MATH 20100 section 3.1

Def. The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{ (provided the limit exists).}$$

The tangent line to the curve at P is the line through P with this slope.

Def. The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided this limit exists.

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

1) The slope of the graph of $y = f(x)$ at $x = x_0$.
2) The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$.
3) The rate of change of $f(x)$ with respect to x at the $x = x_0$.
4) The derivative $f'(x_0)$ at $x = x_0$.

6) $y = (x-1)^2 + 1$, $(1,1)$

$f(x) = (x-1)^2 + 1$

$f(1) = ((1)-1)^2 + 1 = (0)^2 + 1 = 1$

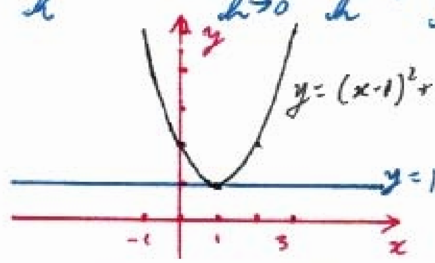
$f(1+h) = ((1-h)-1)^2 + 1 = (-h)^2 + 1 = h^2 + 1$

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(h^2+1) - (1)}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

$y - (1) = 0(x - (1))$

$y - 1 = 0$

$y = 1$



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8) $y = \frac{1}{x^2}$, $(-1,1)$

$f(x) = \frac{1}{x^2}$

$f(-1) = \frac{1}{(-1)^2} = \frac{1}{1} = 1$

$f(-1+h) = \frac{1}{(-1+h)^2}$

$$m = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{(-1+h)^2}\right) - \left(\frac{1}{1}\right)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\frac{1}{(-1+h)^2} - \frac{1}{1}}{\frac{h}{1}}\right)\left(\frac{\frac{(-1+h)^2}{1}}{\frac{(-1+h)^2}{1}}\right) = \lim_{h\to 0} \frac{(1) - (-1+h)^2}{h(-1+h)^2}$$

$$= \lim_{h\to 0} \frac{1 - (1 - 2h + h^2)}{h(-1+h)^2} = \lim_{h\to 0} \frac{2h - h^2}{h(-1+h)^2}$$

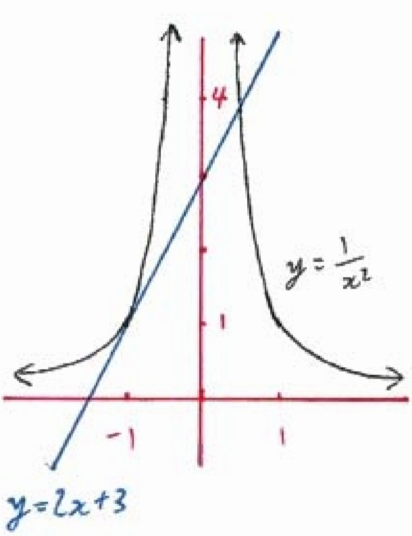
$$= \lim_{h\to 0} \frac{h(2-h)}{h(-1+h)^2} = \lim_{h\to 0} \frac{2-h}{(-1+h)^2} = \frac{2-(0)}{(-1+(0))^2} = \frac{2}{1} = 2$$

$y - (1) = 2(x - (-1))$

$y - 1 = 2(x+1)$

$y - 1 = 2x + 2$

$y = 2x + 3$



10) $y = \frac{1}{x^3}$, $\left(-2, \frac{-1}{8}\right)$

$f(x) = \frac{1}{x^3}$

$f(-2) = \frac{1}{(-2)^3}$ $\qquad$ $f(-2+h) = \frac{1}{(-2+h)^3}$

$$m = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{(-2+h)^3}\right) - \left(\frac{1}{(-2)^3}\right)}{h} = \lim_{h\to 0} \left(\frac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^2}}{\frac{h}{1}}\right)\left(\frac{\frac{(-2)^3(-2+h)^3}{1}}{\frac{(-2)^3(-2+h)^3}{1}}\right)$$

$$= \lim_{h\to 0} \frac{(-2)^3 - (-2+h)^3}{h(-2)^3(-2+h)^3} = \lim_{h\to 0} \frac{(-8) - (-8+12h-6h^2+h^3)}{h(-8)(-2+h)^3} = \lim_{h\to 0} \frac{-12h+6h^2-h^3}{h(-8)(-2+h)^3}$$

$$= \lim_{h\to 0} \frac{h(-12+6h-h^2)}{h(-8)(-2+h)^3} = \lim_{h\to 0} \frac{-12+6h-h^2}{(-8)(-2+h)^3} = \frac{-12+6(0)-(0)^2}{(-8)(-2+(0))^3} = \frac{-12}{(-8)(-8)}$$

$$= \frac{-3}{(2)(8)} = \frac{-3}{16}$$

$y - \left(\frac{-1}{8}\right) = \frac{-3}{16}\left(x - (-2)\right)$

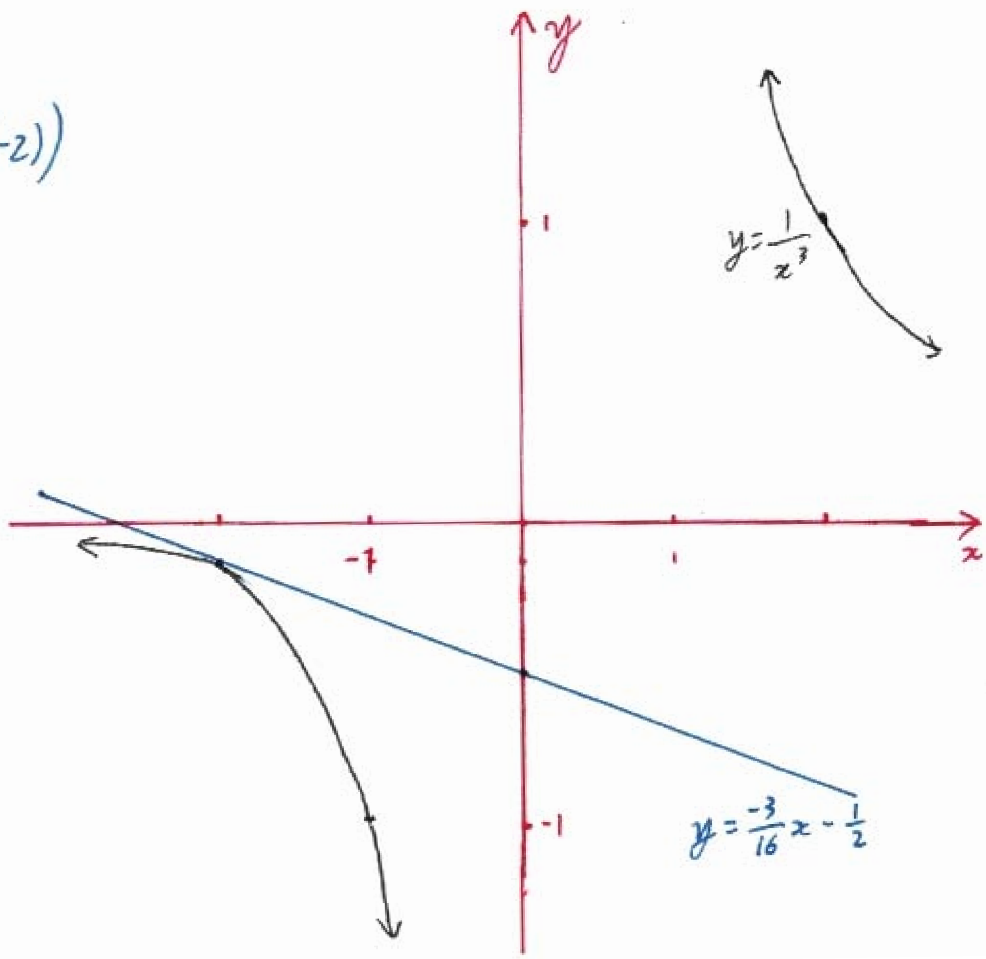
$y + \frac{1}{8} = \frac{-3}{16}(x+2)$

$y + \frac{1}{8} = \frac{-3}{16}x - \frac{3}{8}$

$y = \frac{-3}{16}x - \frac{3}{8} - \frac{1}{8}$

$y = \frac{-3}{16}x - \frac{4}{8}$

$y = \frac{-3}{16}x - \frac{1}{2}$



12) $f(x) = x - 2x^2 \ , \ (1,-1)$

$f(1) = (1) - 2(1)^2 = 1 - 2 = -1$

$f(1+h) = (1+h) - 2(1+h)^2$
$= (1+h) - 2(1+2h+h^2)$
$= (-1-3h-h^2)$

$$m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(-1-3h-h^2)-(-1)}{h} = \lim_{h\to 0} \frac{-3h-h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-3-h)}{h} = \lim_{h\to 0} (-3-h) = -3-(0) = -3$$

$y-(-1) = -3(x-(1)) \implies y+1 = -3x+3$
$y+1 = -3(x-1) \qquad y = -3x+2$

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14) $g(x) = \frac{8}{x^2} \ , \ (2,2)$

$g(2) = \frac{8}{(2)^2} \qquad g(2+h) = \frac{8}{(2+h)^2}$

$$m = \lim_{h\to 0} \frac{g(2+h)-g(2)}{h} = \lim_{h\to 0} \frac{\left(\frac{8}{(2+h)^2}\right)-\left(\frac{8}{(2)^2}\right)}{h} = \lim_{h\to 0} \left(\frac{\frac{8}{(2+h)^2}-\frac{8}{(2)^2}}{\frac{h}{1}}\right)\left(\frac{\frac{(2)^2(2+h)^2}{1}}{\frac{(2)^2(2+h)^2}{1}}\right)$$

$$= \lim_{h\to 0} \frac{8(2)^2-8(2+h)^2}{h(2)^2(2+h)^2} = \lim_{h\to 0} \frac{8(4)-8(4+4h+h^2)}{h(4)(2+h)^2} = \lim_{h\to 0} \frac{8(-4h-h^2)}{h(4)(2+h)^2}$$

$$= \lim_{h\to 0} \frac{8h(-4-h)}{h(4)(2+h)^2} = \lim_{h\to 0} \frac{8(-4-h)}{(4)(2+h)^2} = \frac{8(-4-(0))}{(4)(2+(0))^2} = \frac{8(-4)}{(4)(4)} = \frac{2(-1)}{(1)(1)} = -2$$

$y-(2) = -2(x-(2))$
$y-2 = -2x+4$
$y = -2x+6$

16) $h(t) = t^3 + 3t \quad , (1,4)$

$h(1) = (1)^3 + 3(1) = 4$

$h(1+h) = (1+h)^3 + 3(1+h) = (1 + 3h + 3h^2 + h^3) + 3 + 3h$

$= (4 + 6h + 3h^2 + h^3)$

$$m = \lim_{h \to 0} \frac{h(1+h) - h(1)}{h} = \lim_{h \to 0} \frac{(4 + 6h + 3h^2 + h^3) - (4)}{h}$$

$$= \lim_{h \to 0} \frac{6h + 3h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(6 + 3h + h^2)}{h} = \lim_{h \to 0} (6 + 3h + h^2)$$

$$= 6 + 3(0) + (0)^2 = 6$$

$y - (4) = 6(x - (1)) \Rightarrow y - 4 = 6x - 6$

$y - 4 = 6(x - 1) \qquad y = 6x - 2$

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18) $f(x) = \sqrt{x+1} \quad , (8,3)$

$f(8) = \sqrt{(8)+1} = \sqrt{9} = 3 \qquad f(8+h) = \sqrt{(8+h)+1} = \sqrt{9+h}$

$$m = \lim_{h \to 0} \frac{f(8+h) - f(8)}{h} = \lim_{h \to 0} \frac{(\sqrt{9+h}) - (3)}{h} = \lim_{h \to 0} \left(\frac{\sqrt{9+h} - 3}{h}\right)\left(\frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}\right)$$

$$= \lim_{h \to 0} \frac{(9+h) - (9)}{h(\sqrt{9+h} + 3)} = \lim_{h \to 0} \frac{h}{h(\sqrt{9+h} + 3)} = \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3}$$

$$= \frac{1}{\sqrt{9+(0)} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

$y - (3) = \frac{1}{6}(x - (8)) \Rightarrow y - 3 = \frac{1}{6}x - \frac{8}{6} \qquad y = \frac{1}{6}x - \frac{4}{3} + 3$

$y - 3 = \frac{1}{6}(x - 8) \qquad y - 3 = \frac{1}{6}x - \frac{4}{3} \Rightarrow y = \frac{1}{6}x + \frac{5}{3}$

20) $y = x^3 - 2x + 7$, $x = -2$ $\quad f(x) = x^3 - 2x + 7$

$$f(-2) = (-2)^3 - 2(-2) + 7 = -8 + 4 + 7 = 3$$

$$f(-2+h) = (-2+h)^3 - 2(-2+h) + 7 = (-8 + 12h - 6h^2 + h^3) + 4 - 2h + 7$$
$$= (h^3 - 6h^2 + 10h + 3)$$

$$m = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{(h^3 - 6h^2 + 10h + 3) - (3)}{h} = \lim_{h \to 0} \frac{h^3 - 6h^2 + 10h}{h}$$

$$= \lim_{h \to 0} \frac{h(h^2 - 6h + 10)}{h} = \lim_{h \to 0} (h^2 - 6h + 10) = (0)^2 - 6(0) + 10 = 10$$

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22) $y = \frac{x-1}{x+1}$, $x = 0$ $\quad f(x) = \frac{x-1}{x+1}$

$$f(0) = \frac{(0)-1}{(1)+1} = \frac{-1}{1} = -1 \qquad f(0+h) = \frac{(0+h)-1}{(0+h)+1} = \frac{h-1}{h+1}$$

$$m = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\left(\frac{h-1}{h+1}\right) - (-1)}{h} = \lim_{h \to 0} \left(\frac{\frac{h-1}{h+1} + \frac{1}{1}}{\frac{h}{1}} \right) \left(\frac{\frac{(h+1)}{1}}{\frac{(h+1)}{1}} \right)$$

$$= \lim_{h \to 0} \frac{(h-1) + (h+1)}{h(h+1)} = \lim_{h \to 0} \frac{2h}{h(h+1)} = \lim_{h \to 0} \frac{2}{h+1}$$

$$= \frac{2}{(0)+1} = \frac{2}{1} = 2$$

26) $g(x) = x^3 - 3x = (x^3 - 3x)$

$$g(x+h) = (x+h)^3 - 3(x+h) = (x^3 + 3x^2h + 3xh^2 + h^3) - 3x - 3h$$
$$= (x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h)$$

$$m(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h}$$
$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 + 3x(0) + (0)^2 - 3 = 3x^2 - 3$$

$0 = 3x^2 - 3$

$0 = 3(x^2 - 1)$

$0 = 3(x+1)(x-1)$

$x+1=0$ | $x-1=0$

$x=-1$ | $x=1$

$x=-1:\ g(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$

$(-1, 2)$

$x=1:\ g(1) = (1)^3 - 3(1) = 1 - 3 = -2$

$(1, -2)$

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28) $y = \sqrt{x}$, $m(x) = \frac{1}{4}$ $\qquad f(x) = \sqrt{x} \qquad f(x+h) = \sqrt{x+h}$

$$m(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h}) - (\sqrt{x})}{h} = \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$
$$= \lim_{h\to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+(0)} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$\frac{1}{4} = \frac{1}{2\sqrt{x}}$

$\left(\frac{8\sqrt{x}}{1}\right)\left(\frac{1}{4}\right) = \left(\frac{1}{2\sqrt{x}}\right)\left(\frac{8\sqrt{x}}{1}\right) \Rightarrow$

$2\sqrt{x} = 4$

$\sqrt{x} = 2$

$x = 4$

$x = 4;\ y = \sqrt{(4)} = 2 \qquad (4, 2)$

$y - (2) = \frac{1}{4}(x - (4))$

$y - 2 = \frac{1}{4}(x - 4) \Rightarrow$

$y - 2 = \frac{1}{4}x - 1$

$y = \frac{1}{4}x + 1$

36) $g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$g(0) = 0 \qquad g(0+h) = (0+h)\sin\left(\frac{1}{(0+h)}\right) = h \sin\left(\frac{1}{h}\right)$

$m = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{\left(h \sin\left(\frac{1}{h}\right)\right) - (0)}{h} = \lim_{h\to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h}$

$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$ D.N.E. because as $h \to 0$, $\frac{1}{h} \to \pm\infty$ depending how we approach 0. Also as $\theta \to \pm\infty$, $\sin\theta$ oscilate from -1 to 1.

Since slope D.N.E, at $x=0$, $g(x)$ has no tangent line at the origin.

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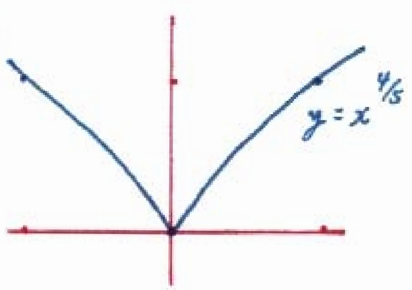
40) $y = x^{4/5}$

a)



at $x=0$, we have a point where it is a pointed end where two curves meet (cusp).

b) $f(x) = y = x^{4/5}$

$f(0) = (0)^{4/5} = 0 \qquad f(0+h) = (0+h)^{4/5} = h^{4/5}$

$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(h^{4/5})-(0)}{h} = \lim_{h\to 0^-} \frac{h^{4/5}}{h}$

$= \lim_{h\to 0^-} \frac{1}{h^{1/5}} = \frac{1}{(0^-)^{1/5}} = \frac{1}{0^-} = -\infty$

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(h^{4/5})-(0)}{h} = \lim_{h\to 0^+} \frac{h^{4/5}}{h}$

$= \lim_{h\to 0^+} \frac{1}{h^{1/5}} = \frac{1}{(0^+)^{1/5}} = \frac{1}{0^+} = +\infty$

Since $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ D.N.E

also $y = x^{4/5}$ does not have a vertical tangent at $x=0$.