Theorem 3 - The Derivative Rule for Inverses. {Numerical}

If f has an interval I as domain and $f'(x) = \frac{df}{dx}$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \text{or} \quad \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$

Derivative of the Natural Logarithm Function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}(1) = \frac{1}{x} \quad , \; x > 0$$

since Natural Logarithm is a function we must apply the Chain Rule.

The Chain Rule extends for positive functions $u(x)$:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad , \; u > 0$$

The Derivative of $\log_b u$. (use the change of base formula)

let $u(x)$

$$\frac{d}{dx}(\log_b u) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{d}{dx}\left(\left(\frac{1}{\ln b}\right)\ln x\right) = \left(\frac{1}{\ln b}\right)\frac{d}{dx}(\ln x)$$
$$= \left(\frac{1}{\ln b}\right)\left[\frac{1}{u}\frac{du}{dx}\right]$$

The Derivative of b^x, $b > 0$ {reminder: it is possible for $b = x$}

this type where the exponent contains a variable, it is best if we use a procedure shown below which deals with all cases (including when, $b = x$, the base is a variable).

step 1: apply natural logarithm on both sides of equal sign and use the properties of logarithm to remove the exponent.

step 2: use implicit differentiation

step 3: solve for the derivative and use appropriate simplification

$$\frac{dy}{dx} = ?$$

$$y = b^x$$

① $\ln y = \ln(b^x)$

$\ln y = x(\ln b)$

② $\left[\frac{1}{y}\frac{dy}{dx}\right] = (\ln b)[1]$

③ $\frac{dy}{dx} = (\ln b)\, y$

$= (\ln b)\, b^x$

here $(\ln b)$ is a constant

$$y = x^x$$

① $\ln y = \ln(x^x)$

$\ln y = x \ln x$

② $\left[\frac{1}{y}\frac{dy}{dx}\right] = (x)\left[\frac{1}{x}(1)\right] + (\ln x)[1]$

③ $\frac{dy}{dx} = \left\{\frac{x}{x} + \ln x\right\} y$

$= \{1 + \ln x\}\, x^x$

product rule is needed here

Theorem 4 - The number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}.$$

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4) $f(x) = 2x^2$, $x \geq 0$, $a = 5$ $\{x = f(a)\}$

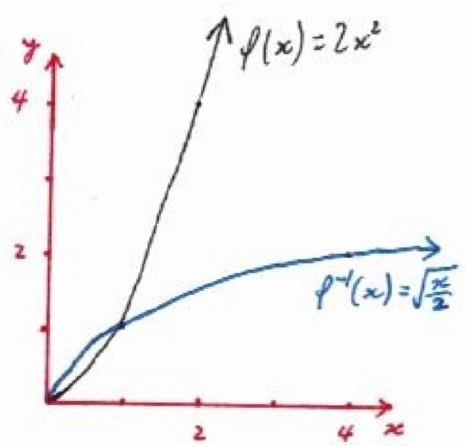
a) $y = 2x^2$

$\frac{y}{2} = x^2$

$\pm\sqrt{\frac{y}{2}} = x$

$f^{-1}(x) = \sqrt{\frac{x}{2}}$

b)



c) $\frac{df}{dx} = 2[2x] = 4x$

$f^{-1}(x) = \sqrt{\frac{x}{2}} = \frac{1}{\sqrt{2}}\sqrt{x} = \frac{1}{\sqrt{2}} x^{\frac{1}{2}}$

$\frac{df^{-1}}{dx} = \frac{1}{\sqrt{2}}\left[\frac{1}{2} x^{-\frac{1}{2}}\right]$

$= \frac{1}{(2\sqrt{2})\sqrt{x}}$

$f(5) = 2(5)^2 = 50$

$\left.\frac{df}{dx}\right|_{x=5} = 4(5) = 20$

$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \frac{1}{(2\sqrt{2})\sqrt{(50)}} = \frac{1}{(2\sqrt{2})(5\sqrt{2})}$

$= \frac{1}{10(2)} = \frac{1}{20}$

8) $f(x) = x^2 - 4x - 5$, $x > 2$; $x = 0 = f(5)$: $\frac{df^{-1}}{dx} = ?$

$f'(x) = \frac{df}{dx} = 2x - 4$ $\qquad\qquad \Downarrow$

$\qquad\qquad\qquad f^{-1}(0) = 5$

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(5)} = \frac{1}{2(5)-4} = \frac{1}{10-4} = \frac{1}{6}$$

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10) $y = g(x)$, $m(0) = 2 = \frac{dy}{dx}\Big|_{x=0} = g'(0)$

origin: $(0,0)$ $\qquad g(0) = 0$

$$\Downarrow$$

$$0 = g^{-1}(0)$$

$$(g^{-1})'(0) = \frac{1}{g'(g^{-1}(0))} = \frac{1}{g'(0)} = \frac{1}{2}$$

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12) $y = \frac{1}{\ln 3x} = (\ln 3x)^{-1}$

$$\frac{dy}{dx} = \left[-1(\ln 3x)^{-2}\left(\frac{1}{3x}(3)\right)\right] = \underline{\underline{\frac{-1}{x(\ln 3x)^2}}}$$

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14) $y = \ln\left(t^{\frac{3}{2}}\right) + \sqrt{t} = \frac{3}{2}\ln t + t^{\frac{1}{2}}$

$$\frac{dy}{dt} = \frac{3}{2}\left[\frac{1}{t}(1)\right] + \left[\frac{1}{2}t^{-\frac{1}{2}}\right] = \underline{\underline{\frac{3}{2t} + \frac{1}{2\sqrt{t}}}}$$

16) $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x}\left(\cos x\,(1)\right) = \frac{\cos x}{\sin x} = \underline{\underline{\cot x}}$$

18) $y = (\cos\theta)\ln(2\theta+2)$

$$\frac{dy}{d\theta} = (\cos\theta)\left[\frac{1}{2\theta+2}(2)\right] + \left(\ln(2\theta+2)\right)\left[-\sin\theta\,(1)\right]$$

$$= \frac{2\cos\theta}{2\theta+2} - (\sin\theta)\ln(2\theta+2) = \underline{\underline{\frac{\cos\theta}{\theta+1} - (\sin\theta)\ln(2\theta+2)}}$$

20) $y = (\ln x)^3$

$$\frac{dy}{dx} = 3(\ln x)^2\left(\frac{1}{x}(1)\right) = \underline{\underline{\frac{3(\ln x)^2}{x} = \frac{3\ln^2 x}{x}}}$$

22) $y = t\ln\sqrt{t} = (t)\left(\ln(t^{\frac{1}{2}})\right) = (t)\left(\frac{1}{2}\ln t\right) = \left(\frac{1}{2}t\right)(\ln t)$

$$\frac{dy}{dt} = \left(\frac{1}{2}t\right)\left[\frac{1}{t}(1)\right] + (\ln t)\left[\frac{1}{2}\right] = \frac{1}{2} + \frac{1}{2}\ln t = \underline{\underline{\frac{1+\ln t}{2}}}$$

24) $y = (x^2 \ln x)^4 = (x^2)^4 (\ln x)^4 = (x^8)((\ln x)^4)$

$$\frac{dy}{dx} = (x^8)\left[4(\ln x)^3\left(\tfrac{1}{x}(1)\right)\right] + ((\ln x)^4)\left[8x^7\right]$$

$$= (x^7)(\ln x)^3 \left\{(x)\left[\tfrac{4}{x}\right] + (\ln x)[8]\right\}$$

$$= \underline{\underline{\{4 + 8\ln x\}(x^7)(\ln x)^3}}$$

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26) $y = \dfrac{t}{\sqrt{\ln t}} = \dfrac{t}{(\ln t)^{\frac{1}{2}}}$

$$\frac{dy}{dt} = \frac{((\ln t)^{\frac{1}{2}})[1] - (t)\left[\frac{1}{2}(\ln t)^{-\frac{1}{2}}\left(\frac{1}{t}(1)\right)\right]}{((\ln t)^{\frac{1}{2}})^2}$$

$$= \frac{\sqrt{\ln t} - \dfrac{1}{2\sqrt{\ln t}}}{(\sqrt{\ln t})^2} = \left(\frac{\dfrac{\sqrt{\ln t}}{1} - \dfrac{1}{2\sqrt{\ln t}}}{\dfrac{(\sqrt{\ln t})^2}{1}}\right)\left(\frac{\dfrac{2\sqrt{\ln t}}{1}}{\dfrac{2\sqrt{\ln t}}{1}}\right)$$

$$= \underline{\underline{\frac{2\ln t - 1}{2(\sqrt{\ln t})^3}}}$$

28) $y = \dfrac{x \ln x}{1+\ln x}$

$$\frac{dy}{dx} = \frac{(1+\ln x)\left[(x)\left[\frac{1}{x}(1)\right]+(\ln x)[1]\right]-(x\ln x)\left[\frac{1}{x}(1)\right]}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x)[1+\ln x]-\ln x}{(1+\ln x)^2}$$

$$= \frac{(1+\ln x)^2-\ln x}{(1+\ln x)^2} = \underline{\underline{1-\frac{\ln x}{(1+\ln x)^2}}}$$

30) $y = \ln(\ln(\ln x))$

$$\frac{dy}{dx} = \left[\frac{1}{\ln(\ln x)}\left(\frac{1}{\ln x}\left(\frac{1}{x}(1)\right)\right)\right] = \frac{1}{x(\ln x)\ln(\ln x)}$$

32) $y = \ln(\sec\theta + \tan\theta)$

$$\frac{dy}{d\theta} = \frac{1}{\sec\theta+\tan\theta}\left(\sec\theta\tan\theta(1)+\sec^2\theta(1)\right) = \frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}$$

$$= \frac{\sec\theta(\tan\theta+\sec\theta)}{\sec\theta+\tan\theta} = \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta} = \underline{\underline{\sec\theta}}$$

34) $y = \frac{1}{2} \ln \frac{1+x}{1-x} = \frac{1}{2} \left\{ \ln(1+x) - \ln(1-x) \right\}$

$= \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x)$

$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{1+x} (1) \right] - \frac{1}{2} \left[\frac{1}{1-x} (-1) \right] = \frac{1}{2} \left(\frac{1}{1+x} \right) + \frac{1}{2} \left(\frac{1}{1-x} \right)$

$= \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) = \frac{1}{2} \left(\frac{1}{1+x} \left(\frac{1-x}{1-x} \right) + \frac{1}{1-x} \left(\frac{1+x}{1+x} \right) \right)$

$= \frac{1}{2} \left(\frac{(1-x)+(1+x)}{(1+x)(1-x)} \right) = \frac{1}{2} \left(\frac{2}{(1+x)(1-x)} \right) = \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2}$ or $= \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$

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36) $y = \sqrt{\ln \sqrt{t}} = \sqrt{\ln (t^{\frac{1}{2}})} = \sqrt{\frac{1}{2} \ln t}$

$= \sqrt{\frac{1}{2}} \sqrt{\ln t} = \frac{1}{\sqrt{2}} (\ln t)^{\frac{1}{2}}$

$\frac{dy}{dt} = \frac{1}{\sqrt{2}} \left[\frac{1}{2} (\ln t)^{-\frac{1}{2}} \left(\frac{1}{t} (1) \right) \right] = \frac{1}{2\sqrt{2}\, t \sqrt{\ln t}}$

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38) $y = \ln \left(\frac{\sqrt{\sin\theta \cos\theta}}{1 + 2\ln\theta} \right) = \ln \left(\frac{\sqrt{\sin\theta}\ \sqrt{\cos\theta}}{1 + 2\ln\theta} \right)$

$= \frac{1}{2} \ln(\sin\theta) + \frac{1}{2} \ln(\cos\theta) - \ln(1 + 2\ln\theta)$

$\frac{dy}{d\theta} = \frac{1}{2} \left[\frac{1}{\sin\theta} (\cos\theta (1)) \right] + \frac{1}{2} \left[\frac{1}{\cos\theta} (-\sin\theta (1)) \right] - \left[\frac{1}{1+2\ln\theta} \left(2 \left(\frac{1}{\theta} (1) \right) \right) \right]$

$= \frac{1}{2} \cot\theta - \frac{1}{2} \tan\theta - \frac{2}{\theta(1 + 2\ln\theta)}$

40) $y = \ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}} = \ln\dfrac{\sqrt{(x+1)^5}}{\sqrt{(x+2)^{20}}} = \ln\left(\dfrac{(x+1)^{\frac{5}{2}}}{(x+2)^{\frac{20}{2}}}\right)$

$= \ln\left(\dfrac{(x+1)^{\frac{5}{2}}}{(x+2)^{10}}\right) = \dfrac{5}{2}\ln(x+1) - 10\ln(x+2)$

$$\frac{dy}{dx} = \frac{5}{2}\left[\frac{1}{x+1}(1)\right] - 10\left[\frac{1}{x+2}(1)\right] = \underline{\underline{\frac{5}{2(x+1)} - \frac{10}{x+2}}}$$

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42) $y = \sqrt{(x^2+1)(x-1)^2}$

$$\ln y = \ln\sqrt{(x^2+1)(x-1)^2} = \tfrac{1}{2}\ln(x^2+1) + \tfrac{1}{2}\ln(x-1)^2$$

$$\ln y = \tfrac{1}{2}\ln(x^2+1) + \ln(x-1)$$

$$\left[\frac{1}{y}\frac{dy}{dx}\right] = \frac{1}{2}\left[\frac{1}{x^2+1}(2x)\right] + \left[\frac{1}{x-1}(1)\right]$$

$$\frac{dy}{dx} = \left\{\frac{x}{x^2+1} + \frac{1}{x-1}\right\} y$$

$$= \underline{\underline{\left\{\frac{x}{x^2+1} + \frac{1}{x-1}\right\}\sqrt{(x^2+1)(x-1)^2}}}$$

44) $y = \sqrt{\dfrac{1}{t(t+1)}}$

$$\ln y = \ln \sqrt{\frac{1}{t(t+1)}}$$

$$\ln y = \tfrac{1}{2}\ln(1) - \tfrac{1}{2}\ln t - \tfrac{1}{2}\ln(t+1)$$

$$\left[\frac{1}{y}\frac{dy}{dt}\right] = [0] - \frac{1}{2}\left[\frac{1}{t}(1)\right] - \frac{1}{2}\left[\frac{1}{t+1}(1)\right]$$

$$\frac{dy}{dt} = \left\{\frac{-1}{2t} - \frac{1}{2(t+1)}\right\} y = \underline{\underline{\left\{\frac{-1}{2t} - \frac{1}{2(t+1)}\right\}\sqrt{\frac{1}{t(t+1)}}}}$$

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46) $y = (\tan\theta)\sqrt{2\theta+1}$

$$\ln y = \ln\left((\tan\theta)\sqrt{2\theta+1}\right)$$

$$\ln y = \ln(\tan\theta) + \tfrac{1}{2}\ln(2\theta+1)$$

$$\left[\frac{1}{y}\frac{dy}{d\theta}\right] = \left[\frac{1}{\tan\theta}(\sec^2\theta(1))\right] + \frac{1}{2}\left[\frac{1}{2\theta+1}(2)\right]$$

$$\frac{dy}{d\theta} = \left\{\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right\} y = \left\{\frac{\left(\frac{1}{\cos^2\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)} + \frac{1}{2\theta+1}\right\}(\tan\theta)\sqrt{2\theta+1}$$

$$= \left\{\frac{1}{\sin\theta\cos\theta} + \frac{1}{2\theta+1}\right\}(\tan\theta)\sqrt{2\theta+1} = \left\{\frac{1}{\frac{1}{2}\sin(2\theta)} + \frac{1}{2\theta+1}\right\}(\tan\theta)\sqrt{2\theta+1}$$

$$= \underline{\underline{\left\{2\csc(2\theta) + \frac{1}{2\theta+1}\right\}(\tan\theta)\sqrt{2\theta+1}}}$$

48) $y = \dfrac{1}{t(t+1)(t+2)}$

$$\ln y = \ln\left(\frac{1}{t(t+1)(t+2)}\right)$$

$$\ln y = \ln(1) - \ln t - \ln(t+1) - \ln(t+2)$$

$$\left[\frac{1}{y}\frac{dy}{dt}\right] = [0] - \left[\frac{1}{t}(1)\right] - \left[\frac{1}{t+1}(1)\right] - \left[\frac{1}{t+2}(1)\right]$$

$$\frac{dy}{dt} = \left\{\frac{-1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right\} y = \underline{\underline{\left\{\frac{-1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right\} \frac{1}{t(t+1)(t+2)}}}$$

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50) $y = \dfrac{\theta \sin\theta}{\sqrt{\sec\theta}}$

$$\ln y = \ln\left(\frac{\theta\sin\theta}{\sqrt{\sec\theta}}\right)$$

$$\ln y = \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta)$$

$$\left[\frac{1}{y}\frac{dy}{d\theta}\right] = \left[\frac{1}{\theta}(1)\right] + \left[\frac{1}{\sin\theta}(\cos\theta(1))\right] - \frac{1}{2}\left[\frac{1}{\sec\theta}(\sec\theta\tan\theta(1))\right]$$

$$\frac{dy}{d\theta} = \left\{\frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{\sec\theta\tan\theta}{2\sec\theta}\right\} y$$

$$= \underline{\underline{\left\{\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right\}\frac{\theta\sin\theta}{\sqrt{\sec\theta}}}}$$

52) $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}} = \dfrac{\sqrt{(x+1)^{10}}}{\sqrt{(2x+1)^5}} = \dfrac{(x+1)^5}{(2x+1)^{5/2}}$

$$\ln y = \ln\left(\frac{(x+1)^5}{(2x+1)^{5/2}}\right)$$

$$\ln y = 5\ln(x+1) - \frac{5}{2}\ln(2x+1)$$

$$\left[\frac{1}{y}\frac{dy}{dx}\right] = 5\left[\frac{1}{x+1}(1)\right] - \frac{5}{2}\left[\frac{1}{2x+1}(2)\right]$$

$$\frac{dy}{dx} = \left\{\frac{5}{x+1} - \frac{5}{2x+1}\right\} y = \underline{\underline{\left\{\frac{5}{x+1} - \frac{5}{2x+1}\right\}\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}}}$$

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54) $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x-3)}}$

$$\ln y = \ln \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x-3)}}$$

$$\ln y = \frac{1}{3}\ln x + \frac{1}{3}\ln(x+1) + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x^2+1) - \frac{1}{3}\ln(2x-3)$$

$$\left[\frac{1}{y}\frac{dy}{dx}\right] = \frac{1}{3}\left[\frac{1}{x}(1)\right] + \frac{1}{3}\left[\frac{1}{x+1}(1)\right] + \frac{1}{3}\left[\frac{1}{x-2}(1)\right] - \frac{1}{3}\left[\frac{1}{x^2+1}(2x)\right] - \frac{1}{3}\left[\frac{1}{2x-3}(2)\right]$$

$$\frac{dy}{dx} = \left\{\frac{1}{3x} + \frac{1}{3(x+1)} + \frac{1}{3(x-2)} - \frac{2}{3(x^2+1)} - \frac{2}{3(2x-3)}\right\} y$$

$$= \underline{\underline{\left\{\frac{1}{3x} + \frac{1}{3(x+1)} + \frac{1}{3(x-2)} - \frac{2}{3(x^2+1)} - \frac{2}{3(2x-3)}\right\}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x-3)}}}}$$

56) $y = \ln(3\theta e^{-\theta}) = \ln(3) + \ln\theta + \ln(e^{-\theta})$

$y = \ln(3) + \ln\theta - \theta$

$$\frac{dy}{d\theta} = [0] + \left[\frac{1}{\theta}(1)\right] - [1] = \underline{\underline{\frac{1}{\theta} - 1}}$$

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58) $y = \ln(2e^{-t}\sin t) = \ln(2) + \ln(e^{-t}) + \ln(\sin t)$

$y = \ln(2) - t + \ln(\sin t)$

$$\frac{dy}{dt} = [0] - [1] + \left[\frac{1}{\sin t}(\cos t(1))\right] = -1 + \frac{\cos t}{\sin t} = \underline{\underline{\cot t - 1}}$$

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60) $y = \ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right) = \ln(\sqrt{\theta}) - \ln(1+\sqrt{\theta})$

$y = \frac{1}{2}\ln\theta - \ln(1+\theta^{\frac{1}{2}})$

$$\frac{dy}{d\theta} = \frac{1}{2}\left[\frac{1}{\theta}(1)\right] - \left[\frac{1}{1+\theta^{1/2}}\left(\frac{1}{2}\theta^{-\frac{1}{2}}\right)\right] = \frac{1}{2\theta} - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right)$$

$$= \underline{\underline{\frac{1}{2\theta} - \frac{1}{2\sqrt{\theta} + 2\theta}}}$$

62) $y = e^{\sin t}(\ln t^2 + 1) = (e^{\sin t})(2\ln t + 1)$

$$\frac{dy}{dt} = (e^{\sin t})\left[2\left[\frac{1}{t}(1)\right] + [0]\right] + (2\ln t + 1)\left[e^{\sin t}(\cos t(1))\right]$$

$$= \underline{\underline{e^{\sin t}\left\{\frac{2}{t} + (\cos t)(2\ln t + 1)\right\}}}$$

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64) $\ln xy = e^{x+y}$

$$\ln x + \ln y = e^{(x+y)}$$

$$\left[\frac{1}{x}(1)\right] + \left[\frac{1}{y}\frac{dy}{dx}\right] = \left[e^{(x+y)}\left(1 + 1\frac{dy}{dx}\right)\right]$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = e^{(x+y)} + e^{(x+y)}\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - e^{(x+y)}\frac{dy}{dx} = e^{(x+y)} - \frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - e^{(x+y)}\right) = e^{(x+y)} - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{e^{(x+y)} - \frac{1}{x}}{\frac{1}{y} - e^{(x+y)}} = \left(\frac{\frac{e^{(x+y)}}{1} - \frac{1}{x}}{\frac{1}{y} - \frac{e^{(x+y)}}{1}}\right)\left(\frac{\frac{xy}{1}}{\frac{xy}{1}}\right)$$

$$= \underline{\underline{\frac{xye^{(x+y)} - y}{x - xye^{(x+y)}}}}$$

66) $\tan y = e^x + \ln x$

$$\left[\sec^2 y \frac{dy}{dx}\right] = \left[e^x(1)\right] + \left[\frac{1}{x}(1)\right]$$

$$\frac{dy}{dx} = \frac{\left\{e^x + \frac{1}{x}\right\}}{\sec^2 y} = \underline{\underline{\left\{e^x + \frac{1}{x}\right\}\cos^2 y = \frac{(xe^x+1)\cos^2 y}{x}}}$$

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68) $y = 3^{-x}$

$$\ln y = \ln(3^{-x})$$

$$\ln y = -x \ln 3$$

$$\ln y = -(\ln 3)x$$

$$\left[\frac{1}{y}\frac{dy}{dx}\right] = -(\ln 3)[1]$$

$$\frac{dy}{dx} = -(\ln 3)y$$

$$= \underline{\underline{-(\ln 3)\,3^{-x}}}$$

$$= \underline{\underline{\frac{-(\ln 3)}{3^x}}}$$

70) $y = 2^{s^2}$

$$\ln y = \ln(2^{s^2})$$

$$\ln y = s^2 \ln 2$$

$$\ln y = (\ln 2)\,s^2$$

$$\left[\frac{1}{y}\frac{dy}{ds}\right] = (\ln 2)[2s]$$

$$\frac{dy}{ds} = \{2(\ln 2)s\}\,y$$

$$= \underline{\underline{\{2(\ln 2)s\}\,2^{s^2}}}$$

72) $y = t^{1-e} = t^{(1-e)}$

$$\frac{dy}{dt} = \left[(1-e)\, t^{\{(1-e)-1\}}\right] = \underline{\underline{(1-e)\,t^{-e} = \frac{1-e}{t^{e}}}}$$

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74) $y = \log_3 (1 + \theta \ln 3) = \dfrac{\ln(1+(\ln 3)\theta)}{\ln 3}$

$$y = \frac{1}{\ln 3} \ln(1+(\ln 3)\theta)$$

$$\frac{dy}{d\theta} = \frac{1}{\ln 3}\left[\frac{1}{1+(\ln 3)\theta}(\ln 3)\right] = \underline{\underline{\frac{1}{1+(\ln 3)\theta}}}$$

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76) $y = \log_{25} e^{x} - \log_5 \sqrt{x} = \dfrac{\ln(e^{x})}{\ln 25} - \dfrac{\ln\sqrt{x}}{\ln 5}$

$$y = \frac{1}{\ln 25}x - \frac{1}{\ln 5}\left(\frac{1}{2}\ln x\right) = \frac{1}{\ln 25}x - \frac{1}{2\ln 5}\ln x$$

$$y = \frac{1}{\ln 25}x - \frac{1}{\ln(5^2)}\ln x = \frac{1}{\ln 25}x - \frac{1}{\ln 25}\ln x$$

$$y = \frac{1}{\ln 25}(x - \ln x)$$

$$\frac{dy}{dx} = \frac{1}{\ln 25}\left([1] - \left[\frac{1}{x}(1)\right]\right) = \underline{\underline{\frac{1}{\ln 25}\left(1 - \frac{1}{x}\right)}} = \frac{1}{\ln 25}\left(\frac{x-1}{x}\right)$$

$$= \underline{\underline{\frac{x-1}{x\ln 25}}}$$

78) $y = \log_3 r \cdot \log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{1}{(\ln 3)(\ln 9)}(\ln r)^2$

$$\frac{dy}{dr} = \frac{1}{(\ln 3)(\ln 9)}\left[2(\ln r)^{1}\left(\frac{1}{r}(1)\right)\right] = \underline{\underline{\frac{2\ln r}{(\ln 3)(\ln 9)r}}}$$

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80) $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5 \left(\frac{7x}{3x+2}\right)^{\frac{\ln 5}{2}}$

$$y = \frac{\ln\left(\frac{7x}{3x+2}\right)^{\frac{\ln 5}{2}}}{\ln 5} = \frac{\left(\frac{\ln 5}{2}\right)\ln\left(\frac{7x}{3x+2}\right)}{\ln 5} = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$$

$$y = \frac{1}{2}\ln(7x) - \frac{1}{2}\ln(3x+2) = \frac{1}{2}\ln(7) + \frac{1}{2}\ln x - \frac{1}{2}\ln(3x+2)$$

$$\frac{dy}{dx} = [0] + \frac{1}{2}\left[\frac{1}{x}(1)\right] - \frac{1}{2}\left[\frac{1}{3x+2}(3)\right] = \underline{\underline{\frac{1}{2x} - \frac{3}{2(3x+2)}}}$$

---

82) $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right) = \frac{\ln\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)}{\ln 7}$

$$y = \frac{1}{\ln 7}\left\{\ln(\sin\theta) + \ln(\cos\theta) - \ln(e^\theta) - \ln(2^\theta)\right\}$$

$$y = \frac{1}{\ln 7}\left\{\ln(\sin\theta) + \ln(\cos\theta) - \theta - (\ln 2)\theta\right\}$$

$$\frac{dy}{d\theta} = \frac{1}{\ln 7}\left\{\left[\frac{1}{\sin\theta}(\cos\theta(1))\right] + \left[\frac{1}{\cos\theta}(-\sin\theta(1))\right] - [1] - (\ln 2)[1]\right\}$$

$$= \underline{\underline{\frac{1}{\ln 7}\left\{\cot\theta - \tan\theta - 1 - \ln 2\right\}}}$$

84) $y = \log_2\left(\frac{x^2 e^2}{2\sqrt{x+1}}\right) = \frac{\ln\left(\frac{x^2 e^2}{2\sqrt{x+1}}\right)}{\ln 2}$

$$y = \frac{1}{\ln 2}\left\{\ln(x^2) + \ln(e^2) - \ln(2) - \ln\sqrt{x+1}\right\}$$

$$y = \frac{1}{\ln 2}\left\{2\ln x + 2 - (\ln 2) - \frac{1}{2}\ln(x+1)\right\}$$

$$\frac{dy}{dx} = \frac{1}{\ln 2}\left\{2\left[\frac{1}{x}(1)\right] + [0] - [0] - \frac{1}{2}\left[\frac{1}{x+1}(1)\right]\right\}$$

$$= \frac{1}{\ln 2}\left\{\frac{2}{x} - \frac{1}{2(x+1)}\right\}$$

---

86) $y = 3\log_8(\log_2 t) = 3\left\{\frac{\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8}\right\}$

$$y = \frac{3}{\ln 8}\ln\left(\frac{1}{\ln 2}\ln t\right)$$

$$\frac{dy}{dt} = \frac{3}{\ln 8}\left[\frac{1}{\left(\frac{1}{\ln 2}\ln t\right)}\left(\frac{1}{\ln 2}\left(\frac{1}{t}(1)\right)\right)\right]$$

$$= \frac{3}{(\ln 8)\, t(\ln t)}$$

88) $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

$$y = t\left(\frac{\ln\left(e^{(\sin t)(\ln 3)}\right)}{\ln 3}\right) = t\left(\frac{(\sin t)(\ln 3)}{\ln 3}\right)$$

$$y = (t)(\sin t)$$

$$\frac{dy}{dt} = (t)[\cos t(1)] + (\sin t)[1] = \underline{\underline{t\cos t + \sin t}}$$

---

90) $y = x^{(x+1)}$

$$\ln y = \ln\left(x^{(x+1)}\right)$$

$$\ln y = (x+1)(\ln x)$$

$$\left[\frac{1}{y}\frac{dy}{dx}\right] = (x+1)\left[\frac{1}{x}(1)\right] + (\ln x)[1]$$

$$\frac{dy}{dx} = \left\{1 + \frac{1}{x} + \ln x\right\}y$$

$$= \underline{\underline{\left\{1 + \frac{1}{x} + \ln x\right\}x^{(x+1)}}}$$

92) $y = t^{\sqrt{t}}$

$$\ln y = \ln\left(t^{\sqrt{t}}\right)$$

$$\ln y = \sqrt{t}\ln t$$

$$\ln y = \left(t^{\frac{1}{2}}\right)(\ln t)$$

$$\left[\frac{1}{y}\frac{dy}{dt}\right] = \left(t^{\frac{1}{2}}\right)\left[\frac{1}{t}(1)\right] + (\ln t)\left[\frac{1}{2}t^{-\frac{1}{2}}\right]$$

$$\frac{dy}{dt} = \left\{\frac{\sqrt{t}}{t} + \frac{\ln t}{2\sqrt{t}}\right\}y$$

$$= \underline{\underline{\left\{\frac{2 + \ln t}{2\sqrt{t}}\right\}t^{\sqrt{t}}}}$$

94) $y = x^{\sin x}$

$\ln y = \ln(x^{\sin x})$

$\ln y = (\sin x)(\ln x)$

$\left[\frac{1}{y}\frac{dy}{dx}\right] = (\sin x)\left[\frac{1}{x}(1)\right] + (\ln x)\left[\cos x(1)\right]$

$\frac{dy}{dx} = \left\{\frac{\sin x}{x} + (\cos x)\ln x\right\} y$

$= \left\{\frac{\sin x}{x} + (\cos x)\ln x\right\} x^{\sin x}$

96) $y = (\ln x)^{\ln x}$

$\ln y = \ln\left((\ln x)^{\ln x}\right)$

$\ln y = (\ln x)(\ln(\ln x))$

$\left[\frac{1}{y}\frac{dy}{dx}\right] = (\ln x)\left[\frac{1}{\ln x}\left(\frac{1}{x}(1)\right)\right] + (\ln(\ln x))\left[\frac{1}{x}(1)\right]$

$\frac{dy}{dx} = \left\{\frac{1}{x} + \frac{\ln(\ln x)}{x}\right\} y$

$= \left\{\frac{1 + \ln(\ln x)}{x}\right\} (\ln x)^{\ln x}$

---

98) $x^{\sin y} = \ln y$

$\ln(x^{\sin y}) = \ln(\ln y)$

$(\sin y)(\ln x) = \ln(\ln y)$

$(\sin y)\left[\frac{1}{x}(1)\right] + (\ln x)\left[\cos y \frac{dy}{dx}\right] = \left[\frac{1}{\ln y}\left(\frac{1}{y}\frac{dy}{dx}\right)\right]$

$\frac{\sin y}{x} + (\cos y)(\ln x)\frac{dy}{dx} = \frac{1}{y \ln y}\frac{dy}{dx}$

$\frac{\sin y}{x} = \frac{1}{y \ln y}\frac{dy}{dx} - (\cos y)(\ln x)\frac{dy}{dx}$

$\frac{\sin y}{x} = \frac{dy}{dx}\left(\frac{1}{y \ln y} - (\cos y)(\ln x)\right)$

98) continued

$$\frac{\frac{\sin y}{x}}{\frac{1}{y \ln y} - (\cos y)(\ln x)} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \left(\frac{\frac{\sin y}{x}}{\frac{1}{y \ln y} - \frac{(\cos y)(\ln x)}{1}}\right)\left(\frac{\frac{xy \ln y}{1}}{\frac{xy \ln y}{1}}\right)$$

$$= \underline{\underline{\frac{y(\ln y)\sin y}{x - xy(\cos y)(\ln x)(\ln y)}}}$$

---

100) $e^y = y^{\ln x}$

$$\ln(e^y) = \ln(y^{\ln x})$$

$$y = (\ln x)(\ln y)$$

$$\left[1\frac{dy}{dx}\right] = (\ln x)\left[\frac{1}{y}\frac{dy}{dx}\right] + (\ln y)\left[\frac{1}{x}(1)\right]$$

$$\frac{dy}{dx} = \frac{\ln x}{y}\frac{dy}{dx} + \frac{\ln y}{x}$$

$$\frac{dy}{dx} - \frac{\ln x}{y}\frac{dy}{dx} = \frac{\ln y}{x}$$

$$\frac{dy}{dx}\left(1 - \frac{\ln x}{y}\right) = \frac{\ln y}{x}$$

$$\frac{dy}{dx} = \frac{\frac{\ln y}{x}}{1 - \frac{\ln x}{y}}$$

$$\frac{dy}{dx} = \left(\frac{\frac{\ln y}{x}}{\frac{1}{1} - \frac{\ln x}{y}}\right)\left(\frac{\frac{xy}{1}}{\frac{xy}{1}}\right)$$

$$= \underline{\underline{\frac{y \ln y}{xy - x\ln x}}}$$