MATH 20100 section 2.6

Def. 1) We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \text{ whenever } x > M.$$

2) We say that $f(x)$ has the limit L as x approaches negative infinity and write

$$\lim_{x\to -\infty} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f.

$$|f(x) - L| < \varepsilon \text{ whenever } x < N.$$

Thm 12: All the Limit Laws in Thm 1 are true when we replace $\lim\limits_{x\to c}$ by $\lim\limits_{x\to\infty}$ or $\lim\limits_{x\to -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

Def. A line $y=b$ is a horizontal asymptote of the graph of a function $y=f(x)$ if either
$$\lim_{x\to\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b.$$

Oblique Asymptotes (Polynomial Asymptotes)

If the degree of the numerator of a rational function is 1 ($n>1$, where n is an positive integer) greater than the degree of the denominator, the graph has an oblique or slant line (Polynomial) asymptote. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear (Polynomial) function plus a remainder that goes to zero as $x\to\pm\infty$.

Def. 1) We say that $f(x)$ approaches infinity as x approaches c, and write
$$\lim_{x\to c} f(x)=\infty,$$
if for every positive real number B there exists a corresponding $\delta>0$ such that $\quad f(x)>B$ whenever $0<|x-c|<\delta$.

2) We say that $f(x)$ approaches negative infinity as x approaches c, and write
$$\lim_{x\to c} f(x)=-\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta>0$ such that $\quad f(x)<-B$ whenever $0<|x-c|<\delta$.

Def. A line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

2-a) $\lim_{x\to 4} f(x) = 2$

2-b) $\lim_{x\to 2^+} f(x) = -3$

2-c) $\lim_{x\to 2^-} f(x) = 1$

2-d) $\lim_{x\to 2} f(x)$ D.N.E.

2-e) $\lim_{x\to -3^+} f(x) = +\infty$

2-f) $\lim_{x\to -3^-} f(x) = +\infty$

2-g) $\lim_{x\to -3} f(x) = +\infty$

2-h) $\lim_{x\to 0^+} f(x) = +\infty$

2-i) $\lim_{x\to 0^-} f(x) = -\infty$

2-j) $\lim_{x\to 0} f(x)$ D.N.E.

2-k) $\lim_{x\to \infty} f(x) = 0$

2-l) $\lim_{x\to -\infty} f(x) = -1$

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4) $f(x) = \pi - \frac{2}{x^2}$

a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\pi - \frac{2}{x^2}\right) = \pi - (0) = \pi$

b) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\pi - \frac{2}{x^2}\right) = \pi - (0) = \pi$

6) $g(x) = \dfrac{1}{8 - \frac{5}{x^2}}$

a) $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty}\left(\dfrac{1}{8-\frac{5}{x^2}}\right) = \dfrac{1}{8-(0)} = \dfrac{1}{8}$

b) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty}\left(\dfrac{1}{8-\frac{5}{x^2}}\right) = \dfrac{1}{8-(0)} = \dfrac{1}{8}$

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8) $h(x) = \dfrac{3 - \frac{2}{x}}{4 + \frac{\sqrt{2}}{x^2}}$

a) $\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to\infty}\left(\dfrac{3-\frac{2}{x}}{4+\frac{\sqrt{2}}{x^2}}\right) = \dfrac{3-(0^+)}{4+(0)} = \dfrac{3}{4}$

b) $\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to-\infty}\left(\dfrac{3-\frac{2}{x}}{4+\frac{\sqrt{2}}{x^2}}\right) = \dfrac{3-(0^-)}{4+(0)} = \dfrac{3}{4}$

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10) $\lim\limits_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$ since $-1 \le \cos\theta \le 1$

$$-1 \le \cos\theta \le 1$$

$$\frac{-1}{3\theta} \le \frac{\cos\theta}{3\theta} \le \frac{1}{3\theta}$$

$$\lim_{\theta\to-\infty}\frac{-1}{3\theta} \le \lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta} \le \lim_{\theta\to-\infty}\frac{1}{3\theta}$$

$\lim\limits_{\theta\to-\infty}\dfrac{-1}{3\theta} = 0$ and $\lim\limits_{x\to-\infty}\dfrac{1}{3\theta} = 0$

$$0 \le \lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta} \le 0$$

by the Sandwich (Squeeze) Theorem,

$$\underline{\underline{\lim_{\theta\to-\infty}\frac{\cos\theta}{3\theta} = 0}}$$

12) $\lim_{n\to\infty} \dfrac{n + \sin n^{(+\infty)}}{2n+7-5\sin n^{(+\infty)}}$

$= \lim_{n\to\infty}\left(\dfrac{\frac{n+\sin n}{1}}{\frac{2n+7-5\sin n}{1}}\right)\left(\dfrac{\frac{1}{n}}{\frac{1}{n}}\right)$

$= \lim_{n\to\infty} \dfrac{1+\frac{\sin n}{n}}{2+\frac{7}{n}-5\frac{\sin n}{n}}$

$= \dfrac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\left(\frac{\sin n}{n}\right)}{\lim_{n\to\infty} 2 + \lim_{n\to\infty}\left(\frac{7}{n}\right) - 5\lim_{n\to\infty}\left(\frac{\sin n}{n}\right)}$

$= \dfrac{1+(0)}{2+(0)-5(0)} = \underline{\underline{\dfrac{1}{2}}}$

since $-1 \le \sin n \le 1$

$\dfrac{-1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$

$\lim_{n\to\infty}\left(\frac{-1}{n}\right) \le \lim_{n\to\infty}\left(\frac{\sin n}{n}\right) \le \lim_{n\to\infty}\left(\frac{1}{n}\right)$

$0 \le \lim_{n\to\infty}\left(\frac{\sin n}{n}\right) \le 0$

by Sandwich Theorem

$\lim_{n\to\infty}\left(\frac{\sin n}{n}\right) = 0$

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14) $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$

method 1: $f(x) = \dfrac{\frac{2x^3}{x^3}+\frac{7}{x^3}}{\frac{x^3}{x^3}-\frac{x^2}{x^3}+\frac{x}{x^3}+\frac{7}{x^3}} = \dfrac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$

a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\dfrac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}\right) = \dfrac{2+(0)}{1-(0)+(0)+(0)} = 2$

b) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\dfrac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}\right) = \dfrac{2+(0)}{1-(0)+(0)+(0)} = 2$

method 2:

$$\begin{array}{r} 2 \\ x^3-x^2+x+7 \,\big)\, \overline{2x^3+0x^2+0x+7} \\ -(2x^3-2x^2+2x+14) \\ \hline +2x^2-2x-7 \end{array}$$

$f(x) = 2 + \dfrac{(+2x^2-2x-7)}{x^3-x^2+x+7} = 2 + \dfrac{\frac{2x^2}{x^3}-\frac{2x}{x^3}-\frac{7}{x^3}}{\frac{x^3}{x^3}-\frac{x^2}{x^3}+\frac{x}{x^3}+\frac{7}{x^3}}$

$= 2 + \dfrac{\frac{2}{x}-\frac{2}{x^2}-\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$

a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(2+\dfrac{\frac{2}{x}-\frac{2}{x^2}-\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}\right)$

$= 2+(0) = \underline{\underline{2}}$

b) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(2+\dfrac{\frac{2}{x}-\frac{2}{x^2}-\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}\right)$

$= 2+(0) = 2$

16) $f(x) = \dfrac{3x+7}{x^2-2} = \dfrac{\frac{3x}{x^2} + \frac{7}{x^2}}{\frac{x^2}{x^2} - \frac{2}{x^2}} = \dfrac{\frac{3}{x} + \frac{7}{x^2}}{1 - \frac{2}{x^2}}$

a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \left(\dfrac{\frac{3}{x} + \frac{7}{x^2}}{1 - \frac{2}{x^2}} \right) = \dfrac{(0)+(0)}{1-(0)} = 0$

b) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \left(\dfrac{\frac{3}{x} + \frac{7}{x^2}}{1 - \frac{2}{x^2}} \right) = \dfrac{(0)+(0)}{1-(0)} = 0$

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18) $h(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$ {method 1 is better}

$= \dfrac{\frac{9x^4}{x^4} + \frac{x}{x^4}}{\frac{2x^4}{x^4} + \frac{5x^2}{x^4} - \frac{x}{x^4} + \frac{6}{x^4}} = \dfrac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}}$

a) $\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to\infty} \left(\dfrac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \right) = \dfrac{9+(0)}{2+(0)-(0)+(0)} = \dfrac{9}{2}$

b) $\lim\limits_{x\to-\infty} h(x) = \lim\limits_{x\to-\infty} \left(\dfrac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \right) = \dfrac{9+(0)}{2+(0)-(0)+(0)} = \dfrac{9}{2}$

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20) $g(x) = \dfrac{x^3+7x^2-2}{x^2-x+1}$ {method 2 is better}

$$\begin{array}{r}
x + 8 \\
x^2-x+1 \overline{\big)\, x^3+7x^2+0x-2} \\
-(x^3-x^2+x) \\
\hline
+8x^2-x-2 \\
-(8x^2-8x+8) \\
\hline
+7x-10
\end{array}$$

$g(x) = x+8+\dfrac{(+7x-10)}{x^2-x+1}$

$= x+8+\dfrac{\frac{7x}{x^2} - \frac{10}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}}$

$= x+8+\dfrac{\frac{7}{x} - \frac{10}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}}$

a) $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \left(x+8+\dfrac{\frac{7}{x} - \frac{10}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}} \right)$

$= (+\infty)+8+\dfrac{(0)-(0)}{1-(0)+(0)} = +\infty$

b) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \left(x+8+\dfrac{\frac{7}{x} - \frac{10}{x^2}}{1 - \frac{1}{x} + \frac{1}{x^2}} \right)$

$= (-\infty)+8+\dfrac{(0)-(0)}{1-(0)+(0)} = -\infty$

"$g(x)$ has Oblique asymptote $y = x+8$"

22) $h(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$ {method 2 is better}

$$\begin{array}{r l}
 & \frac{-5}{4}x^3 \\
-4x^5+0x^4+0x^3+0x^2+x+3 \; \big) & 5x^8+0x^7+0x^6+0x^5+0x^4-2x^3+0x^2+0x+9 \\
 & -\left(5x^8+0x^7+0x^6+0x^5-\frac{5}{4}x^4+\frac{15}{4}x^3\right) \\
\hline
 & \frac{+5}{4}x^4+\frac{7}{4}x^3+0x^2+0x+9
\end{array}$$

$$h(x) = \frac{-5}{4}x^3 + \frac{\left(\frac{+5}{4}x^4+\frac{7}{4}x^3+9\right)}{3+x-4x^5} = \frac{-5}{4}x^3 + \frac{\frac{5x^4}{4x^5}+\frac{7x^3}{x^5}+\frac{9}{x^5}}{\frac{3}{x^5}+\frac{x}{x^5}-\frac{4x^5}{x^5}}$$

$$= \frac{-5}{4}x^3 + \frac{\frac{5}{4x}+\frac{7}{x^2}+\frac{9}{x^5}}{\frac{3}{x^5}+\frac{1}{x^4}-4}$$

a) $$\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left(\frac{-5}{4}x^3 + \frac{\frac{5}{4x}+\frac{7}{x^2}+\frac{9}{x^5}}{\frac{3}{x^5}+\frac{1}{x^4}-4}\right)$$

$$= \frac{-5}{4}(+\infty)^3 + \frac{(0)+(0)+(0)}{(0)+(0)-4} = -\infty$$

b) $$\lim_{x\to-\infty} = \lim_{x\to-\infty}\left(\frac{-5}{4}x^3 + \frac{\frac{5}{4x}+\frac{7}{x^2}+\frac{9}{x^5}}{\frac{3}{x^5}+\frac{1}{x^4}-4}\right)$$

$$= \frac{-5}{4}(-\infty)^3 + \frac{(0)+(0)+(0)}{(0)+(0)-4} = +\infty$$

"$h(x)$ has Polynomial asymptote $y = \frac{-5}{4}x^3$"

24) $$\lim_{x\to-\infty}\left(\frac{x^2+x-1}{8x^2-3}\right)^{\frac{1}{3}} = \lim_{x\to-\infty}\sqrt[3]{\frac{x^2+x-1}{8x^2-3}} = \lim_{x\to-\infty}\frac{\overset{(+\infty)}{\sqrt[3]{x^2+x-1}}}{\underset{(+\infty)}{\sqrt[3]{8x^2-3}}}$$

$$= \lim_{x\to-\infty}\left(\frac{\frac{\sqrt[3]{x^2+x-1}}{1}}{\frac{\sqrt[3]{8x^2-3}}{1}}\right)\left(\frac{\frac{1}{\sqrt[3]{x^2}}}{\frac{1}{\sqrt[3]{x^2}}}\right) = \lim_{x\to-\infty}\frac{\frac{\sqrt[3]{x^2+x-1}}{\sqrt[3]{x^2}}}{\frac{\sqrt[3]{8x^2-3}}{\sqrt[3]{x^2}}}$$

$$= \lim_{x\to-\infty}\frac{\sqrt[3]{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}}}{\sqrt[3]{\frac{8x^2}{x^2}-\frac{3}{x^2}}} = \lim_{x\to\infty}\frac{\sqrt[3]{1+\frac{1}{x}-\frac{1}{x^2}}}{\sqrt[3]{8-\frac{3}{x^2}}} = \frac{\sqrt[3]{1+(0)-(0)}}{\sqrt[3]{8-(0)}} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}}$$

$$= \underline{\underline{\frac{1}{2}}}$$

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26) $$\lim_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x\to\infty}\frac{\overset{(+\infty)}{\sqrt{x^2-5x}}}{\underset{(+\infty)}{\sqrt{x^3+x-2}}}$$

$$= \lim_{x\to\infty}\left(\frac{\frac{\sqrt{x^2-5x}}{1}}{\frac{\sqrt{x^3+x-2}}{1}}\right)\left(\frac{\frac{1}{\sqrt{x^3}}}{\frac{1}{\sqrt{x^3}}}\right) = \lim_{x\to\infty}\frac{\frac{\sqrt{x^2-5x}}{\sqrt{x^3}}}{\frac{\sqrt{x^3+x-2}}{\sqrt{x^3}}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{\frac{x^2}{x^3}-\frac{5x}{x^3}}}{\sqrt{\frac{x^3}{x^3}+\frac{x}{x^3}-\frac{2}{x^3}}} = \lim_{x\to\infty}\frac{\sqrt{\frac{1}{x}-\frac{5}{x^2}}}{\sqrt{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \frac{\sqrt{(0)-(0)}}{\sqrt{1+(0)-(0)}}$$

$$= \frac{\sqrt{0}}{\sqrt{1}} = \underline{\underline{0}}$$

28) $$\lim_{x\to\infty} \frac{\overset{(+\infty)}{2+\sqrt{x}}}{\underset{(-\infty)}{2-\sqrt{x}}} = \lim_{x\to\infty}\left(\frac{\frac{2+\sqrt{x}}{1}}{\frac{2-\sqrt{x}}{1}}\right)\left(\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{x}}}{\frac{2}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{2}{\sqrt{x}}+1}{\frac{2}{\sqrt{x}}-1} = \frac{(0)+1}{(0)-1} = \underline{\underline{-1}}$$

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30) $$\lim_{x\to\infty} \frac{x^{-1}+\overset{0}{x^{-4}}}{\underset{0}{x^{-2}}-x^{-3}} = \lim_{x\to\infty}\left(\frac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}\right)\left(\frac{\frac{x^4}{1}}{\frac{x^4}{1}}\right) = \lim_{x\to\infty}\frac{x^3+1}{x^2-x}$$

$$\begin{array}{r} x+1 \\ x^2-x+0\,\overline{)\,x^3+0x^2+0x+1} \\ -(x^3-x^2+0x) \\ \hline +x^2+0x+1 \\ -(x^2-x+0) \\ \hline +x+1 \end{array}$$

$$= \lim_{x\to\infty}\left(x+1+\frac{(+x+1)}{x^2-x}\right) = \lim_{x\to\infty}\left(x+1+\frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{x}{x^2}}\right)$$

$$= \lim_{x\to\infty}\left(x+1+\frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x}}\right) = (+\infty)+1+\frac{(0)+(0)}{1-(0)}$$

$$= \underline{\underline{+\infty}}$$

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32) $$\lim_{x\to-\infty} \frac{\overset{(-\infty)}{\sqrt[3]{x}-5x+3}}{\underset{(-\infty)}{2x+x^{\frac{2}{3}}-4}} = \lim_{x\to-\infty}\left(\frac{\frac{\sqrt[3]{x}-5x+3}{1}}{\frac{2x+(\sqrt[3]{x})^2-4}{1}}\right)\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt[3]{x}}{x}-\frac{5x}{x}+\frac{3}{x}}{\frac{2x}{x}+\frac{(\sqrt[3]{x})^2}{x}-\frac{4}{x}} = \lim_{x\to-\infty} \frac{\frac{1}{(\sqrt[3]{x})^2}-5+\frac{3}{x}}{2+\frac{1}{\sqrt[3]{x}}-\frac{4}{x}}$$

$$= \frac{(0)-5+(0)}{2+(0)-(0)} = \underline{\underline{\frac{-5}{2}}}$$

34) $$\lim_{x\to-\infty} \frac{\overset{(+\infty)}{\sqrt{x^2+1}}}{\underset{(-\infty)}{x+1}} = \lim_{x\to-\infty}\left(\frac{\frac{\sqrt{x^2+1}}{1}}{\frac{x+1}{1}}\right)\left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}\right)$$

$$= \lim_{x\to-\infty}\left(\frac{\frac{\sqrt{x^2+1}}{1}}{\frac{x+1}{1}}\right)\left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}\right) = \lim_{x\to-\infty}\frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to-\infty}\frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}}$$

$$= \frac{\sqrt{1+(0)}}{1+(0)} = \frac{\sqrt{1}}{1} = \underline{\underline{1}}$$

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36) $$\lim_{x\to-\infty}\frac{\overset{(+\infty)}{4-3x^3}}{\underset{(+\infty)}{\sqrt{x^6+9}}} = \lim_{x\to-\infty}\left(\frac{\frac{4-3x^3}{1}}{\frac{\sqrt{x^6+9}}{1}}\right)\left(\frac{\frac{1}{\sqrt{x^6}}}{\frac{1}{\sqrt{x^6}}}\right)$$

$$= \lim_{x\to-\infty}\left(\frac{\frac{4-3x^3}{1}}{\frac{\sqrt{x^6+9}}{1}}\right)\left(\frac{\frac{1}{(-x^3)}}{\frac{1}{\sqrt{x^6}}}\right) = \lim_{x\to-\infty}\frac{\frac{4}{(-x^3)}-\frac{3x^3}{(-x^3)}}{\sqrt{\frac{x^6}{x^6}+\frac{9}{x^6}}}$$

$$= \lim_{x\to-\infty}\frac{\frac{-4}{x^3}+3}{\sqrt{1+\frac{9}{x^6}}} = \frac{(0)+3}{\sqrt{1+(0)}} = \frac{3}{\sqrt{1}} = \underline{\underline{3}}$$

38) $\lim_{x\to 0^-} \frac{5}{2x} = \frac{5}{2(0^-)} = \frac{5}{(0^-)} = -\infty$

40) $\lim_{x\to 3^+} \frac{1}{x-3} = \frac{1}{(3^+)-3} = \frac{1}{(0^+)} = +\infty$

42) $\lim_{x\to 5^-} \frac{3x}{2x+10} = \frac{3(5^-)}{2(5^-)+10} = \frac{15}{(10^-)+10} = \frac{15}{(0^+)} = +\infty$

44) $\lim_{x\to 0} \frac{-1}{x^2(x+1)} = \frac{-1}{(0)^2((0)+1)} = \frac{-1}{(0^+)(1)} = -\infty$

46-a) $\lim_{x\to 0^+} \frac{2}{x^{\frac{1}{5}}} = \lim_{x\to 0^+} \frac{2}{\sqrt[5]{x}} = \frac{2}{\sqrt[5]{(0^+)}} = \frac{2}{(0^+)} = +\infty$

46-b) $\lim_{x\to 0^-} \frac{2}{x^{\frac{1}{5}}} = \lim_{x\to 0^-} \frac{2}{\sqrt[5]{x}} = \frac{2}{\sqrt[5]{(0^-)}} = \frac{2}{(0^-)} = -\infty$

48) $\lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{(\sqrt[3]{x})^2} = \frac{1}{(\sqrt[3]{(0)})^2} = \frac{1}{(0^+)} = +\infty$

54) $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$

a) $\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{(x+1)(x-1)} = \frac{(1^+)}{((1^+)+1)((1^+)-1)} = \frac{1}{(2)(0^+)} = +\infty$

b) $\lim_{x\to 1^-} \frac{x}{(x+1)(x-1)} = \frac{(1^-)}{((1^-)+1)((1^-)-1)} = \frac{1}{(2)(0^-)} = -\infty$

c) $\lim_{x\to -1^+} \frac{x}{(x+1)(x-1)} = \frac{(-1^+)}{((-1^+)+1)((-1^+)-1)} = \frac{-1}{(0^+)(-2)} = +\infty$

d) $\lim_{x\to -1^-} \frac{x}{(x+1)(x-1)} = \frac{(-1^-)}{((-1^-)+1)((-1^-)-1)} = \frac{-1}{(0^-)(-2)} = -\infty$

56) $\dfrac{x^2-1}{2x+4} = \dfrac{x^2-1}{2(x+2)} = \dfrac{(x+1)(x-1)}{2(x+2)}$

a) $\lim\limits_{x\to -2^+} \dfrac{(x+1)(x-1)}{2(x+2)} = \dfrac{((-2^+)+1)((-2^+)-1)}{2((-2^+)+2)} = \dfrac{(-1)(-3)}{2(0^+)} = +\infty$

b) $\lim\limits_{x\to -2^-} \dfrac{(x+1)(x-1)}{2(x+2)} = \dfrac{((-2^-)+1)((-2^-)-1)}{2((-2^-)+2)} = \dfrac{(-1)(-3)}{2(0^-)} = -\infty$

c) $\lim\limits_{x\to 1^+} \dfrac{(x+1)(x-1)}{2(x+2)} = \dfrac{((1^+)+1)((1^+)-1)}{2((1^+)+2)} = \dfrac{(2)(0^+)}{2(3^+)} = 0$

d) $\lim\limits_{x\to 0^-} \dfrac{(x+1)(x-1)}{2(x+2)} = \dfrac{((0^-)+1)((0^-)-1)}{2((0^-)+2)} = \dfrac{(1^-)(-1^-)}{2(2^-)} = \dfrac{-1}{4}$

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58) $\dfrac{x^2-3x+2}{x^3-4x} = \dfrac{(x-1)(x-2)}{x(x+2)(x-2)} = \dfrac{x-1}{x(x+2)}$

a) $\lim\limits_{x\to 2^+} \dfrac{x-1}{x(x+2)} = \dfrac{(2^+)-1}{(2^+)((2^+)+2)} = \dfrac{1}{(2)(4)} = \dfrac{1}{8}$

b) $\lim\limits_{x\to -2^+} \dfrac{x-1}{x(x+2)} = \dfrac{(-2^+)-1}{(-2^+)((-2^+)+2)} = \dfrac{-3}{(-2)(0^+)} = +\infty$

c) $\lim\limits_{x\to 0^-} \dfrac{x-1}{x(x+2)} = \dfrac{(0^-)-1}{(0^-)((0^-)+2)} = \dfrac{-1}{(0^-)(2)} = +\infty$

d) $\lim\limits_{x\to 1^+} \dfrac{x-1}{x(x+2)} = \dfrac{(1^+)-1}{(1^+)((1^+)+2)} = \dfrac{(0^+)}{(1)(3)} = 0$

e) $\lim\limits_{x\to 0^+} \dfrac{x-1}{x(x+2)} = \dfrac{(0^+)-1}{(0^+)((0^+)+2)} = \dfrac{-1}{(0^+)(2)} = -\infty$ and from part c,

$\lim\limits_{x\to 0^-} \dfrac{x-1}{x(x+2)} = +\infty$ $\quad$ $\lim\limits_{x\to 0} \dfrac{x-1}{x(x+2)} = \lim\limits_{x\to 0} \dfrac{x^2-3x+2}{x^3-4x}$ D.N.E.

64) $y = \frac{1}{x+1} = \frac{1}{x-(-1)} + (0)$

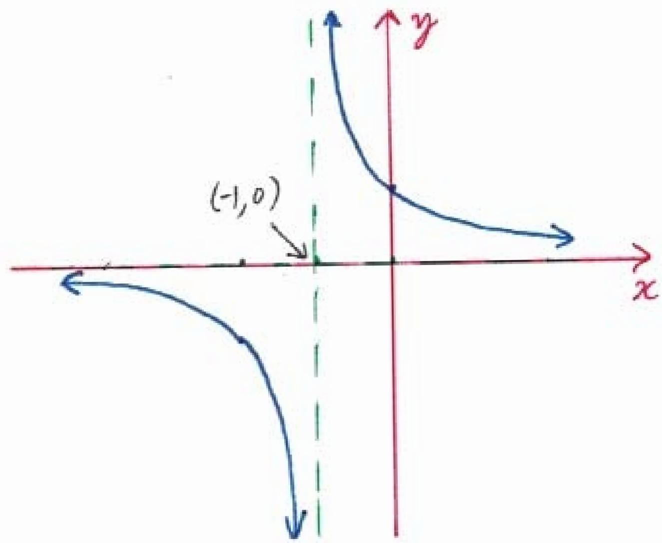
"center": $(-1, 0)$
[like $y = \frac{1}{x}$]

V.A.: $x+1=0$
$x = -1$

H.A.:

$y = \lim_{x\to\infty} \frac{1}{x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{x}{x}+\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x}} = \frac{(0)}{1+(0)} = 0$



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66) $y = \frac{-3}{x-3} = \frac{-3}{x-(3)} + (0)$

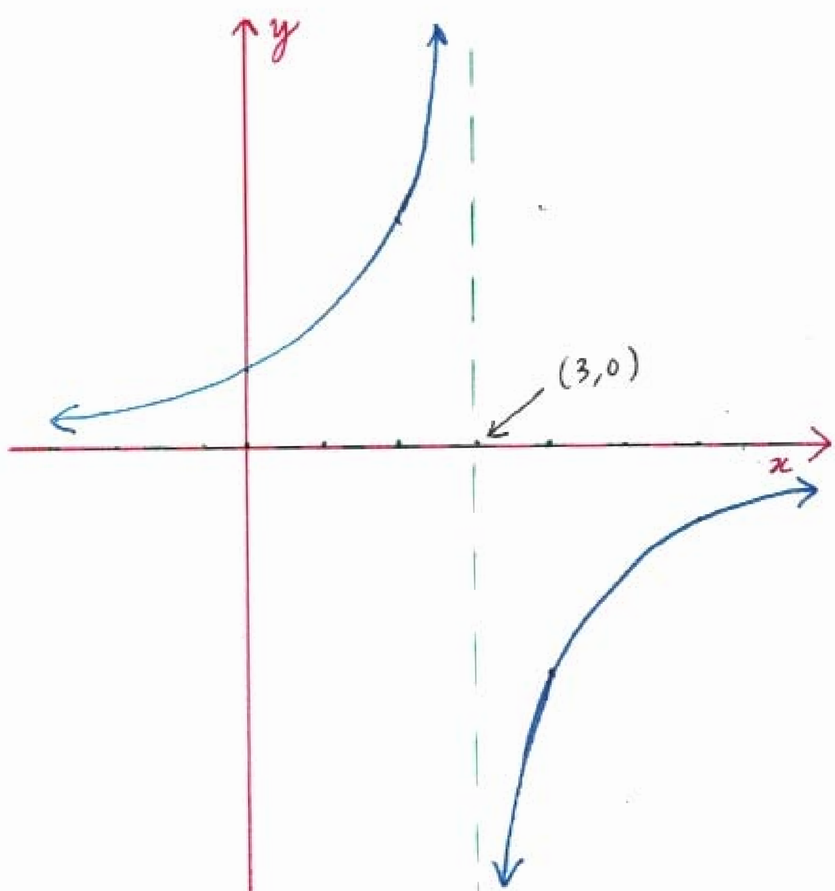
"center": $(3, 0)$
[like $y = \frac{-3}{x}$]

V.A.: $x-3=0$
$x = 3$

H.A.:

$y = \lim_{x\to\infty} \frac{-3}{x-3} = \lim_{x\to\infty} \frac{\frac{-3}{x}}{\frac{x}{x}-\frac{3}{x}}$

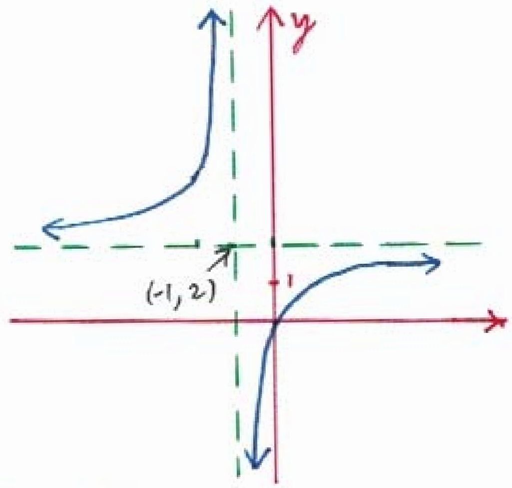
$= \lim_{x\to\infty} \frac{\frac{-3}{x}}{1-\frac{3}{x}} = \frac{(0)}{1-(0)} = 0$



68) $y = \frac{2x}{x+1} = 2 - \frac{2}{x+1} = \frac{-2}{x-(-1)} + (2)$ "center": $(-1, 2)$ [like $y = \frac{-2}{x}$]

$$\begin{array}{r} 2 \\ x+1 \overline{)\, 2x + 0} \\ -(2x+2) \\ \hline -2 \end{array}$$

V.A.:
$x + 1 = 0$
$x = -1$



H.A.: $y = \lim_{x \to \infty} \frac{2x}{x+1} = \lim_{x \to \infty} \left(2 - \frac{2}{x+1}\right)$

$= 2 - (0) = 2$

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70) $y = \frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)}$

H.A.: $\lim_{x \to \infty} \frac{2x}{x^2-1} = \lim_{x \to \infty} \frac{\frac{2x}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \frac{(0)}{1-(0)} = 0$

$\lim_{x \to -\infty} \frac{2x}{x^2-1} = \lim_{x \to -\infty} \frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \frac{(0)}{1-(0)} = 0$

$\therefore y = 0$

V.A.: $x^2 - 1 = 0$
$(x+1)(x-1) = 0$

$x+1=0$	$x-1=0$
$x=-1$	$x=1$

domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

when $x = 0$, $y = \frac{2(0)}{(0)^2-1} = 0$, when $x = 0^+$, $y = \frac{2(0^+)}{(0^+)^2-1} = \frac{(0^+)}{-1} = 0^-$

when $x = 0^-$, $y = \frac{2(0^-)}{(0^-)^2-1} = \frac{(0^-)}{-1} = 0^+$

this shows that the graph crosses the x-axis.

$\lim_{x \to 1^+} \frac{2x}{(x+1)(x-1)} = \frac{2(1^+)}{((1^+)+1)((1^+)-1)} = \frac{2}{(2)(0^+)} = +\infty$

$\lim_{x \to 1^-} \frac{2x}{(x+1)(x-1)} = \frac{2(1^-)}{((1^-)+1)((1^-)-1)} = \frac{2}{(2)(0^-)} = -\infty$

$\lim_{x \to -1^+} \frac{2x}{(x+1)(x-1)} = \frac{2(-1^+)}{((-1^+)+1)((-1^+)-1)} = \frac{-2}{(0^+)(-2)} = +\infty$

$\lim_{x \to -1^-} \frac{2x}{(x+1)(x-1)}$

$= \frac{2(-1^-)}{((-1^-)+1)((-1^-)-1)} = \frac{-2}{(0^-)(-2)}$

$= -\infty$

range: $(-\infty, \infty)$

72) $y = \dfrac{4e^x + e^{2x}}{e^x + e^{2x}}$

V.A.: $e^x + e^{2x} = 0$

$e^x(1+e^x) = 0$

$e^x = 0 \quad 1 + e^x = 0$

$\Downarrow \qquad e^x = -1$

$x = \ln(0) \qquad \Downarrow$

$x = \ln(-1)$

no solution

no V.A.

H.A.:

$$\lim_{x\to\infty} \frac{4e^x + e^{2x}}{e^x + e^{2x}} = \lim_{x\to\infty} \frac{\frac{4e^x}{e^{2x}} + \frac{e^{2x}}{e^{2x}}}{\frac{e^x}{e^{2x}} + \frac{e^{2x}}{e^{2x}}}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{e^x} + 1}{\frac{1}{e^x} + 1} = \frac{(0)+1}{(0)+1} = 1 \; : y = 1$$

$$\lim_{x\to-\infty} \frac{4e^x + e^{2x}}{e^x + e^{2x}} = \lim_{x\to-\infty} \frac{\frac{4e^x}{e^x} + \frac{e^{2x}}{e^x}}{\frac{e^x}{e^x} + \frac{e^{2x}}{e^x}}$$

$$= \lim_{x\to-\infty} \frac{4 + e^x}{1 + e^x} = \frac{4+(0)}{1+(0)} = 4 \; : y = 4$$

domain: $(-\infty, \infty)$ range: $(1, 4)$

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74) $y = \dfrac{x^3}{x^3 - 8} = \dfrac{x^3}{(x-2)(x^2+2x+4)}$

V.A.: $x^3 - 8 = 0$

$(x-2)(x^2+2x+4) = 0$

$x - 2 = 0 \quad x^2 + 2x + 4 = 0$

$x = 2 \quad$ no real # solution

H.A.:

$$\lim_{x\to\infty} \frac{x^3}{x^3-8} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}}{\frac{x^3}{x^3} - \frac{8}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1}{1 - \frac{8}{x^3}} = \frac{1}{1-(0)} = 1$$

$$\lim_{x\to-\infty} \frac{x^3}{x^3-8} = \lim_{x\to-\infty} \frac{1}{1-\frac{8}{x^3}} = \frac{1}{1-(0)} = 1$$

$y = 1$

domain: $(-\infty, 2) \cup (2, \infty)$

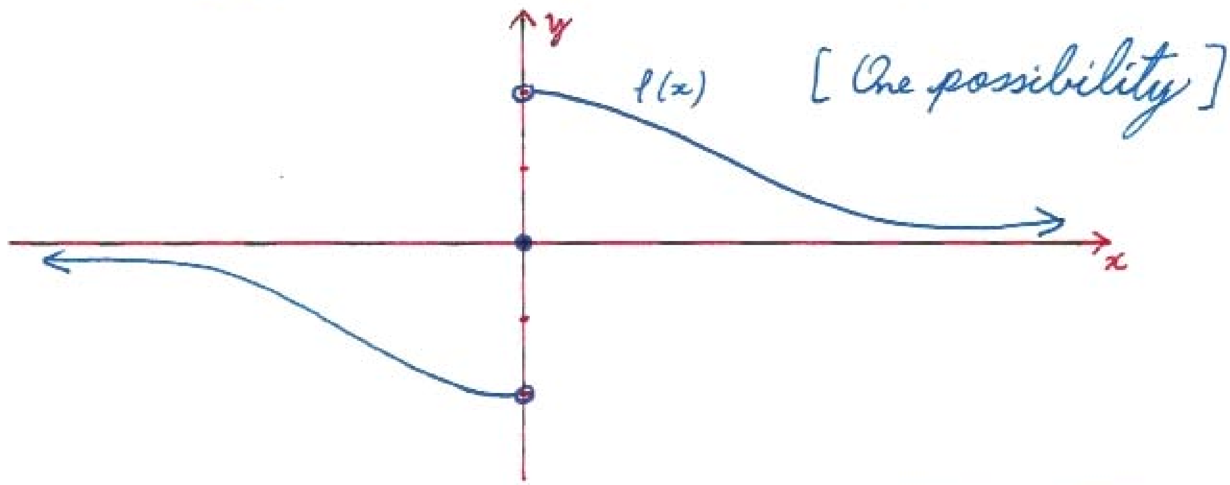
$$\lim_{x\to 2^+} \frac{x^3}{(x-2)(x^2+2x+4)} = \frac{(2^+)^3}{((2^+)-2)((2^+)^2 + 2(2^+)+4)} = \frac{8}{(0^+)(12)} = +\infty$$

$$\lim_{x\to 2^-} \frac{x^3}{(x-2)(x^2+2x+4)} = \frac{(2^-)^3}{((2^-)-2)((2^-)^2 + 2(2^-)+4)} = \frac{8}{(0^-)(12)} = -\infty$$

also $\dfrac{x^3}{x^3+8} \neq 1$ so range: $(-\infty, 1) \cup (1, \infty)$

76) $f(0) = 0$, $\lim_{x \to \pm\infty} f(x) = 0$,

$\lim_{x \to 0^+} f(x) = 2$, and $\lim_{x \to 0^-} f(x) = -2$



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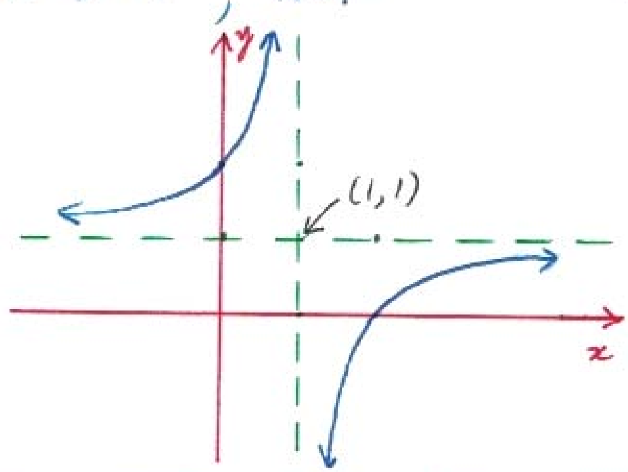
82) $\lim_{x \to \pm\infty} k(x) = 1$, $\lim_{x \to 1^-} k(x) = \infty$, $\lim_{x \to 1^+} k(x) = -\infty$



[One possibility

$y = \frac{1}{x-(1)} + (1)$

$y = \frac{1}{x-1} + 1$]

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86) $$\lim_{x\to\infty} \left(\overset{(+\infty)}{\sqrt{x+9}} - \overset{(+\infty)}{\sqrt{x+4}}\right) = \lim_{x\to\infty} \frac{\left(\sqrt{x+9} - \sqrt{x+4}\right)}{1}\left(\frac{\sqrt{x+9} + \sqrt{x+4}}{\sqrt{x+9} + \sqrt{x+4}}\right)$$

$$= \lim_{x\to\infty} \frac{(x+9) - (x+4)}{\sqrt{x+9} + \sqrt{x+4}} = \lim_{x\to\infty} \frac{5}{\sqrt{x+9} + \sqrt{x+4}} = \lim_{x\to\infty} \left(\frac{\frac{5}{1}}{\frac{\sqrt{x+9}}{1} + \frac{\sqrt{x+4}}{1}}\right)\left(\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{5}{\sqrt{x}}}{\sqrt{\frac{x}{x} + \frac{9}{x}} + \sqrt{\frac{x}{x} + \frac{4}{x}}} = \lim_{x\to\infty} \frac{\frac{5}{\sqrt{x}}}{\sqrt{1 + \frac{9}{x}} + \sqrt{1 + \frac{4}{x}}} = \frac{(0)}{\sqrt{1+(0)} + \sqrt{1+(0)}} = \frac{0}{1+1} = \underline{\underline{0}}$$

88) $$\lim_{x\to-\infty}\left(\overset{(+\infty)}{\sqrt{x^2+3}}\ \overset{+\,(-\infty)}{+\,x}\right) = \lim_{x\to-\infty}\left(\frac{(\sqrt{x^2+3}+x)}{1}\right)\left(\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}\right)$$

$$= \lim_{x\to-\infty}\frac{(x^2+3)-x^2}{\sqrt{x^2+3}-x} = \lim_{x\to-\infty}\frac{3}{\sqrt{x^2+3}-x} = \lim_{x\to-\infty}\left(\frac{\frac{3}{1}}{\frac{\sqrt{x^2+3}}{1}-\frac{x}{1}}\right)\left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}\right)$$

$$= \lim_{x\to-\infty}\frac{\frac{3}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{\sqrt{x^2}}} = \lim_{x\to-\infty}\frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1+\frac{3}{x^2}}-\frac{x}{(-x)}} = \lim_{x\to-\infty}\frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{(0)}{\sqrt{1+(0)}+1} = \frac{0}{1+1} = 0$$ because $x\to-\infty$

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90) $$\lim_{x\to\infty}\left(\overset{(+\infty)}{\sqrt{9x^2-x}}\ \overset{-\,(+\infty)}{-\,3x}\right) = \lim_{x\to\infty}\left(\frac{(\sqrt{9x^2-x}-3x)}{1}\right)\left(\frac{\sqrt{9x^2-x}+3x}{\sqrt{9x^2-x}+3x}\right)$$

$$= \lim_{x\to\infty}\frac{(9x^2-x)-9x^2}{\sqrt{9x^2-x}+3x} = \lim_{x\to\infty}\frac{-x}{\sqrt{9x^2-x}+3x} = \lim_{x\to\infty}\left(\frac{\frac{-x}{1}}{\frac{\sqrt{9x^2-x}}{1}+\frac{3x}{1}}\right)\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{-x}{x}}{\frac{\sqrt{9x^2-x}}{x}+\frac{3x}{x}} = \lim_{x\to\infty}\frac{-1}{\sqrt{\frac{9x^2}{x^2}-\frac{x}{x^2}}+3} = \lim_{x\to\infty}\frac{-1}{\sqrt{9-\frac{1}{x}}+3}$$

$$= \frac{-1}{\sqrt{9-(0)}+3} = \frac{-1}{\sqrt{9}+3} = \frac{-1}{3+3} = \frac{-1}{6}$$

$$92)\ \lim_{x\to\infty}\left(\overset{(+\infty)}{\sqrt{x^2+x}} - \overset{(+\infty)}{\sqrt{x^2-x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)}{1}\right)\left(\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)$$

$$= \lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty}\left(\frac{\frac{2x}{1}}{\frac{\sqrt{x^2+x}}{1}+\frac{\sqrt{x^2-x}}{1}}\right)\left(\frac{\frac{1}{x}}{\frac{1}{x}}\right) = \lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+x}}{x}+\frac{\sqrt{x^2-x}}{x}}$$

$$= \lim_{x\to\infty}\frac{2}{\frac{\sqrt{x^2+x}}{\sqrt{x^2}}+\frac{\sqrt{x^2-x}}{\sqrt{x^2}}} = \lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{x}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{\sqrt{1+(0)}+\sqrt{1-(0)}} = \frac{2}{\sqrt{1}+\sqrt{1}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1$$