MATH 20100 section 3.6

Theorem 2 - The Chain Rule

If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f\circ g)(x)=f(g(x))$ is differentiable at x, and

$$\frac{d}{dx}\big((f\circ g)(x)\big)=\frac{d}{dx}\big(f(g(x)\big)\cdot\left(\frac{dg}{dx}\right).$$

In Leibniz's notation, if $y=f(u)$ and $u=g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx},$$

where $\frac{dy}{du}$ is evaluated at $u=g(x)$.

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note: $$\frac{d}{dx}(e^u)=e^u\frac{du}{dx}$$

also $\frac{d}{dx}(\sin u)=\cos u\frac{du}{dx}$ $\quad\frac{d}{dx}(\csc u)=-\csc u\cot u\frac{du}{dx}$

$\frac{d}{dx}(\cos u)=-\sin u\frac{du}{dx}$ $\quad\frac{d}{dx}(\sec u)=\sec u\tan u\frac{du}{dx}$

$\frac{d}{dx}(\tan u)=\sec^2 u\frac{du}{dx}$ $\quad\frac{d}{dx}(\cot u)=-\csc^2 u\frac{du}{dx}$

4) $y = \cos u = f(u) \qquad u = e^{-x} = g(x)$

$y = f(g(x)) = f(e^{-x}) = \cos(e^{-x})$

$\frac{dy}{dx} = -\sin(e^{-x})(e^{-x}(-1)) = e^{-x}\sin(e^{-x})$

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2) $y = 2u^3 = f(u) \qquad u = 8x-1 = g(x)$

$y = f(g(x)) = f(8x-1) = 2(8x-1)^3$

$\frac{dy}{dx} = 2\left[3(8x-1)^2(8)\right] = 48(8x-1)^2$

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6) $y = \sin u = f(u) \qquad u = x - \cos x = g(x)$

$y = f(g(x)) = f(x-\cos x) = \sin(x-\cos x)$

$\frac{dy}{dx} = \cos(x-\cos x)(1-[-\sin x(1)]) = (1+\sin x)\cos(x-\cos x)$

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8) $y = -\sec u = f(u) \qquad u = \frac{1}{x} + 7x = g(x) = x^{-1} + 7x$

$y = f(g(x)) = f(x^{-1}+7x) = -\sec(x^{-1}+7x)$

$\frac{dy}{dx} = -\left[\sec(x^{-1}+7x)\tan(x^{-1}+7x)\ (-1x^{-2}+7)\right]$

$= -\left(\frac{1}{x^2}+7\right)\sec\left(\frac{1}{x}+7x\right)\tan\left(\frac{1}{x}+7x\right)$

10) $y = (4-3x)^9$ $u = 4-3x$, $y = u^9$

$\frac{du}{dx} = -3$ $\frac{dy}{du} = 9u^8$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (9u^8) \cdot (-3) = -27(4-3x)^8$

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12) $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$ $y = u^{-10}$ $u = \frac{\sqrt{x}}{2} - 1 = \frac{1}{2}x^{\frac{1}{2}} - 1$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\frac{dy}{du} = -10u^{-11}$ $\frac{du}{dx} = \frac{1}{2}\left[\frac{1}{2}x^{-\frac{1}{2}}\right] = \frac{1}{4\sqrt{x}}$

$= (-10u^{-11})\left(\frac{1}{4\sqrt{x}}\right) = \frac{-1}{4\sqrt{x}}\left(\frac{\sqrt{x}}{2} - 1\right)^{-11}$

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14) $y = \sqrt{3x^2 - 4x + 6}$ $y = \sqrt{u} = u^{\frac{1}{2}}$ $u = 3x^2 - 4x + 6$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$ $\frac{du}{dx} = 6x - 4$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \left(\frac{1}{2\sqrt{u}}\right)(6x-4) = \frac{6x-4}{2\sqrt{3x^2-4x+6}} = \frac{2(3x-2)}{2\sqrt{3x^2-4x+6}} = \frac{3x-2}{\sqrt{3x^2-4x+6}}$

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16) $y = \cot\left(\pi - \frac{1}{x}\right)$ $y = \cot u$ $u = \pi - \frac{1}{x} = \pi - x^{-1}$

$\frac{dy}{du} = -\csc^2 u$ $\frac{du}{dx} = -\left[-1x^{-2}\right] = \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (-\csc^2 u)\left(\frac{1}{x^2}\right) = \frac{-\csc^2\left(\pi - \frac{1}{x}\right)}{x^2}$

18) $y = 5\cos^{-4} x$ $\qquad y = 5u^{-4}$ $\qquad u = \cos x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\qquad \frac{dy}{du} = -20u^{-5}$ $\qquad \frac{du}{dx} = -\sin x$

$$= (-20u^{-5})(-\sin x) = 20\cos^{-5} x \sin x = \frac{20 \sin x}{\cos^5 x}$$

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20) $y = e^{\frac{2x}{3}}$ $\qquad y = e^u$ $\qquad u = \frac{2x}{3} = \frac{2}{3}x$

$\frac{dy}{du} = e^u$ $\qquad \frac{du}{dx} = \frac{2}{3}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= (e^u)\left(\frac{2}{3}\right) = \frac{2}{3} e^{\frac{2x}{3}}$$

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22) $y = e^{(4\sqrt{x} + x^2)}$ $\qquad y = e^u$ $\qquad u = 4\sqrt{x} + x^2 = 4x^{\frac{1}{2}} + x^2$

$\frac{dy}{du} = e^u$ $\qquad \frac{du}{dx} = 4\left[\frac{1}{2}x^{-\frac{1}{2}}\right] + [2x] = \frac{2}{\sqrt{x}} + 2x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= (e^u)\left(\frac{2}{\sqrt{x}} + 2x\right) = \left(\frac{2}{\sqrt{x}} + 2x\right) e^{(4\sqrt{x} + x^2)}$$

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24) $q = \sqrt[3]{2r - r^2} = (2r - r^2)^{1/3}$

$$\frac{dq}{dr} = \frac{1}{3}(2r - r^2)^{-\frac{2}{3}}(2 - 2r) = \frac{2 - 2r}{3(2r - r^2)^{2/3}}$$

$$= \frac{2 - 2r}{3\left(\sqrt[3]{2r - r^2}\right)^2}$$

26) $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

$$\frac{ds}{dt} = \left[\cos\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right)\right] + \left[-\sin\left(\frac{3\pi t}{2}\right)\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\sin\left(\frac{3\pi t}{2}\right) = \frac{3\pi}{2}\left(\cos\left(\frac{3\pi t}{2}\right) - \sin\left(\frac{3\pi t}{2}\right)\right)$$

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28) $r = 6(\sec\theta - \tan\theta)^{3/2}$

$$\frac{dr}{d\theta} = 6\left[\frac{3}{2}(\sec\theta - \tan\theta)^{\frac{1}{2}}(\sec\theta\tan\theta(1) - \sec^2\theta)\right]$$

$$= 9(\sec\theta\tan\theta - \sec^2\theta)\sqrt{\sec\theta - \tan\theta}$$

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30) $y = \frac{1}{x}\sin^{-5}x - \frac{x}{3}\cos^3 x = (x^{-1})(\sin^{-5}x) - (\frac{1}{3}x)(\cos^3 x)$

$$\frac{dy}{dx} = \left\{(x^{-1})\left[-5\sin^{-6}x\,(\cos x(1))\right] + (\sin^{-5}x)\left[-1x^{-2}\right]\right\}$$

$$- \left\{\left(\frac{1}{3}x\right)\left[3\cos^2 x\,(-\sin x(1))\right] + (\cos^3 x)\left[\frac{1}{3}\right]\right\}$$

$$= \frac{-5}{x}\sin^{-6}x\cos x - \frac{1}{x^2}\sin^5 x + x\cos^2 x\sin x - \frac{1}{3}\cos^3 x$$

$$= x\cos^2 x\sin x - \frac{5\cos x}{x\sin^6 x} - \frac{\sin^5 x}{x^2} - \frac{1}{3}\cos^3 x$$

32) $y=(5-2x)^{-3}+\frac{1}{8}\left(\frac{2}{x}+1\right)^4=(5-2x)^{-3}+\frac{1}{8}\left(2x^{-1}+1\right)^4$

$$\frac{dy}{dx}=\left[-3(5-2x)^{-4}(-2)\right]+\frac{1}{8}\left[4(2x^{-1}+1)^3(-2x^{-2})\right]$$

$$=\frac{6}{(5-2x)^4}-\frac{\left(\frac{2}{x}+1\right)^3}{x^2}$$

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34) $y=(2x-5)^{-1}(x^2-5x)^6=\frac{(x^2-5x)^6}{(2x-5)}$ "better method"

$$\frac{dy}{dx}=\left((2x-5)^{-1}\right)\left[6(x^2-5x)^5(2x-5)\right]+\left((x^2-5x)^6\right)\left[-1(2x-5)^{-2}(2)\right]$$

$$=\frac{6(2x-5)(x^2-5x)^5}{(2x-5)}-\frac{2(x^2-5x)^6}{(2x-5)^2}=6(x^2-5x)^5-\frac{2(x^2-5x)^6}{(2x-5)^2}$$

$$=(x^2-5x)^5\left\{6-\frac{2(x^2-5x)}{(2x-5)^2}\right\}=(x^2-5x)^5\left\{\frac{6(2x-5)^2}{(2x-5)^2}-\frac{2(x^2-5x)}{(2x-5)^2}\right\}$$

$$=(x^2-5x)^5\left\{\frac{6(4x^2-20x+25)-2x^2+10x}{(2x-5)^2}\right\}$$

$$=(x^2-5x)^5\left\{\frac{24x^2-120x+150-2x^2+10x}{(2x-5)^2}\right\}=\frac{\{22x^2-110x+150\}(x^2-5x)^5}{(2x-5)^2}$$

$$\frac{dy}{dx}=\frac{(2x-5)\left[6(x^2-5x)^5(2x-5)\right]-\left((x^2-5x)^6\right)[2]}{(2x-5)^2}$$

$$=\frac{(x^2-5x)^5\left\{6(2x-5)^2-2(x^2-5x)\right\}}{(2x-5)^2}=\frac{(x^2-5x)^5\left\{6(4x^2-20x+25)-2x^2+10x\right\}}{(2x-5)^2}$$

$$=\frac{(x^2-5x)^5\left\{24x^2-120x+150-2x^2+10x\right\}}{(2x-5)^2}=\frac{\{22x^2-110x+150\}(x^2-5x)^2}{(2x-5)^2}$$

36) $y=(1+2x)\,e^{-2x}$

$$\frac{dy}{dx}=(1+2x)\left[e^{-2x}(-2)\right]+\left(e^{-2x}\right)[2]$$

$$=-2e^{-2x}-4xe^{-2x}+2e^{-2x}=-4xe^{-2x}$$

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38) $y=(9x^2-6x+2)\,e^{x^3}$

$$\frac{dy}{dx}=(9x^2-6x+2)\left[e^{x^3}(3x^2)\right]+\left(e^{x^3}\right)[18x-6]$$

$$=e^{x^3}\left\{(9x^2-6x+2)(3x^2)+[18x-6](1)\right\}$$

$$=e^{x^3}\left\{27x^4-18x^3+6x^2+18x-6\right\}$$

$$=\left\{27x^4-18x^3+6x^2+18x-6\right\}e^{x^3}$$

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40) $k(x)=x^2\sec\left(\frac{1}{x}\right)=x^2\sec\left(x^{-1}\right)$

$$\frac{dk}{dx}=(x^2)\left[\sec(x^{-1})\tan(x^{-1})(-1x^{-2})\right]+\left(\sec(x^{-1})\right)[2x]$$

$$=\frac{-x^2\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}{x^2}+2x\sec\left(\frac{1}{x}\right)$$

$$=2x\sec\left(\frac{1}{x}\right)-\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)=\sec\left(\frac{1}{x}\right)\left\{2x-\tan\left(\frac{1}{x}\right)\right\}$$

42) $g(x) = \dfrac{\tan 3x}{(x+7)^4}$

$$\frac{dg}{dx} = \frac{((x+7)^4)[\sec^2(3x)(3)] - (\tan(3x))[4(x+7)^3(1)]}{((x+7)^4)^2}$$

$$= \frac{(x+7)^3\{(x+7)[3\sec^2(3x)] - (\tan(3x))[4]\}}{(x+7)^8}$$

$$= \frac{3x\sec^2(3x) + 21\sec^2(3x) - 4\tan(3x)}{(x+7)^5}$$

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44) $g(t) = \left(\dfrac{1+\sin 3t}{3-2t}\right)^{-1} = \dfrac{3-2t}{1+\sin(3t)}$

$$\frac{dg}{dt} = \frac{(1+\sin(3t))[-2] - (3-2t)[\cos(3t)(3)]}{(1+\sin(3t))^2}$$

$$= \frac{-2 - 2\sin(3t) - 9\cos(3t) + 6t\cos(3t)}{(1+\sin(3t))^2}$$

$$= \frac{6t\cos(3t) - 9\cos(3t) - 2\sin(3t) - 2}{(1+\sin(3t))^2}$$

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46) $r = \sec\sqrt{\theta}\,\tan\left(\frac{1}{\theta}\right) = \sec(\theta^{1/2})\tan(\theta^{-1})$

$$\frac{dr}{d\theta} = (\sec(\theta^{\frac{1}{2}}))\left[\sec^2(\theta^{-1})(-1\theta^{-2})\right] + (\tan(\theta^{-1}))\left[\sec(\theta^{\frac{1}{2}})\tan(\theta^{\frac{1}{2}})\left(\tfrac{1}{2}\theta^{-\frac{1}{2}}\right)\right]$$

$$= \sec\sqrt{\theta}\left\{(1)\left[\frac{-\sec^2(\frac{1}{\theta})}{\theta^2}\right] + \left[\frac{\tan\sqrt{\theta}}{2\sqrt{\theta}}\right]\left(\tan\left(\tfrac{1}{\theta}\right)\right)\right\}$$

$$= \sec\sqrt{\theta}\left\{\frac{\tan\sqrt{\theta}\,\tan(\frac{1}{\theta})}{2\sqrt{\theta}} - \frac{\sec^2(\frac{1}{\theta})}{\theta^2}\right\}$$

48) $q = \cot\left(\frac{\sin t}{t}\right)$

$$\frac{dq}{dt} = -\csc^2\left(\frac{\sin t}{t}\right)\left\{\frac{(t)[\cos t(1)] - (\sin t)[1]}{(t)^2}\right\}$$

$$= -\left\{\frac{t\cos t - \sin t}{t^2}\right\}\csc^2\left(\frac{\sin t}{t}\right) = \left(\frac{\sin t - t\cos t}{t^2}\right)\csc^2\left(\frac{\sin t}{t}\right)$$

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50) $y = \theta^3 e^{-2\theta}\cos 5\theta = \frac{\theta^3\cos(5\theta)}{e^{2\theta}}$

$$\frac{dy}{d\theta} = \frac{(e^{2\theta})\left[\{(\theta^3)[-\sin(5\theta)(5)] + [3\theta^2](\cos(5\theta))\}\right] - (\theta^3\cos(5\theta))[e^{2\theta}(2)]}{(e^{2\theta})^2}$$

$$= \frac{\theta^2 e^{2\theta}\left\{(1)[-5\theta\sin(5\theta) + 3\cos(5\theta)] - (\theta\cos(5\theta))[2]\right\}}{(e^{2\theta})^2}$$

$$= \frac{\theta^2\{3\cos(5\theta) - 5\theta\sin(5\theta) - 2\theta\cos(5\theta)\}}{e^{2\theta}}$$

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52) $y = \sec^2 \pi t$

$$\frac{dy}{dt} = 2\sec(\pi t)\left(\sec(\pi t)\tan(\pi t)(\pi)\right)$$

$$= 2\pi\sec^2(\pi t)\tan(\pi t)$$

54)

$$y = \left(1 + \cot\left(\frac{t}{2}\right)\right)^{-2}$$

$$\frac{dy}{dt} = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3}\left(-\csc^2\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\right) = \frac{\csc^2\left(\frac{t}{2}\right)}{\left(1+\cot\left(\frac{t}{2}\right)\right)^3}$$

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56) $$y = \left(t^{-3/4}\sin t\right)^{4/3} = \left(t^{-1}\right)\left(\sin^{4/3} t\right) = \frac{\sin^{4/3} t}{t}$$

$$\frac{dy}{dt} = \frac{(t)\left[\frac{4}{3}\sin^{1/3} t\cos t\,(1)\right] - \left(\sin^{4/3} t\right)[1]}{(t)^2}$$

$$= \frac{\sin^{1/3} t\left\{(t)\left[\frac{4}{3}\cos t\right] - \left(\sin^{3/3} t\right)[1]\right\}}{t^2}$$

$$= \frac{\sqrt[3]{\sin t}\left\{\frac{4}{3}t\cos t - \sin t\right\}}{t^2} = \frac{\{4t\cos t - 3\sin t\}\sqrt[3]{\sin t}}{3t^2}$$

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58) $$y = \left(e^{\sin\left(\frac{t}{2}\right)}\right)^3 = e^{3\sin\left(\frac{t}{2}\right)}$$

$$\frac{dy}{dt} = e^{3\sin\left(\frac{t}{2}\right)}\left(3\cos\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\right) = \frac{3}{2}\cos\left(\frac{t}{2}\right)e^{3\sin\left(\frac{t}{2}\right)}$$

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60) $$y = \left(\frac{3t-4}{5t+2}\right)^{-5} = \left(\frac{5t+2}{3t-4}\right)^5 = \frac{(5t+2)^5}{(3t-4)^5}$$

$$\frac{dy}{dt} = \frac{\left((3t-4)^5\right)\left[5(5t+2)^4(5)\right] - \left((5t+2)^5\right)\left[5(3t-4)^4(3)\right]}{\left((3t-4)^5\right)^2}$$

⋮

60) continued

$$\frac{dy}{dt} = \frac{5(5t+2)^4(3t-4)^4\{(3t-4)[5]-(5t+2)[3]\}}{(3t-4)^{10}}$$

$$= \frac{5\{15t-20-15t-6\}(5t+2)^4}{(3t-4)^6} = \frac{5\{-26\}(5t+2)^4}{(3t-4)^6}$$

$$= \frac{-130(5t+2)^4}{(3t-4)^6}$$

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62) $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

$$\frac{dy}{dx} = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\cos\left(\frac{t}{3}\right)\left(\frac{1}{3}\right)\right)$$

$$= \frac{-5}{3}\cos\left(\frac{t}{3}\right)\sin\left(5\sin\left(\frac{t}{3}\right)\right)$$

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64) $y = \frac{1}{6}\left(1+\cos^2(7t)\right)^3$

$$\frac{dy}{dt} = \frac{1}{6}\left[3\left(1+\cos^2(7t)\right)^2\left(2\cos(7t)(-\sin(7t))(7)\right)\right]$$

$$= -7\sin(7t)\cos(7t)\left(1+\cos^2(7t)\right)^2$$

$$= \frac{-7}{2}\sin(2(7t))\left(1+\cos^2(7t)\right)^2$$

$$= \frac{-7}{2}\sin(14t)\left(1+\cos^2(7t)\right)^2$$

66) $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right) = 4\sin(1+t^{1/2})^{\frac{1}{2}}$

$$\frac{dy}{dt} = 4\left[\cos(1+t^{1/2})^{1/2}\left(\tfrac{1}{2}(1+t^{1/2})^{-\frac{1}{2}}\left(\tfrac{1}{2}t^{-\frac{1}{2}}\right)\right)\right]$$

$$= \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\left(\sqrt{1+\sqrt{t}}\right)\left(\sqrt{t}\right)} = \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{(1+\sqrt{t})t}} = \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}}$$

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68) $y = \cos^4(\sec^2(3t))$

$$\frac{dy}{dt} = 4\cos^3(\sec^2(3t))\left(-\sin(\sec^2(3t))\,2\sec(3t)\left(\sec(3t)\tan(3t)(3)\right)\right)$$

$$= -24\sec^2(3t)\tan(3t)\sin(\sec^2(3t))\cos^3(\sec^2(3t))$$

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70) $y = \sqrt{3t+\sqrt{2+\sqrt{1-t}}} = \left(3t+\left(2+(1-t)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$\frac{dy}{dt} = \frac{1}{2}\left(3t(2+(1-t)^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(3+\frac{1}{2}\left(2+(1-t)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}(1-t)^{-\frac{1}{2}}(-1)\right)\right)$$

$$= \left(\frac{1}{2\sqrt{3t+\sqrt{2+\sqrt{1-t}}}}\right)\left(3+\frac{1}{2\sqrt{2+\sqrt{1-t}}}\right)\left(\frac{-1}{2\sqrt{1-t}}\right)$$

$$= \frac{-3}{4\sqrt{1-t}\,\sqrt{3t+\sqrt{2+\sqrt{1-t}}}} - \frac{1}{8\sqrt{1-t}\,\sqrt{2+\sqrt{1-t}}\,\sqrt{3t+\sqrt{2+\sqrt{1-t}}}}$$

72) $y = (1-\sqrt{x})^{-1} = (1-x^{\frac{1}{2}})^{-1}$

$$\frac{dy}{dx} = -1\left(1-x^{\frac{1}{2}}\right)^{-2}\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2}\left(1-x^{\frac{1}{2}}\right)^{-2}\left(x^{-\frac{1}{2}}\right)$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2}\left(1-x^{\frac{1}{2}}\right)^{-2}\right)\left[\frac{-1}{2}x^{-\frac{3}{2}}\right] + \left(x^{-\frac{1}{2}}\right)\left[\frac{1}{2}\left(-2\left(1-x^{\frac{1}{2}}\right)^{-3}\left(\frac{-1}{2}x^{-\frac{1}{2}}\right)\right)\right]$$

$$= \frac{-1}{4x^{\frac{3}{2}}(1-\sqrt{x})^2} + \frac{1}{2(\sqrt{x})(\sqrt{x})(1-\sqrt{x})^3}$$

$$= \frac{-1}{4(\sqrt{x})^3(1-\sqrt{x})^2}\left(\frac{1-\sqrt{x}}{1-\sqrt{x}}\right) + \frac{1}{2(\sqrt{x})^2(1-\sqrt{x})^3}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)$$

$$= \frac{-(1-\sqrt{x}) + 2\sqrt{x}}{4(\sqrt{x})^3(1-\sqrt{x})^3} = \frac{-1+\sqrt{x}+2\sqrt{x}}{4(\sqrt{x})^3(1-\sqrt{x})^3} = \frac{3\sqrt{x}-1}{4(\sqrt{x})^3(1-\sqrt{x})^3}$$

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74) $y = 9\tan\left(\frac{x}{3}\right)$

$$\frac{dy}{dx} = 9\left[\sec^2\left(\frac{x}{3}\right)\left(\frac{1}{3}\right)\right] = 3\sec^2\left(\frac{x}{3}\right)$$

$$\frac{d^2y}{dx^2} = 3\left[2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\left(\frac{1}{3}\right)\right)\right]$$

$$= 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$$

76) $y = x^2 (x^3-1)^5$

$$\frac{dy}{dx} = (x^2)\left[5(x^3-1)^4(3x^2)\right] + \left((x^3-1)^5\right)[2x]$$

$$= x(x^3-1)^4\left\{(x^2)[15x] + (x^3-1)[2]\right\}$$

$$= x\ (x^3-1)^4\left\{15x^3 + 2x^3 - 2\right\} = x(x^3-1)^4\left\{17x^3-2\right\}$$

$$= (17x^4-2x)(x^3-1)^4$$

$$\frac{d^2y}{dx^2} = (17x^4-2x)\left[4(x^3-1)^3(3x^2)\right] + \left((x^3-1)^4\right)\left[68x^3-2\right]$$

$$= 2(x^3-1)^3\left\{(17x^4-2x)[6x^2] + (x^3-1)[34x^3-1]\right\}$$

$$= 2(x^3-1)^3\left\{102x^6 - 12x^3 + 34x^6 - 35x^3 + 1\right\}$$

$$= 2\left\{136x^6 - 47x^3 + 1\right\}(x^3-1)^3$$

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78) $y = \sin(x^2 e^x)$

$$\frac{dy}{dx} = \cos(x^2e^x)\left(\left\{(x^2)[e^x(1)] + (e^x)[2x]\right\}\right)$$

$$= \left\{e^x(x^2+2x)\right\}\cos(x^2e^x)$$

78) continued

$$\frac{d^2y}{dx^2} = \left(\{e^x(x^2+2x)\}\right)\left[-\sin(x^2e^x)\left(\{(x^2)[e^x(1)]+(e^x)[2x]\}\right)\right]$$
$$+\left(\cos(x^2e^x)\right)\left[(e^x)[2x+2]+(x^2+2x)[e^x(1)]\right]$$

$$= \left(e^x(x^2+2x)\right)\left[-e^x\sin(x^2e^x)\ (x^2+2x)\right]$$
$$+\left[e^x(x^2+4x+2)\right]\left(\cos(x^2e^x)\right)$$

$$= -e^{2x}(x^4+4x^3+4x^2)\sin(x^2e^x)+e^x(x^2+4x+2)\cos(x^2e^x)$$
$$= (x^2+4x+2)e^x\cos(x^2e^x)-(x^4+4x^3+4x^2)e^{2x}\sin(x^2e^x)$$

---

80) $f(x) = \sec^2 x - 2\tan x$ for $0 \le x \le 2\pi$

$$\frac{df}{dx} = \left[2\sec x\,(\sec x\tan x(1))\right]-2\left[\sec^2 x\,(1)\right]$$
$$= 2\sec^2 x\tan x - 2\sec^2 x = 2\sec^2 x\,(\tan x - 1)$$

$$\frac{d^2f}{dx^2} = (2\sec^2 x)\left[\sec^2 x\,(1)\right]+(\tan x - 1)\left[2\left(2\sec x(\sec x\tan x\,(1))\right)\right]$$
$$= 2\sec^4 x + 4\sec^2 x\tan^2 x - 4\sec^2 x\tan x$$
$$= 2\sec^2 x\,(\sec^2 x + 2\tan^2 x - 2\tan x)$$

80) continued

$$0 = \frac{df}{dx} = 2\sec^2 x\,(\tan x - 1)$$

$2\sec^2 x = 0$	$\tan x - 1 = 0$
$\sec^2 x = 0$	$\tan x = 1$
$\sec x = 0$	$x = \frac{\pi}{4},\ x = \frac{5\pi}{4}$
no real number solution	

$$0 = \frac{d^2 f}{dx^2} = 2\sec^2 x\,(\sec^2 x + 2\tan^2 x - 2\tan x)$$

$2\sec^2 x = 0$	$\sec^2 x + 2\tan^2 x - 2\tan x = 0$
$\sec^2 x = 0$	$(1+\tan^2 x) + 2\tan^2 x - 2\tan x = 0$
$\sec x = 0$	$3\tan^2 x - 2\tan x + 1 = 0$
no real number solution	$\tan x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)}$
	$= \frac{2 \pm \sqrt{4-12}}{6} = \frac{2 \pm \sqrt{-8}}{6}$ {complex number}
	no real number solution

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82) $f(u) = u^5 + 1$, $u = g(x) = \sqrt{x}$, $x = 1$

$$y = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^5 + 1 = x^{\frac{5}{2}} + 1$$

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} = \frac{5}{2}(\sqrt{x})^3 \qquad \left.\frac{dy}{dx}\right|_{x=1} = \frac{5}{2}\left(\sqrt{(1)}\right)^3 = \frac{5}{2}(1) = \frac{5}{2}$$

84) $f(u) = u + \dfrac{1}{\cos^2 u}$, $u = g(x) = \pi x$, $x = \frac{1}{4}$

$$y = f(g(x)) = f(\pi x) = (\pi x) + \frac{1}{\cos^2(\pi x)} = \pi x + \sec^2(\pi x)$$

$$\frac{dy}{dx} = \pi[1] + [2\sec(\pi x)(\sec(\pi x)\tan(\pi x)(\pi))]$$

$$= \pi + 2\pi\sec^2(\pi x)\tan(\pi x)$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \pi + 2\pi\sec^2(\pi(\tfrac{1}{4}))\tan(\pi(\tfrac{1}{4}))$$

$$= \pi + 2\pi(\sqrt{2})^2(1) = \pi + 2\pi(2) = \pi + 4\pi = 5\pi$$

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86) $f(u) = \left(\dfrac{u-1}{u+1}\right)^2$, $u = g(x) = \dfrac{1}{x^2} - 1$, $x = -1$

$$u = g(x) = \frac{1}{x^2} - \frac{x^2}{x^2} = \frac{1-x^2}{x^2}$$

$$y = f(g(x)) = f\left(\frac{1-x^2}{x^2}\right) = \left(\frac{\left(\frac{1-x^2}{x^2}\right) - 1}{\left(\frac{1-x^2}{x^2}\right) + 1}\right)^2$$

$$= \left(\left(\frac{\left(\frac{1-x^2}{x^2}\right) - \frac{1}{1}}{\left(\frac{1-x^2}{x^2}\right) + \frac{1}{1}}\right)\left(\frac{\frac{x^2}{1}}{\frac{x^2}{1}}\right)\right)^2 = \left(\frac{(1-x^2) - x^2}{(1-x^2) + x^2}\right)^2 = \left(\frac{1-2x^2}{1}\right)^2 = (1-2x^2)^2$$

$$\frac{dy}{dx} = 2(1-2x^2)(-4x) = -8x(1-2x^2)$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -8(-1)(1-2(-1)^2)$$

$$= 8(1-2) = 8(-1) = -8$$

88) $r = \sin(\theta(t))$ $\quad \theta(0) = \frac{\pi}{3}$ $\quad \theta'(0) = \left.\frac{d\theta}{dt}\right|_{t=0} = 4$

$\left.\frac{dr}{dt}\right|_{t=0} = ?$ $\quad \frac{dr}{dt} = \cos(\theta(t))\frac{d\theta}{dt}$

$$\left.\frac{dr}{dt}\right|_{t=0} = \cos(\theta(0))\left(\left.\frac{d\theta}{dt}\right|_{t=0}\right) = \cos\left(\frac{\pi}{3}\right)(4) = \left(\frac{1}{2}\right)(4) = 2$$

90-a) $y = 5f(x) - g(x)$, $x = 1$

$\frac{dy}{dx} = 5\frac{df}{dx} - \frac{dg}{dx}$ $\quad \left.\frac{dy}{dx}\right|_{x=1} = 5\left.\frac{df}{dx}\right|_{x=1} - \left.\frac{dg}{dx}\right|_{x=1} = 5\left(\frac{-1}{3}\right) - \left(\frac{-8}{3}\right)$

$$= \frac{-5}{3} + \frac{8}{3} = \frac{3}{3} = 1$$

90-b) $y = f(x)\, g^3(x)$, $x = 0$

$$\frac{dy}{dx} = (f(x))\left[3g^2(x)\frac{dg}{dx}\right] + (g^3(x))\left[\frac{df}{dx}\right]$$

$$\left.\frac{dy}{dx}\right|_{x=0} = (f(0))\left[3(g(0))^2\left.\frac{dg}{dx}\right|_{x=0}\right] + \left((g(0))^3\right)\left[\left.\frac{df}{dx}\right|_{x=0}\right]$$

$$= (1)\left[3(1)^2\left(\frac{1}{3}\right)\right] + \left((1)^3\right)\left[(5)\right] = 1 + 5 = 6$$

90-c) $y = \frac{f(x)}{g(x)+1}$, $x = 1$

$$\frac{dy}{dx} = \frac{(g(x)+1)\left[\frac{df}{dx}\right] - (f(x))\left[\frac{dg}{dx}\right]}{(g(x)+1)^2}$$

90-c) continued

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{(g(1)+1)\left[\left.\frac{df}{dx}\right|_{x=1}\right] - (f(1))\left[\left.\frac{dg}{dx}\right|_{x=1}\right]}{(g(1)+1)^2}$$

$$= \frac{((-4)+1)\left[\frac{-1}{3}\right] - (3)\left[\frac{-8}{3}\right]}{((-4)+1)^2} = \frac{(-3)\left(\frac{-1}{3}\right) - (3)\left(\frac{-8}{3}\right)}{(-3)^2}$$

$$= \frac{1+8}{9} = \frac{9}{9} = 1$$

90-d) $y = f(g(x))$, $x = 0$

$$\frac{dy}{dx} = \frac{d}{dx}(f(g(x)))\left(\frac{dg}{dx}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{d}{dx}(f(g(0)))\left(\left.\frac{dg}{dx}\right|_{x=0}\right) = \frac{d}{dx}(f(1))\left(\frac{1}{3}\right) = \left(\frac{-1}{3}\right)\left(\frac{1}{3}\right) = \frac{-1}{9}$$

90-e) $y = g(f(x))$, $x = 0$

$$\frac{dy}{dx} = \frac{d}{dx}(g(f(x)))\left(\frac{df}{dx}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{d}{dx}(g(f(0)))\left(\left.\frac{df}{dx}\right|_{x=0}\right) = \frac{d}{dx}(g(1))(5) = \left(\frac{-8}{3}\right)(5)$$

$$= \frac{-40}{5}$$

90-f) $y = \left(x^{11} + f(x)\right)^{-2}$, $x = 1$

$$\frac{dy}{dx} = -2\left(x^{11} + f(x)\right)^{-3}\left(11x^{10} + \frac{df}{dx}\right) = \frac{-2\left(11x^{10} + \frac{df}{dx}\right)}{\left(x^{11} + f(x)\right)^3}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-2\left(11(1)^{10} + \left.\frac{df}{dx}\right|_{x=1}\right)}{\left((1)^{11} + f(1)\right)^3} = \frac{-2\left(11 + \left(\frac{-1}{3}\right)\right)}{\left(1 + (3)\right)^3} = \frac{-2\left(\frac{32}{3}\right)}{(4)^3} = \frac{-1}{3}$$

90-g) $y = f(x + g(x))$, $x = 0$

$$\frac{dy}{dx} = \frac{d}{dx}\left(f\left(x + g(x)\right)\right)\left(1 + \frac{dg}{dx}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{d}{dx}\left(f\left((0) + g(0)\right)\right)\left(1 + \left.\frac{dg}{dx}\right|_{x=0}\right) = \frac{d}{dx}\left(f\left(0 + (1)\right)\right)\left(1 + \left(\frac{1}{3}\right)\right)$$

$$= \frac{d}{dx}\left(f(1)\right)\left(\frac{4}{3}\right) = \left(\frac{-1}{3}\right)\left(\frac{4}{3}\right) = \frac{-4}{9}$$

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112)

$$f(x) = \begin{cases} x^2\cos\left(\frac{2}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

a) $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$ (for $x \neq 0$)

$$-x^2 \leq x^2\cos\left(\frac{2}{x}\right) \leq x^2$$

$$0 = \lim_{x\to 0}\left(-x^2\right) \leq \lim_{x\to 0}\left(x^2\cos\left(\frac{2}{x}\right)\right) \leq \lim_{x\to 0}\left(x^2\right) = 0$$

1(2-a) continued

so by Sandwich Theorem $\lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right)=0$

$\lim_{x\to 0} f(x)=0$

$\lim_{x\to 0} f(x)=f(0)=0$, and $f(x)$ is continuous at $x=0$.

b) for $x\neq 0$, $\frac{df}{dx}=(x^2)\left[-\sin\left(\frac{2}{x}\right)(-2x^{-2})\right]+\left(\cos\left(\frac{2}{x}\right)\right)[2x]$

$$=\frac{2x^2\sin\left(\frac{2}{x}\right)}{x^2}+2x\cos\left(\frac{2}{x}\right)$$

$$=2\sin\left(\frac{2}{x}\right)+2x\cos\left(\frac{2}{x}\right)$$

c) $f(0)=0$ $\quad f(0+h)=(0+h)^2\cos\left(\frac{2}{(0+h)}\right)=h^2\cos\left(\frac{2}{h}\right)$

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{\left(h^2\cos\left(\frac{2}{h}\right)\right)-(0)}{h}=\lim_{h\to 0}\left(h\cos\left(\frac{2}{h}\right)\right)$$

$$-1\le\cos\left(\frac{2}{h}\right)<1$$

$-h\le h\cos\left(\frac{2}{h}\right)\le h$ for $h>0$ $\qquad -h\ge h\cos\left(\frac{2}{h}\right)\ge h$ for $h<0$

$0=\lim_{h\to 0}(-h)\le\lim_{h\to 0}\left(h\cos\left(\frac{2}{h}\right)\right)\le\lim_{h\to 0}(h)=0$ $\qquad 0=\lim_{h\to 0}(-h)\ge\lim_{h\to 0}\left(h\cos\left(\frac{2}{h}\right)\right)\ge\lim_{h\to 0}(h)=0$

so by Sandwich Theorem

$$f'(0)=\lim_{h\to 0}\left(h\cos\left(\frac{2}{h}\right)\right)=0$$

112-d) $$-1 \le \cos\left(\frac{2}{x}\right) \le 1$$

$$-2x \le 2x\cos\left(\frac{2}{x}\right) \le 2x$$

$$0 = \lim_{x\to 0}(-2x) \le \lim_{x\to 0}\left(2x\cos\left(\frac{2}{x}\right)\right) \le \lim_{x\to 0}(2x) = 0$$

so by Sandwich Theorem

$$\lim_{x\to 0}\left(2x\cos\left(\frac{2}{x}\right)\right) = 0$$

and $\lim_{x\to 0}\left(2\sin\left(\frac{2}{x}\right)\right)$ D.N.E.

so $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{df}{dx} = \lim_{x\to 0}\left(2\sin\left(\frac{2}{x}\right) + 2x\cos\left(\frac{2}{x}\right)\right)$

$$= \lim_{x\to 0}\left(2\sin\left(\frac{2}{x}\right)\right) + \lim_{x\to 0}\left(2x\cos\left(\frac{2}{x}\right)\right)$$

$$= \lim_{x\to 0}\left(2\sin\left(\frac{2}{x}\right)\right) + (0)$$

since $\lim_{x\to 0}\left(2\sin\left(\frac{2}{x}\right)\right)$ D.N.E., $\lim_{x\to 0}\frac{df}{dx}$ also D.N.E.

so $\frac{df}{dx}$ is not continuous at $x=0$.