 5$

$f(5) = |(5)-5| = |0| = 0$

$f(4) = |(4)-5| = |-1| = 1 \qquad f(7) = |(7)-5| = |2| = 2$

Abs Max is 2 at $t = 7$ abs min is 0 at $t = 0$

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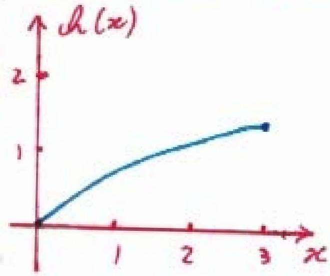
38)



$h(x) = \ln(x+1) \qquad 0 \le x \le 3$

$$\frac{dh}{dx}\left[\frac{1}{x+1}(1)\right] = \frac{1}{x+1}$$

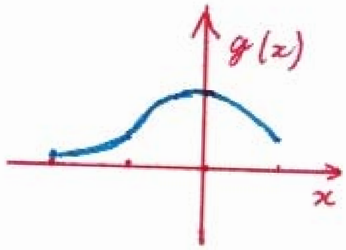
no critical point in $0 \le x \le 3$

$h(0) = \ln((0)+1) = \ln(1) = 0$

$h(3) = \ln((3)+1) = \ln(4)$

Abs Max is $\ln(4)$ at $x = 3$ abs min is 0 at $x = 1$

40)



$$g(x) = e^{-x^2} = \frac{1}{e^{x^2}} \qquad -2 \le x \le 1$$

$$\frac{dg}{dx} = \left[e^{-x^2}(-2x)\right] = 2x\,e^{-x^2} = \frac{2x}{e^{x^2}}$$

critical point

$$0 = \frac{dg}{dx} = \frac{2x}{e^{x^2}} \qquad g(0) = \frac{1}{e^{(0)^2}} = \frac{1}{(1)} = 1$$

$$0 = \frac{2x}{e^{x^2}} \qquad g(-2) = \frac{1}{e^{(-2)^2}} = \frac{1}{e^4}$$

$$0 = 2x \qquad g(1) = \frac{1}{e^{(1)^2}} = \frac{1}{e}$$

$0 = x$ Abs Max is 1 at $x=0$ abs min is $\frac{1}{e^4}$ at $x=-2$

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42) $f(x) = x^{5/3} = (\sqrt[3]{x})^5 \qquad -1 \le x \le 8$

$$\frac{df}{dx} = \left[\frac{5}{3}x^{2/3}\right] = \frac{5}{3}(\sqrt[3]{x})^2$$

critical point

$$0 = \frac{df}{dx} = \frac{5}{3}(\sqrt[3]{x})^2 \Rightarrow x = 0$$

$f(0) = (\sqrt[3]{(0)})^5 = (0)^5 = 0 \quad f(-1) = (\sqrt[3]{(-1)})^5 = (-1)^5 = -1 \quad f(8) = (\sqrt[3]{(8)})^5 = (2)^5 = 32$

Abs Max is 32 at $x=8$ abs min is -1 at $x=-1$

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44) $h(\theta) = 3\theta^{2/3} = 3(\sqrt[3]{\theta})^2 \qquad -27 \le \theta \le 8$

$$\frac{dh}{d\theta} = 3\left[\frac{2}{3}\theta^{-\frac{1}{3}}\right] = \frac{2}{\sqrt[3]{\theta}}$$

critical point

$\theta = 0$ because $\frac{dh}{d\theta}$ undefined

$h(0) = 3(\sqrt[3]{(0)})^2 = 3(0)^2 = 0 \qquad h(8) = 3(\sqrt[3]{(8)})^2 = 3(2)^2 = 12$

$h(-27) = 3(\sqrt[3]{(-27)})^2 = 3(-3)^2 = 3(9) = 27$

Abs Max is 27 at $\theta = -27$ abs min is 0 at $\theta = 0$

46) $f(x) = 6x^2 - x^3$

$$\frac{df}{dx} = 6[2x] - [3x^2] = 12x - 3x^2$$

Critical Points:

$$0 = 12x - 3x^2$$
$$0 = 3x(x-4)$$

$3x = 0$ | $x - 4$

$\underline{\underline{x = 0}}$ | $\underline{\underline{x = 4}}$

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48) $g(x) = (x-1)^2(x-3)^2$

$$\frac{dg}{dx} = ((x-1)^2)[2(x-3)'(1)] + ((x-3)^2)[2(x-1)'(1)]$$
$$= 2(x-1)(x-3)\{(x-1)[1] + (x-3)[1]\}$$
$$= 2(x-1)(x-3)\{x-1+x-3\} = 2(x-1)(x-3)\{2x-4\}$$
$$= 4(x-1)(x-2)(x-3)$$

Critical Points

$$0 = 4(x-1)(x-2)(x-3)$$

$x - 1 = 0$ | $x - 2 = 0$ | $x - 3 = 0$

$\underline{\underline{x = 1}}$ | $\underline{\underline{x = 2}}$ | $\underline{\underline{x = 3}}$

50) $f(x) = \dfrac{x^2}{x-2}$ domain: $(-\infty, 2) \cup (2, \infty)$

$$\frac{df}{dx} = \frac{(x-2)[2x] - (x^2)[1]}{(x-2)^2} = \frac{2x^2 - 4x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

Critical Points

$$0 = \frac{x^2-4x}{(x-2)^2} \Rightarrow 0 = x(x-4)$$

$0 = x^2 - 4x$

$\underline{\underline{x = 0}}$ | $x - 4 = 0$, $\underline{\underline{x = 4}}$

$\left.\begin{array}{l}(x-2)^2 = 0\\ x-2=0\\ x=2\end{array}\right\}$ $\frac{df}{dx}$ undefined

discard "not in domain"

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52) $g(x) = \sqrt{2x - x^2}$
$= (2x - x^2)^{\frac{1}{2}}$

$2x - x^2 \geq 0$
$x(2-x) \geq 0$
$x = \boxed{0}$ | $2 - x = 0$, $\boxed{2} = x$

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
x	neg	POS	POS
$(2-x)$	POS	POS	neg
$x(2-x)$	neg	POS	neg

domain: $[0, 2]$

$$\frac{dg}{dx} = \left[\frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2-2x)\right]$$

$$= \frac{2-2x}{2\sqrt{2x-x^2}} = \frac{2(1-x)}{2\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$$

Critical Points

$$0 = \frac{1-x}{\sqrt{2x-x^2}} \Rightarrow \underline{\underline{x = 1}}$$

$0 = 1 - x$

$\frac{df}{dx}$ undefined

$\underline{\underline{x = 0}}$, $\underline{\underline{x = 2}}$

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54) $y = 2\sqrt{1-x^2} + \sin^{-1}x$

domain of $\sqrt{1-x^2}$: $[-1, 1]$

domain of $\sin^{-1}x$: $[-1, 1]$

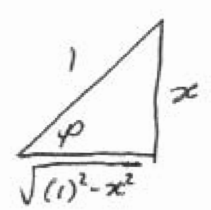
domain of y : $[-1, 1]$

$\varphi = \sin^{-1}x$
$\Downarrow$
$\sin\varphi = x$

$\left[\cos\varphi \frac{d\varphi}{dx}\right] = [1]$

$$\frac{d\varphi}{dx} = \frac{1}{\cos\varphi} = \frac{1}{\left(\frac{\sqrt{(1)^2 - x^2}}{1}\right)}$$

$$\frac{d\varphi}{dx} = \frac{1}{\sqrt{1-x^2}}$$



54) continued

$$y = 2(1-x^2)^{\frac{1}{2}} + \sin^{-1}x$$

$$\frac{dy}{dx} = 2\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right] + \left[\frac{1}{\sqrt{1-x^2}}\right] = \frac{-2x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = \frac{1-2x}{\sqrt{1-x^2}}$$

Critical Points

$$0 = \frac{1-2x}{\sqrt{1-x^2}}$$

$$0 = 1-2x$$

$$2x = 1$$

$$\underline{\underline{x = \frac{1}{2}}}$$

$\frac{dy}{dx}$ undefined

$$0 = \sqrt{1-x^2}$$

$$0 = 1-x^2$$

$$0 = (1+x)(1-x)$$

$1+x=0$, $\underline{\underline{x=-1}}$ | $1-x=0$, $\underline{\underline{1=x}}$

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56) $y = x - 3x^{2/3}$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = [1] - 3\left[\frac{2}{3}x^{-\frac{1}{3}}\right] = 1 - \frac{2}{\sqrt[3]{x}}$$

Critical Points

$$0 = 1 - \frac{2}{\sqrt[3]{x}}$$

$$\frac{2}{\sqrt[3]{x}} = 1$$

$$2 = \sqrt[3]{x}$$

$$\underline{\underline{8 = x}}$$

$\frac{dy}{dx}$ undefined

$$\sqrt[3]{x} = 0$$

$$\underline{\underline{x = 0}}$$

58) $y = x^{\frac{2}{3}}(x^2-4) = (\sqrt[3]{x})^2(x^2-4)$ domain: $(-\infty, \infty)$

$$\frac{dy}{dx} = \left(x^{\frac{2}{3}}\right)\left[2x\right] + (x^2-4)\left[\tfrac{2}{3}x^{-\frac{1}{3}}\right] = (\sqrt[3]{x})^2(2x) + \frac{2(x^2-4)}{3(\sqrt[3]{x})}$$

$$= \left(\frac{(\sqrt[3]{x})^2(2x)}{1}\right)\left(\frac{3(\sqrt[3]{x})}{3(\sqrt[3]{x})}\right) + \frac{2(x^2-4)}{3(\sqrt[3]{x})} = \frac{6x^2+2x^2-8}{3(\sqrt[3]{x})} = \frac{8x^2-8}{3(\sqrt[3]{x})} = \frac{8(x^2-1)}{3(\sqrt[3]{x})}$$

Critical Points

$$0 = \frac{8(x^2-1)}{3(\sqrt[3]{x})}$$

$$0 = 8(x^2-1)$$

$\Rightarrow$ $0 = 8(x+1)(x-1)$

$x+1=0 \;|\; x-1=0$

$x=-1 \;|\; x=1$

$\frac{dy}{dx}$ undefined

$3(\sqrt[3]{x}) = 0$

$\sqrt[3]{x} = 0$

$\underline{\underline{x=0}}$

critical point	derivative	value	extremum
$x=-1$	$\frac{dy}{dx}\Big\|_{x=-1} = \frac{8((-1)^2-1)}{3(\sqrt[3]{(-1)})} = 0$	$y\Big\|_{x=-1} = (\sqrt[3]{(-1)})^2((-1)^2-4)$ $= (-1)^2(1-4) = -3$	minimum
$x=0$	$\frac{dy}{dx}\Big\|_{x=0} = \frac{8((0)^2-1)}{3(\sqrt[3]{(0)})}$ undefined	$y\Big\|_{x=0} = (\sqrt[3]{(0)})^2((0)^2-4) = 0$	Local Max.
$x=1$	$\frac{dy}{dx}\Big\|_{x=1} = \frac{8((1)^2-1)}{3(\sqrt[3]{(1)})} = 0$	$y\Big\|_{x=1} = (\sqrt[3]{(1)})^2((1)^2-4)$ $= (1)^2(1-3) = -3$	minimum

60) $y = x^2\sqrt{3-x} = x^2(3-x)^{\frac{1}{2}}$

domain of x^2 : $(-\infty, \infty)$
domain of $\sqrt{3-x}$: $(-\infty, 3]$
domain of y : $(-\infty, 3]$

$$\frac{dy}{dx} = (x^2)\left[\tfrac{1}{2}(3-x)^{-\frac{1}{2}}(-1)\right] + \left((3-x)^{\frac{1}{2}}\right)[2x]$$

$$= \frac{-x^2}{2\sqrt{3-x}} + 2x\sqrt{3-x} = \frac{-x^2}{2\sqrt{3-x}} + \left(\frac{2x\sqrt{3-x}}{1}\right)\left(\frac{2\sqrt{3-x}}{2\sqrt{3-x}}\right)$$

$$= \frac{-x^2+4x(3-x)}{2\sqrt{3-x}} = \frac{-x^2+12x-4x^2}{2\sqrt{3-x}} = \underline{\underline{\frac{12x-5x^2}{2\sqrt{3-x}}}}$$

Critical Points

$$0 = \frac{12x-5x^2}{2\sqrt{3-x}} \Rightarrow \quad 0 = x(12-5x)$$

$x = 0$ | $12-5x = 0$, $12 = 5x$, $\frac{12}{5} = x$

$0 = 12x - 5x^2$

$\frac{dy}{dx}$ undefined

$3 - x = 0$

$3 = x$

Critical Point	derivative	value	extremum
$x=0$	$\left.\frac{dy}{dx}\right\rvert_{x=0} = \frac{12(0)-5(0)^2}{2\sqrt{3-(0)}} = 0$	$y\rvert_{x=0} = (0)^2\sqrt{3-(0)} = 0$	minimum
$x=\frac{12}{5}$	$\left.\frac{dy}{dx}\right\rvert_{x=\frac{12}{5}} = \frac{12\left(\frac{12}{5}\right)-5\left(\frac{12}{5}\right)^2}{2\sqrt{3-\left(\frac{12}{5}\right)}} = \frac{\left(\frac{12}{5}\right)\left\{12-5\left(\frac{12}{5}\right)\right\}}{2\sqrt{\frac{15}{5}-\frac{12}{5}}} = \frac{\left(\frac{12}{5}\right)\{0\}}{2\sqrt{\frac{3}{5}}} = 0$	$y\rvert_{x=\frac{12}{5}} = \left(\frac{12}{5}\right)^2\sqrt{3-\left(\frac{12}{5}\right)} = \left(\frac{144}{25}\right)\sqrt{\frac{15}{5}-\frac{12}{5}} = \left(\frac{144}{25}\right)\sqrt{\frac{3}{5}} > 0$	Local Max.
$x=3$	$\left.\frac{dy}{dx}\right\rvert_{x=3} = \frac{12(3)-5(3)^2}{2\sqrt{3-(3)}}$ undefined	$y\rvert_{x=3} = (3)^2\sqrt{3-(3)} = (9)(\sqrt{0}) = 0$	minimum

62) $y = \begin{cases} 3 - x & x < 0 \\ 3 + 2x - x^2 & x \geq 0 \end{cases}$ $\quad \frac{dy}{dx} = \begin{cases} [-1] = -1 & x < 0 \\ 2[1] - [2x] = 2 - 2x & x > 0 \end{cases}$

Critical Points:

$0 = \frac{dy}{dx} = 2 - 2x$

$0 = 2 - 2x$

$2x = 2$

$x = 1$

$\lim_{x \to 0^-} \frac{dy}{dx} = \lim_{x \to 0^-} (-1) = -1$

$\lim_{x \to 0^+} \frac{dy}{dx} = \lim_{x \to 0^+} (2 - 2x) = 2 - 2(0^+) = 2 - 0 = 2$

since $x = 0$ is the border of condition of our piecewise function and $\lim_{x \to 0^-} \frac{dy}{dx} \neq \lim_{x \to 0^+} \frac{dy}{dx}$, $x = 0$ is a critical point

Critical point	derivative	value	extremum
$x = 0$	$\left.\frac{dy}{dx}\right\|_{x=0} = ?$ undefined $-1 = \lim_{x \to 0^-} \frac{dy}{dx}$ $\quad \lim_{x \to 0^+} \frac{dy}{dx} = 2$	$y\|_{x=0} = 3 + 2(0) - (0)^2 = 3$	local min
$x = 1$	$\left.\frac{dy}{dx}\right\|_{x=1} = 2 - 2(1) = 2 - 2 = 0$	$y\|_{x=1} = 3 + 2(1) - (1)^2$ $= 3 + 2 - 1 = 4$	Local Max.

64)

$$y = \begin{cases} \frac{-1}{4}x^2 - \frac{1}{2}x + \frac{15}{4} & x \le 1 \\ x^3 - 6x^2 + 8x & x > 1 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} \frac{-1}{4}[2x] - \frac{1}{2}[1] + [0] \\ = \frac{-1}{2}x - \frac{1}{2} & x < 1 \\ [3x^2] - 6[2x] + 8[1] \\ = 3x^2 - 12x + 8 & x > 1 \end{cases}$$

Critical Points

$$0 = \frac{dy}{dx}$$

$$0 = \frac{-1}{2}x - \frac{1}{2}$$

$$0 = \frac{-1}{2}(x+1)$$

$$0 = x + 1$$

$$x = -1$$

$$0 = 3x^2 - 12x + 8$$

not factorable

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(8)}}{2(3)} = \frac{12 \pm \sqrt{(4)^2((3)^2 - (3)(2))}}{2(3)}$$

$$= \frac{12 \pm \sqrt{(4)^2}\sqrt{9-6}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = \frac{2(6 \pm 2\sqrt{3})}{6} = \frac{6 \pm 2\sqrt{3}}{3}$$

$$x = \frac{6 - 2\sqrt{3}}{3} < 1 \qquad x = \frac{6 + 2\sqrt{3}}{3}$$

discard because it is outside of $x > 1$

testing on border value $x = 1$

$$\lim_{x \to 1^-} \frac{dy}{dx} = \lim_{x \to 1^-} \left(\frac{-1}{2}x - \frac{1}{2}\right) = \frac{-1}{2}(1^-) - \frac{1}{2} = \frac{-1}{2} - \frac{1}{2} = -1$$

$$\lim_{x \to 1^+} \frac{dy}{dx} = \lim_{x \to 1^+} (3x^2 - 12x + 8) = 3(1^+)^2 - 12(1^+) + 8 = 3 - 12 + 8 = -1$$

since $-1 = \lim_{x \to 1^-} \frac{dy}{dx} = \lim_{x \to 1^+} \frac{dy}{dx} = -1$, $\lim_{x \to 1} \frac{dy}{dx} = -1$

and $\left.\frac{dy}{dx}\right|_{x=1} = -1$

64) continued

$$\lim_{x\to 1^-} y = \lim_{x\to 1^-}\left(\frac{-1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}\right) = \frac{-1}{4}(1^-)^2 - \frac{1}{2}(1^-) + \frac{15}{4} = \frac{-1}{4} - \frac{1}{2} + \frac{15}{4}$$

$$= \frac{-1}{4} - \frac{2}{4} + \frac{15}{4} = \frac{12}{4} = 3$$

$$\lim_{x\to 1^+} y = \lim_{x\to 1^+}\left(x^3 - 6x^2 + 8x\right) = (1^+)^3 - 6(1^+)^2 + 8(1^+) = 1 - 6 + 8 = 3$$

$$y\Big|_{x=1} = (1)^3 - 6(1)^2 + 8(1) = 1 - 6 + 8 = 3$$

$$3 = \lim_{x\to 1^-} y = y\Big|_{x=1} = \lim_{x\to 1^+} y = 3$$, y is continuous at $x=1$

Since y is continuous at $x=1$ and $\frac{dy}{dx}\Big|_{x=1} = -1$ (which is not 0 or undefined), $x=1$ is not a critical point.

critical point	derivative	value	extremum
$x=-1$	$\frac{dy}{dx}\Big\|_{x=-1} = \frac{-1}{2}(-1) - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$	$y\Big\|_{x=-1} = \frac{-1}{4}(-1)^2 - \frac{1}{2}(-1) + \frac{15}{4}$ $= \frac{-1}{4} + \frac{1}{2} + \frac{15}{4} = \frac{-1}{4} + \frac{2}{4} + \frac{15}{4} = \frac{16}{4} = 4$	Local Max.
$x = \frac{6+2\sqrt{3}}{3}$	$\frac{dy}{dx}\Big\|_{x=\frac{6+2\sqrt{3}}{3}} = 3\left(\frac{6+2\sqrt{3}}{3}\right)^2 - 12\left(\frac{6+2\sqrt{3}}{3}\right) + 8$ $= \frac{1}{3}(6+2\sqrt{3})^2 - 4(6+2\sqrt{3}) + 8$ $= \frac{1}{3}(36+24\sqrt{3}+12) - 24 - 8\sqrt{3} + 8$ $= (12+8\sqrt{3}+4) - 16 - 8\sqrt{3}$ $= (16+8\sqrt{3}) - 16 - 8\sqrt{3} = 0$	$y\Big\|_{x=\frac{6+2\sqrt{3}}{3}} = \left(\frac{6+2\sqrt{3}}{3}\right)^3 - 6\left(\frac{6+2\sqrt{3}}{3}\right)^2 + 8\left(\frac{6+2\sqrt{3}}{3}\right)$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{1}{9}(6+2\sqrt{3})^2 - 3(6+2\sqrt{3})^2 + 8\right\}$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{1}{9}(36+24\sqrt{3}+12) - 18 - 6\sqrt{3} + 8\right\}$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{1}{9}(48+24\sqrt{3}) - 10 - 6\sqrt{3}\right\}$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{48}{9} + \frac{24\sqrt{3}}{9} - 10 - 6\sqrt{3}\right\}$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{16}{3} + \frac{8\sqrt{3}}{3} - \frac{30}{3} - \frac{18\sqrt{3}}{3}\right\}$ $= \left(\frac{6+2\sqrt{3}}{3}\right)\left\{\frac{-14}{3} - \frac{10\sqrt{3}}{3}\right\} < 0$	local min.

72) If $f(c)$ is a Local Maximum value of f, then $f(x) \leq f(c)$ for all x in some open interval (a, b) containing c. Since f is even, $f(-x) = f(x) \leq f(c) = f(-c)$ for all $-x$ in the open interval $(-b, -a)$ containing $-c$. Therefore it is clear that $f(-c)$ is a Local Maximum value. This is also justified from the graph of f because an even function is symmetric about the y-axis.

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74) If there are no boundary points or critical points, the function will have no extreme values in its domain.

Such functions are linear functions $f(x) = mx + b$, where $m \neq 0$, for $(-\infty, \infty)$. Note: this function's $\frac{df}{dx} \neq 0$ and not undefined; also there are no boundary points on simple linear function.