MATH 20100 section 3.11

Definition: If f is differentiable at $x=a$, then the approximating function

$$L(x) = f(a) + \left\{ \frac{df}{dx}\Big|_{x=a} \right\}(x-a) \qquad L(x) = f(a) + f'(a)(x-a)$$

is the linearization of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the standard linear approximation of f at a. The point $x=a$ is the center of the approximation.

Definition Let $y=f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = \left\{ \frac{df}{dx} \right\} dx \qquad dy = f'(x)\,dx$$

Change in $y=f(x)$ near $x=a$

If $y=f(x)$ is differentiable at $x=a$ and x changes from a to $a+\Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

in which $\varepsilon \to 0$ as $\Delta x \to 0$.

2) $f(x) = \sqrt{x^2+9}$, $a = -4$

$f(x) = (x^2+9)^{\frac{1}{2}}$

$$\frac{df}{dx} = \left[\frac{1}{2}(x^2+9)^{-\frac{1}{2}}(2x)\right] = \frac{2x}{2\sqrt{x^2+9}} = \frac{x}{\sqrt{x^2+9}}$$

$$f(-4) = \sqrt{(-4)^2+9} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\left.\frac{df}{dx}\right|_{x=-4} = \frac{(-4)}{\sqrt{(-4)^2+9}} = \frac{-4}{\sqrt{16+9}} = \frac{-4}{\sqrt{25}} = \frac{-4}{5}$$

$$L(x) = f(-4) + \left\{\left.\frac{df}{dx}\right|_{x=-4}\right\}(x-(-4)) = (5) + \left\{\frac{-4}{5}\right\}(x+4)$$

$$= 5 - \frac{4}{5}x - \frac{16}{5} = \frac{-4}{5}x + \frac{9}{5}$$

---

4) $f(x) = \sqrt[3]{x}$ $a = -8$

$f(x) = x^{\frac{1}{3}}$

$$\frac{df}{dx} = \left[\frac{1}{3}x^{-\frac{2}{3}}\right] = \frac{1}{3(\sqrt[3]{x})^2}$$

$$f(-8) = \sqrt[3]{(-8)} = -2$$

$$\left.\frac{df}{dx}\right|_{x=-8} = \frac{1}{3(\sqrt[3]{(-8)})^2} = \frac{1}{3(-2)^2} = \frac{1}{3(4)} = \frac{1}{12}$$

$$L(x) = f(-8) + \left\{\left.\frac{df}{dx}\right|_{x=-8}\right\}(x-(-8)) = (-2) + \left\{\frac{1}{12}\right\}(x+8)$$

$$= -2 + \frac{1}{12}x + \frac{8}{12} = -2 + \frac{1}{12}x + \frac{2}{3} = \underline{\underline{\frac{1}{12}x - \frac{4}{3}}}$$

6) at $x=0$

a) $f(x) = \sin x$ $\qquad f(0) = \sin(0) = 0$

$$\frac{df}{dx} = [\cos x(1)] = \cos x \qquad \left.\frac{df}{dx}\right|_{x=0} = \cos(0) = 1$$

$$L(x) = f(0) + \left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0)) = (0) + \{1\}(x) = \underline{\underline{x}}$$

---

b) $f(x) = \cos x$ $\qquad f(0) = \cos(0) = 1$

$$\frac{df}{dx} = [-\sin x(1)] = -\sin x \qquad \left.\frac{df}{dx}\right|_{x=0} = -\sin(0) = 0$$

$$L(x) = f(0) + \left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0)) = (1) + \{0\}(x) = \underline{\underline{1}}$$

---

c) $f(x) = \tan x$ $\qquad f(0) = \tan(0) = 0$

$$\frac{df}{dx} = [\sec^2 x(1)] = \sec^2 x \qquad \left.\frac{df}{dx}\right|_{x=0} = \sec^2(0) = (1)^2 = 1$$

$$L(x) = f(0) + \left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0)) = (0) + \{1\}(x) = \underline{\underline{x}}$$

---

d) $f(x) = e^x$ $\qquad f(0) = e^{(0)} = 1$

$$\frac{df}{dx} = [e^x(1)] = e^x \qquad \left.\frac{df}{dx}\right|_{x=0} = e^{(0)} = 1$$

$$L(x) = f(0) + \left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0)) = (1) + \{1\}(x) = \underline{\underline{1+x = x+1}}$$

---

e) $f(x) = \ln(1+x)$ $\qquad f(0) = \ln(1+(0)) = \ln(1) = 0$

$$\frac{df}{dx} = \left[\frac{1}{1+x}(1)\right] = \frac{1}{1+x} \qquad \left.\frac{df}{dx}\right|_{x=0} = \frac{1}{1+(0)} = \frac{1}{1} = 1$$

$$L(x) = f(0) + \left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0)) = (0) + \{1\}(x) = \underline{\underline{x}}$$

8) $f(x)=x^{-1}=\frac{1}{x}$ $\quad a=0.9=\frac{9}{10}\Rightarrow$ use $a=1$

$\frac{df}{dx}=\left[-1x^{-2}\right]=\frac{-1}{x^2}$ $\quad f(1)=\frac{1}{(1)}=1$ $\quad \left.\frac{df}{dx}\right|_{x=1}=\frac{-1}{(1)^2}=\frac{-1}{1}=-1$

$$L(x)=f(1)+\left\{\left.\frac{df}{dx}\right|_{x=1}\right\}(x-(1))=(1)+\{-1\}(x-1)=1-x+1=\underline{\underline{-x+2}}$$

---

10) $f(x)=1+x$ $\quad a=8.1\Rightarrow$ use $a=8$

$\frac{df}{dx}=[1]=1$ $\quad f(8)=1+(8)=9$ $\quad \left.\frac{df}{dx}\right|_{x=8}=1$

$$L(x)=f(8)+\left\{\left.\frac{df}{dx}\right|_{x=8}\right\}(x-(8))=(9)+\{1\}(x-8)=9+x-8=\underline{\underline{x+1}}$$

---

12) $f(x)=\frac{x}{x+1}$ $\quad a=1.3\Rightarrow$ use $a=1$

$$\frac{df}{dx}=\frac{(x+1)[1]-(x)[1]}{(x+1)^2}=\frac{x+1-x}{(x+1)^2}=\frac{1}{(x+1)^2}$$

$f(1)=\frac{(1)}{(1)+1}=\frac{1}{2}$ $\quad \left.\frac{df}{dx}\right|_{x=1}=\frac{1}{((1)+1)^2}=\frac{1}{(2)^2}=\frac{1}{4}$

$$L(x)=f(1)+\left\{\left.\frac{df}{dx}\right|_{x=1}\right\}(x-(1))=\left(\tfrac{1}{2}\right)+\left\{\tfrac{1}{4}\right\}(x-1)=\tfrac{1}{2}+\tfrac{1}{4}x-\tfrac{1}{4}=\underline{\underline{\tfrac{1}{4}x+\tfrac{1}{4}}}$$

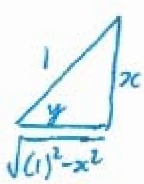
---

14) $f(x)=\sin^{-1}x$ $\quad a=\frac{\pi}{12}\Rightarrow$ use $a=0$

$y=\sin^{-1}x$

$\Downarrow$

$\sin y=x$



$\left[\cos y\frac{dy}{dx}\right]=[1]$

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\left(\frac{\sqrt{(1)^2-x^2}}{1}\right)}=\frac{1}{\sqrt{1-x^2}}$$

$\frac{df}{dx}=\frac{1}{\sqrt{1-x^2}}$ $\quad f(0)=\sin^{-1}(0)=0$

$\left.\frac{df}{dx}\right|_{x=0}=\frac{1}{\sqrt{1-(0)^2}}=\frac{1}{\sqrt{1}}=1$

$$L(x)=f(0)+\left\{\left.\frac{df}{dx}\right|_{x=0}\right\}(x-(0))$$
$$=(0)+\{1\}(x)=\underline{\underline{x}}$$

16) $(1+x)^k \approx 1+kx$

a) $f(x) = (1-x)^6 = (1+(-x))^6 \approx 1+(6)(-x) = \underline{\underline{1-6x}}$

b) $f(x) = \dfrac{2}{1-x} = 2\{(1-x)^{-1}\} = 2\{(1+(-x))^{-1}\}$

$\approx 2\{1+(-1)(-x)\} = 2\{1+x\} = \underline{\underline{2+2x}}$

c) $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} \approx 1+\left(\frac{-1}{2}\right)(x) = \underline{\underline{1-\frac{1}{2}x}}$

d) $f(x) = \sqrt{2+x^2} = \sqrt{2\left(1+\frac{x^2}{2}\right)} = \sqrt{2}\sqrt{1+\frac{x^2}{2}} = \sqrt{2}\left\{\left(1+\left(\frac{x^2}{2}\right)\right)^{\frac{1}{2}}\right\}$

$\approx \sqrt{2}\left\{1+\left(\frac{1}{2}\right)\left(\frac{x^2}{2}\right)\right\} = \sqrt{2}\left\{1+\frac{1}{4}x^2\right\} = \underline{\underline{\sqrt{2}+\frac{\sqrt{2}}{4}x^2}}$

e) $f(x) = (4+3x)^{\frac{1}{3}} = \sqrt[3]{4+3x} = \sqrt[3]{4\left(1+\frac{3}{4}x\right)} = \sqrt[3]{4}\sqrt[3]{1+\frac{3}{4}x}$

$= \sqrt[3]{4}\left\{\left(1+\left(\frac{3}{4}x\right)\right)^{\frac{1}{3}}\right\} \approx \sqrt[3]{4}\left\{1+\left(\frac{1}{3}\right)\left(\frac{3}{4}x\right)\right\} = \sqrt[3]{4}\left\{1+\frac{1}{4}x\right\}$

$= \underline{\underline{\sqrt[3]{4}+\frac{\sqrt[3]{4}}{4}x = \sqrt[3]{4}+\frac{1}{(\sqrt[3]{4})^2}x}}$

f) $f(x) = \sqrt[3]{\left(1-\frac{x}{2+x}\right)^2} = \left(1+\left(\frac{-x}{2+x}\right)\right)^{\frac{2}{3}}$

$\approx 1+\left(\frac{2}{3}\right)\left(\frac{-x}{2+x}\right) = \underline{\underline{1-\frac{2x}{6+3x}}}$

18) $f(x) = \sqrt{x+1} + \sin x \quad \text{at } x=0$

let $g(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}} \qquad g(0) = \sqrt{(0)+1} = \sqrt{1} = 1$

$\frac{dg}{dx} = \left[\frac{1}{2}(x+1)^{-\frac{1}{2}}(1)\right] = \frac{1}{2\sqrt{x+1}} \qquad \frac{dg}{dx}\Big|_{x=0} = \frac{1}{2\sqrt{(0)+1}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$L_g(x) = g(0) + \left\{\frac{dg}{dx}\Big|_{x=0}\right\}(x-(0)) = (1) + \left\{\frac{1}{2}\right\}(x) = 1 + \frac{1}{2}x$

let $h(x) = \sin x \qquad h(0) = \sin(0) = 0$

$\frac{dh}{dx} = [\cos x(1)] = \cos x \qquad \frac{dh}{dx}\Big|_{x=0} = \cos(0) = 1$

$L_h(x) = h(0) + \left\{\frac{dh}{dx}\Big|_{x=0}\right\}(x-(0)) = (0) + \{1\}(x) = x$

now for $L_f(x) \qquad f(0) = \sqrt{(0)+1} + \sin(0) = \sqrt{1} + 0 = 1$

$\frac{df}{dx} = \left[\frac{1}{2\sqrt{x+1}}\right] + [\cos x] = \frac{1}{2\sqrt{x+1}} + \cos x$

$\frac{df}{dx}\Big|_{x=0} = \frac{1}{2\sqrt{(0)+1}} + \cos(0) = \frac{1}{2\sqrt{1}} + (1) = \frac{1}{2} + 1 = \frac{3}{2}$

$L_f(x) = f(0) + \left\{\frac{df}{dx}\Big|_{x=0}\right\}(x-(0)) = (1) + \left\{\frac{3}{2}\right\}(x) = 1 + \frac{3}{2}x$

$L_g(x) + L_h(x) = \left(1 + \frac{1}{2}x\right) + (x) = 1 + \frac{3}{2}x = L_f(x)$

20) $y = x\sqrt{1-x^2} = (x)\left((1-x^2)^{\frac{1}{2}}\right)$

$$\frac{dy}{dx} = (x)\left[\frac{1}{2}(1-x^2)(-2x)\right] + \left((1-x^2)^{\frac{1}{2}}\right)\left[1\right] = \frac{-2x^2}{2\sqrt{1-x^2}} + \sqrt{1-x^2}$$

$$= \frac{-x^2}{\sqrt{1-x^2}} + \left(\frac{\sqrt{1-x^2}}{1}\right)\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right) = \frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

$$dy = \left\{\frac{dy}{dx}\right\} dx = \underline{\underline{\left\{\frac{1-2x^2}{\sqrt{1-x^2}}\right\} dx}}$$

---

22) $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})} = \frac{2x^{\frac{1}{2}}}{3(1+x^{\frac{1}{2}})} = \frac{2}{3}\left\{\frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}}\right\}$

$$\frac{dy}{dx} = \frac{2}{3}\left\{\frac{(1+x^{\frac{1}{2}})\left[\frac{1}{2}x^{-\frac{1}{2}}\right] - (x^{\frac{1}{2}})\left[\frac{1}{2}x^{-\frac{1}{2}}\right]}{(1+x^{\frac{1}{2}})^2}\right\}$$

$$= \frac{2}{3}\left\{\frac{\frac{(1+\sqrt{x})}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}}}{(1+\sqrt{x})^2}\right\} = \frac{2}{3}\left\{\frac{\frac{(1+\sqrt{x})-\sqrt{x}}{2\sqrt{x}}}{(1+\sqrt{x})^2}\right\} = \frac{2}{3}\left\{\frac{1+\sqrt{x}-\sqrt{x}}{2\sqrt{x}(1+\sqrt{x})^2}\right\}$$

$$= \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}$$

$$dy = \left\{\frac{dy}{dx}\right\} dx = \underline{\underline{\left\{\frac{1}{3\sqrt{x}(1+\sqrt{x})^2}\right\} dx}}$$

---

24) $xy^2 - 4x^{\frac{3}{2}} - y = 0$

$$\left\{(x)\left[2y\frac{dy}{dx}\right] + (y^2)[1]\right\} - 4\left[\frac{3}{2}x^{\frac{1}{2}}\right] - \left[1\frac{dy}{dx}\right] = 0$$

$$2xy\frac{dy}{dx} + y^2 - 6\sqrt{x} - \frac{dy}{dx} = 0$$

24) continued

$$y^2 - 6\sqrt{x} = \frac{dy}{dx} - 2xy\frac{dy}{dx} \qquad \frac{y^2-6\sqrt{x}}{1-2xy} = \frac{dy}{dx}$$

$$y^2 - 6\sqrt{x} = \frac{dy}{dx}(1-2xy)$$

$$dy = \left\{\frac{dy}{dx}\right\}dx = \underline{\underline{\left\{\frac{y^2-6\sqrt{x}}{1-2xy}\right\}dx}}$$

---

26) $y = \cos(x^2)$

$$\frac{dy}{dx} = [-\sin(x^2)(2x)] = -2x\sin(x^2)$$

$$dy = \left\{\frac{dy}{dx}\right\}dx = \underline{\underline{\{-2x\sin(x^2)\}dx}}$$

---

28) $y = \sec(x^2-1)$

$$\frac{dy}{dx} = [\sec(x^2-1)\tan(x^2-1)(2x)] = 2x\sec(x^2-1)\tan(x^2-1)$$

$$dy = \left\{\frac{dy}{dx}\right\}dx = \underline{\underline{\{2x\sec(x^2-1)\tan(x^2-1)\}dx}}$$

---

30) $y = 2\cot\left(\frac{1}{\sqrt{x}}\right) = 2\cot\left(x^{-\frac{1}{2}}\right)$

$$\frac{dy}{dx} = 2\left[-\csc^2\left(x^{-\frac{1}{2}}\right)\left(\frac{-1}{2}x^{-\frac{3}{2}}\right)\right] = \frac{2\csc^2\left(\frac{1}{\sqrt{x}}\right)}{2(\sqrt{x})^3} = \frac{\csc^2\left(\frac{1}{\sqrt{x}}\right)}{(\sqrt{x})^3}$$

$$dy = \left\{\frac{dy}{dx}\right\}dx = \underline{\underline{\left\{\frac{\csc^2\left(\frac{1}{\sqrt{x}}\right)}{(\sqrt{x})^3}\right\}dx}}$$

32) $y = xe^{-x}$

$$\frac{dy}{dx} = (x)\left[e^{-x}(-1)\right] + \left(e^{-x}\right)[1] = e^{-x}\{-x+1\} = e^{-x}(1-x) = \frac{1-x}{e^{x}}$$

$$dy = \left\{\frac{dy}{dx}\right\} dx = \underline{\underline{\left\{\frac{1-x}{e^{x}}\right\} dx}}$$

---

34) $y = \ln\left(\frac{x+1}{\sqrt{x-1}}\right) = \ln(x+1) - \ln\left(\sqrt{x-1}\right) = \ln(x+1) - \frac{1}{2}\ln(x-1)$

$$\frac{dy}{dx} = \left[\frac{1}{x+1}(1)\right] - \frac{1}{2}\left[\frac{1}{x-1}(1)\right] = \frac{1}{x+1} - \frac{1}{2(x-1)}$$

$$dy = \left\{\frac{dy}{dx}\right\} dx = \underline{\underline{\left\{\frac{1}{x+1} - \frac{1}{2(x-1)}\right\} dx}}$$

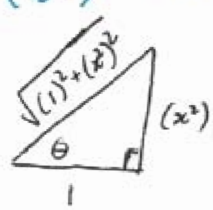
---

36) $y = \cot^{-1}\left(\frac{1}{x^2}\right) + \cos^{-1} 2x$

$\theta = \cot^{-1}\left(\frac{1}{x^2}\right)$

$\Downarrow$

$\cot\theta = \frac{1}{x^2} = x^{-2}$



$$\left[-\csc^2\theta \frac{d\theta}{dx}\right] = \left[-2x^{-3}\right]$$

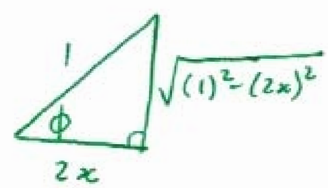
$$\frac{d\theta}{dx} = \frac{-2}{x^3(-\csc^2\theta)} = \frac{2}{x^3\left(\frac{\sqrt{(1)^2+(x^2)^2}}{x^2}\right)^2}$$

$$= \frac{2}{\frac{((1)^2+(x^2))}{x}} = \frac{2x}{1+x^4}$$

$\phi = \cos^{-1} 2x$

$\Downarrow$

$\cos\phi = 2x$



$$\left[-\sin\phi \frac{d\phi}{dx}\right] = [2]$$

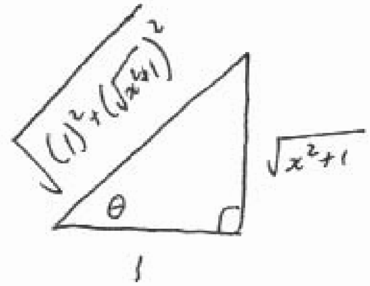
$$\frac{d\phi}{dx} = \frac{2}{-\sin\phi} = \frac{-2}{\left(\frac{\sqrt{(1)^2-(2x)^2}}{1}\right)} = \frac{-2}{\sqrt{1-4x^2}}$$

$$\frac{dy}{dx} = \left[\frac{2x}{1+x^4}\right] + \left[\frac{-2}{\sqrt{1-4x^2}}\right] = \frac{2x}{1+x^4} - \frac{2}{\sqrt{1-4x^2}}$$

$$dy = \left\{\frac{dy}{dx}\right\} dx = \underline{\underline{\left\{\frac{2x}{1+x^4} - \frac{2}{\sqrt{1-4x^2}}\right\} dx}}$$

38) $y = e^{\tan^{-1}\sqrt{x^2+1}}$

$\theta = \tan^{-1}\sqrt{x^2+1}$



$\Downarrow$

$\tan\theta = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$$\left[\sec^2\theta \frac{d\theta}{dx}\right] = \left[\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\right]$$

$$\frac{d\theta}{dx} = \frac{2x}{2\sqrt{x^2+1}\ \sec^2\theta} = \frac{x}{\sqrt{x^2+1}\left(\frac{\sqrt{(1)^2+(\sqrt{x^2+1})^2}}{1}\right)^2} = \frac{x}{\sqrt{x^2+1}\,(1+(x^2+1))}$$

$$= \frac{x}{(x^2+2)\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \left[e^{\tan^{-1}\sqrt{x^2+1}}\left(\frac{x}{(x^2+2)\sqrt{x^2+1}}\right)\right] = \frac{x\,e^{\tan^{-1}\sqrt{x^2+1}}}{(x^2+2)\sqrt{x^2+1}}$$

$$dy = \left\{\frac{dy}{dx}\right\}dx = \underline{\underline{\left\{\frac{x\,e^{\tan^{-1}\sqrt{x^2+1}}}{(x^2+2)\sqrt{x^2+1}}\right\}dx}}$$

---

46) $V = x^3$ $\qquad x_0$ to $x_0 + dx$

$$\frac{dV}{dx} = [3x^2] = 3x^2 \qquad \left.\frac{dV}{dx}\right|_{x=x_0} = 3(x_0)^2$$

$$dV = \left\{\left.\frac{dV}{dx}\right|_{x=x_0}\right\}dx = \{3(x_0)^2\}dx = \underline{\underline{3(x_0)^2 dx}}$$

48) $S = \pi r\sqrt{r^2+h^2} = \pi r(r^2+h^2)^{\frac{1}{2}}$ r_0 to r_0+dr (h, constant)

$$\frac{dS}{dr} = (\pi r)\left[\tfrac{1}{2}(r^2+h^2)^{-\frac{1}{2}}(2r)\right] + \left((r^2+h^2)^{\frac{1}{2}}\right)[\pi]$$

$$= \frac{2\pi r^2}{2\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2} = \frac{\pi r^2}{\sqrt{r^2+h^2}} + \left(\frac{\pi\sqrt{r^2+h^2}}{1}\right)\left(\frac{\sqrt{r^2+h^2}}{\sqrt{r^2+h^2}}\right)$$

$$= \frac{\pi r^2 + \pi(r^2+h^2)}{\sqrt{r^2+h^2}} = \frac{2\pi r^2 + \pi h^2}{\sqrt{r^2+h^2}} \qquad \left.\frac{dS}{dr}\right|_{r=r_0} = \frac{2\pi(r_0)^2+\pi h^2}{\sqrt{(r_0)^2+h^2}}$$

$$dS = \left\{\left.\frac{dS}{dr}\right|_{r=r_0}\right\}dr = \underline{\underline{\left\{\frac{2\pi(r_0)^2+\pi h^2}{\sqrt{(r_0)^2+h^2}}\right\}}}dr \quad (h,\text{ constant})$$

---

50) $S = 2\pi r h$ h_0 to h_0+dh (r, constant)

$$\frac{dS}{dh} = 2\pi r[1] = 2\pi r \qquad \left.\frac{dS}{dh}\right|_{h=h_0} = 2\pi r$$

$$dS = \left\{\left.\frac{dS}{dh}\right|_{h=h_0}\right\}dh = \{2\pi r\}dh = \underline{\underline{2\pi r\, dh}}$$

---

52) $2r = 10$ in $\quad dC = 2$ in

$C = 2\pi r$

$\frac{dC}{dr} = 2\pi[1] = 2\pi$

$dC = \left\{\frac{dC}{dr}\right\}dr$

$dC = 2\pi\, dr$

$(2) = 2\pi\, dr$

$\frac{2}{2\pi} = dr$

$\frac{1}{\pi} = dr$

so radius increased $dr = \frac{1}{\pi}$ in

therefore diameter increased $2\,dr = 2\left(\frac{1}{\pi}\right)$ in $= \underline{\underline{\frac{2}{\pi}\text{ in}}}$

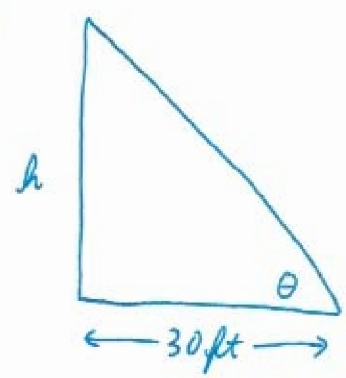
for cross-sectional area; $A = \pi r^2$

$2r = 10$ in $\Rightarrow r = 5$ in $\qquad \frac{dA}{dr} = \pi[2r] = 2\pi r$

$dA = \left\{\frac{dA}{dr}\right\}dr = \{2\pi r\}dr = 2\pi r\, dr$

$dA = 2\pi(5\text{ in})\left(\frac{1}{\pi}\text{ in}\right) = \underline{\underline{\frac{10}{\pi}\text{ in}^2}}$

54)



$\theta = 75° = (75°)\left(\frac{\pi}{180°}\right) = \frac{75\pi}{180} = \frac{15\pi}{36} = \frac{5\pi}{12}$

$\frac{5\pi}{12} = \frac{1}{2}\left(\frac{5\pi}{6}\right)$

$\frac{h}{30} = \tan\theta$

$h = 30\tan\theta$

$\frac{dh}{d\theta} = 30\left[\sec^2\theta(1)\right]$

$= 30\sec^2\theta$

$\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$

$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{1}{2}\left(\frac{5\pi}{6}\right)\right) = \sqrt{\frac{1}{2}\left(1-\cos\left(\frac{5\pi}{6}\right)\right)} = \sqrt{\frac{1}{2}\left(1-\left(\frac{-\sqrt{3}}{2}\right)\right)} = \sqrt{\frac{1}{2}\left(\frac{2+\sqrt{3}}{2}\right)}$

$= \frac{\sqrt{2+\sqrt{3}}}{\sqrt{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{1}{2}\left(\frac{5\pi}{6}\right)\right) = \sqrt{\frac{1}{2}\left(1+\cos\left(\frac{5\pi}{6}\right)\right)} = \sqrt{\frac{1}{2}\left(1+\left(\frac{-\sqrt{3}}{2}\right)\right)} = \sqrt{\frac{1}{2}\left(\frac{2-\sqrt{3}}{2}\right)}$

$= \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

4% error: $4\% = 0.04 \Rightarrow (4\%)h = 0.04h = 0.04(30\tan\theta)$

we must find $|dh| < 0.04h$

$|30\sec^2\theta\, d\theta| < 0.04(30\tan\theta)$

$|\sec^2\theta\, d\theta| < 0.04\tan\theta$

$\frac{1}{\cos^2\theta}|d\theta| < 0.04\left(\frac{\sin\theta}{\cos\theta}\right)$

$\left(\frac{\cos^2\theta}{1}\right)\left(\frac{1}{\cos^2\theta}|d\theta|\right) < \left(0.04\left(\frac{\sin\theta}{\cos\theta}\right)\right)\left(\frac{\cos^2\theta}{1}\right)$

$|d\theta| < 0.04\sin\theta\cos\theta$

$|d\theta| < 0.04\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right)$

$|d\theta| < 0.04\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right)$

$|d\theta| < 0.04\left(\frac{\sqrt{(2+\sqrt{3})(2-\sqrt{3})}}{4}\right)$

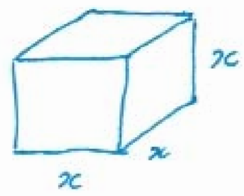
$|d\theta| < 0.01\left(\sqrt{4-3}\right)$

$|d\theta| < 0.01\left(\sqrt{1}\right)$

$|d\theta| < 0.01$ rad

The angle should be measured within an error of less than $|d\theta| < 0.01$ radians.

56)



% error in the edge is $0.5\% = 0.005$

$$\frac{\frac{dx}{dt}}{x} \leq 0.005$$

a) surface area: $S = 6x^2$ $\qquad \frac{dS}{dt} = 6\left[2x\frac{dx}{dt}\right] = 12x\frac{dx}{dt}$

% error in surface area is

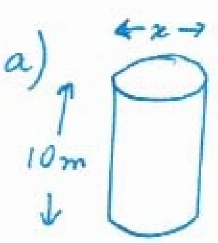
$$\frac{dS}{S} = \frac{\left(12x\frac{dx}{dt}\right)}{(6x^2)} = \frac{2\frac{dx}{dt}}{x} = 2\left(\frac{\frac{dx}{dt}}{x}\right) \leq 2(0.005) = \underline{\underline{0.01 = 1\%}}$$

b) volume: $V = x^3$ $\qquad \frac{dV}{dt} = 3x^2\frac{dx}{dt}$

% error in volume is

$$\frac{dV}{V} = \frac{\left(3x^2\frac{dx}{dt}\right)}{(x^3)} = \frac{3\frac{dx}{dt}}{x} = 3\left(\frac{\frac{dx}{dt}}{x}\right) \leq 3(0.005) = \underline{\underline{0.015 = 1.5\%}}$$

58-a)



let x be the interior diameter

so interior radius is $\left(\frac{x}{2}\right)$

$V = \pi r^2 h \Rightarrow$ at $(h = 10m)$, $V = \pi\left(\frac{x}{2}\right)^2(10) = \frac{5\pi}{2}x^2$

$\frac{dV}{dx} = \frac{5\pi}{2}[2x] = 5\pi x$ $\qquad dV = \left\{\frac{dV}{dx}\right\}dx = \{5\pi x\}dx$ $\qquad \Delta V \approx dV$

$|\Delta V| \leq (1\%)V$

$|dV| \leq 0.01\left(\frac{5\pi}{2}x^2\right)$

$|5\pi x\, dx| \leq 0.01\,\frac{5\pi}{2}x^2$

$5\pi x|dx| \leq \left(\frac{1}{100}\right)\frac{5\pi}{2}x^2$

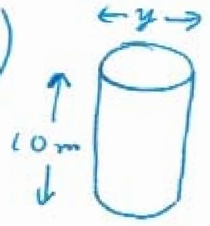
$5\pi x|dx| \leq \left(\frac{1}{200}\right)5\pi x^2$

$\frac{5\pi x^2}{x}|dx| \leq \left(\frac{1}{200}\right)5\pi x^2$

$\frac{|dx|}{x} \leq \left(\frac{1}{200}\right)\frac{5\pi x^2}{5\pi x^2} = \left(\frac{1}{200}\right)$

$\frac{|dx|}{x} \leq \underline{\underline{0.005 = 0.5\%}}$

interior diameter within 0.5%.

58-b) 

let y be the exterior diameter
so exterior radius is $\left(\frac{y}{2}\right)$

paint goes on surface area; so $S = 2\pi r h$, when $h = 10\text{m}$
and $r = \frac{y}{2}$ $\quad S = 2\pi\left(\frac{y}{2}\right)(10) = 10\pi y$

$\frac{dS}{dy} = 10\pi[1] = 10\pi \qquad dS = \left\{\frac{dS}{dy}\right\}dy = \{10\pi\}dy \quad |\Delta S| \approx |dS|$

$|\Delta S| \le (5\%)\,S$

$|dS| \le (0.05)(10\pi y)$

$|10\pi\, dy| \le (0.05)(10\pi y)$

$10\pi|dy| \le (0.05)(10\pi y)$

$\frac{10\pi|dy|}{y} \le (0.05)(10\pi y)$

$\frac{|dy|}{y} \le (0.05)\frac{10\pi y}{10\pi y}$

$\frac{|dy|}{y} \le \underline{\underline{0.05 = 5\%}}$

exterior diameter within 5%

---

60) 

let x be diameter of a sphere $\Rightarrow r = \frac{x}{2}$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{\pi}{6}x^3 \qquad \Delta V \approx dV$

$\frac{dV}{dx} = \frac{\pi}{6}[3x^2] = \frac{\pi}{2}x^2 \qquad dV = \left\{\frac{dV}{dx}\right\}dx = \left\{\frac{\pi}{2}x^2\right\}dx$

$|\Delta V| \le (3\%)\,V$

$|dV| \le (0.03)\left(\frac{\pi}{6}x^3\right)$

$\left|\frac{\pi}{2}x^2\,dx\right| \le (0.03)\left(\frac{\pi}{6}x^3\right)$

$\frac{\pi}{2}x^2|dx| \le (0.01)\left(\frac{\pi}{2}x^3\right)$

$\frac{\frac{\pi}{2}x^3|dx|}{x} \le (0.01)\left(\frac{\pi}{2}x^3\right)$

$\frac{|dx|}{x} \le 0.01\frac{\left(\frac{\pi}{2}x^3\right)}{\left(\frac{\pi}{2}x^3\right)}$

$\frac{|dx|}{x} \le \underline{\underline{0.01 = 1\%}}$

error of diameter within 1%.

62) $C(t) = 1 + \dfrac{4t}{1+t^3} - e^{-0.06t}$ (t in hours)

$t_0 = 20\text{ min} = \frac{20}{60} = \frac{1}{3}\text{ hr} \qquad t = 30\text{ min} = \frac{30}{60} = \frac{1}{2}\text{ hr}$

$$\frac{dC}{dt} = [0] + \left\{\frac{(1+t^3)[4] - (4t)[3t^2]}{(1+t^3)^2}\right\} - \left[e^{-0.06t}(-0.06)\right]$$

$$= \left\{\frac{4+4t^3-12t^3}{(1+t^3)^2}\right\} + 0.06e^{-0.06t} = \frac{4-8t^3}{(1+t^3)^2} + \frac{0.06}{e^{0.06t}}$$

$$\left.\frac{dC}{dt}\right|_{t=\frac{1}{3}} = \frac{4-8(\frac{1}{3})^3}{(1+(\frac{1}{3})^3)^2} + \frac{0.06}{e^{0.06(\frac{1}{3})}} = \frac{4-\frac{8}{27}}{(1+\frac{1}{27})^2} + \frac{0.06}{e^{0.02}}$$

$$= \left(\frac{4-\frac{8}{27}}{(1+\frac{1}{27})^2}\right)\left(\frac{\frac{(27)^2}{1}}{\frac{(27)^2}{1}}\right) + \frac{0.06}{e^{0.02}} = \frac{4(27)^2 - 8(27)}{(27+1)^2} + \frac{0.06}{e^{0.02}}$$

$$= \frac{4(27)(27-2)}{(28)^2} + \frac{0.06}{e^{0.02}} = \frac{4(27)(25)}{(28)^2} + \frac{0.06}{e^{0.02}} = \frac{2700}{(28)^2} + \frac{0.06}{e^{0.02}}$$

$dC = \left\{\frac{dC}{dt}\right\} dt \qquad dt = (\frac{1}{2}) - (\frac{1}{3}) = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$

$$dC = \left\{\left.\frac{dC}{dt}\right|_{t=\frac{1}{3}}\right\} dt = \underline{\underline{\left\{\frac{2700}{(28)^2} + \frac{0.06}{e^{0.02}}\right\}\left(\frac{1}{6}\right)\ \text{mg/mL}}}$$

---

64) $T = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}} = 2\pi L^{\frac{1}{2}} g^{-\frac{1}{2}} = 2\pi\sqrt{\frac{L}{g}}$

a) $\dfrac{dT}{dg} = 2\pi L^{\frac{1}{2}}\left[\frac{-1}{2} g^{-\frac{3}{2}}\right] = -\pi\dfrac{\sqrt{L}}{(\sqrt{g})^3}$

$$dT = \left\{\frac{dT}{dg}\right\} dg = \left\{-\pi\frac{\sqrt{L}}{(\sqrt{g})^3}\right\} dg$$

64) continued

b) as g increases, $dg > 0$

then $dT = \left\{-\pi \frac{\sqrt{L}}{(\sqrt{g})^3}\right\} dg < 0$

this means that $dT < 0$ which the period T decreases and clock thicks more frequently. Both the pendilum and clock speed increase.

c) $L = 100\,cm$, $g = 980\, cm/sec^2 = g_{old}$

$dT = 0.001/sec = \frac{1}{1000} sec$

$$\left(\frac{1}{1000}\right) = \left\{-\pi \frac{\sqrt{(100)}}{(\sqrt{(980)})^3}\right\} dg$$

$$\frac{-(\sqrt{980})^3}{1000\pi\sqrt{100}} = dg$$

change in gravity

$$dg = \frac{-980\sqrt{980}}{1000\pi\sqrt{100}} = \frac{-98(10)\sqrt{10}\sqrt{98}}{1000\pi\sqrt{100}} = \frac{-98\sqrt{98}}{100\pi\sqrt{10}}\ cm/sec^2$$

new gravity:

$$g_{new} \approx g_{old} + dg = (980) + \left(\frac{-98\sqrt{98}}{100\pi\sqrt{10}}\right)$$

$$g_{new} \approx \underline{\left(980 - \frac{98\sqrt{98}}{100\pi\sqrt{10}}\right) cm/sec^2}$$

66) $y = f(x)$ is differentiable at $x = a$

$g(x) = m(x-a) + c$ m and c are constants

error $E(x) = f(x) - g(x)$ small enough near $x = a$

$L(x) = f(a) + f'(a)(x-a)$ 1) $E(a) = 0$ 2) $\lim\limits_{x \to a} \frac{E(x)}{x-a} = 0$

1) $E(a) = 0$

$E(x) = f(x) - g(x) = f(x) - (m(x-a)+c) = f(x) - m(x-a) - c$

$0 = E(a) = f(a) - m((a) - a) - c$ $\Rightarrow$ $0 = f(a) = c$

$0 = f(a) - m(0) - c$ $c = f(a)$

2) $\lim\limits_{x \to a} \frac{E(x)}{x-a} = 0$ to calculate m:

$$0 = \lim_{x \to a} \frac{E(x)}{x-a} = \lim_{x \to a} \left\{ \frac{f(x) - m(x-a) - c}{x-a} \right\} = \lim_{x \to a} \left\{ \frac{f(x) - c}{x-a} - \frac{m(x-a)}{x-a} \right\}$$

$$0 = \lim_{x \to a} \left\{ \frac{f(x) - c}{x-a} - m \right\} = \lim_{x \to a} \left\{ \frac{f(x) - c}{x-a} \right\} - \lim_{x \to a} m$$

$$0 = \lim_{x \to a} \left\{ \frac{f(x) - f(a)}{x-a} \right\} - m$$

$0 = f'(a) - m$

$m = f'(a)$

so, $g(x) = m(x-a) + c$

$g(x) = f'(a)(x-a) + f(a)$ which is the linear approximation