MATH 20100 section 2.5

Def. Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

The function f is right-continuous at c (or continuous from the right) if

$$\lim_{x \to c^+} f(x) = f(c).$$

The function f is left-continuous at c (or continuous from the left) if

$$\lim_{x \to c^-} f(x) = f(c).$$

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1) $f(c)$ exists (c lies in the domain of f).

2) $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).

3) $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Thm 8: Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1) [Sums] $f+g$ 2) [Differences] $f-g$

3) [Constant multiples] $k \cdot f$, for any number k

4) [Products] $f \cdot g$ 5) [Quotients] $\frac{f}{g}$, provided $g(c) \neq 0$

6) [Powers] f^n, n a positive integer

7) [Roots] $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

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Thm 9: Compositions of Continuous Functions

If f is continuous at c and g continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Thm 10: Limits of Continuous Functions

If $\lim_{x \to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x \to c} g(f(x)) = g(b).$$

Thm 11: The Intermediate Value Theorem for Continuous Functions (see figure on page 97)

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

6-a) yes

6-b) yes because $\lim_{x \to 1} f(x) = 2$

6-c) no, because $2 = \lim_{x \to 1} f(x) \neq f(1) = 1$

6-d) no, because part c is not satisfied

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8) $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

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10) $f(1)$ should be changed to the value of $\lim_{x \to 1} f(x)$

so $f(1) = \lim_{x \to 1} f(x) = 2$

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14) $y = \frac{1}{(x+2)^2} + 4$

V.A.: $(x+2)^2 = 0$

$x + 2 = 0$

$x = -2$

domain: $(-\infty, -2) \cup (-2, \infty)$

continuous: $(-\infty, -2) \cup (-2, \infty)$

16) $y = \frac{x+3}{x^2 - 3x - 10}$

V.A.: $x^2 - 3x - 10 = 0$

$(x+2)(x-5) = 0$

$x + 2 = 0 \mid x - 5 = 0$

$x = -2 \mid x = 5$

domain: $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$

continuous: $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$

18) $y = \frac{1}{|x|+1} - \frac{x^2}{2}$

V.A.: $|x|+1=0$

$|x| = -1$

no solution

no V.A.

domain: $(-\infty, \infty)$

continuous: $(-\infty, \infty)$

20) $y = \frac{x+2}{\cos x}$

V.A.: $\cos x = 0$

$x = \frac{\pi}{2}$ ⋮ $x = \frac{3\pi}{2}$

$x = \frac{\pi}{2} + 2k\pi$ ⋮ $x = \frac{3\pi}{2} + 2k\pi$ k is any integer

domain: $\{x \mid x \neq \frac{\pi}{2} + 2k\pi, x \neq \frac{3\pi}{2} + 2k\pi$, where k is any integer$\}$

continuous: $\{x \mid x \neq \frac{\pi}{2} + 2k\pi, x \neq \frac{3\pi}{2} + 2k\pi$, where k is any integer$\}$

22) $y = \tan \frac{\pi x}{2}$

V.A.: $\tan \frac{\pi x}{2} =$ D.N.E.

$\frac{\pi x}{2} = \frac{\pi}{2}$

$\frac{\pi x}{2} = \frac{\pi}{2} + k\pi$, k is any integer

$x = 1 + 2k$

domain: $\{x \mid x \neq 1+2k$, where k is any integer$\}$

continuous: $\{x \mid x \neq 1+2k$, where k is any integer$\}$

24) $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$

$x^4+1 \geq 1$ because $0 \leq x^4 < \infty$ for all x

therefore $\sqrt{x^4+1}$ exists for all x.

Since $-1 \leq \sin x \leq 1$, when we square

$0 \leq \sin^2 x \leq 1$

and $1 \leq 1+\sin^2 x \leq 2$

which will never make $1+\sin^2 x = 0$ for all real number of x.

domain: $(-\infty, \infty)$

continuous: $(-\infty, \infty)$

26) $y = \sqrt[4]{3x-1}$

since we have an even root function,

domain: $3x-1 \geq 0$

$3x \geq 1$

$x \geq \frac{1}{3}$

$\frac{1}{3} \leq x$

$[\frac{1}{3}, \infty)$

continuous: $[\frac{1}{3}, \infty)$

28) $y = (2-x)^{\frac{1}{5}}$

$y = \sqrt[5]{2-x}$

since we have odd root function,

domain: $(-\infty, \infty)$

continuous: $(-\infty, \infty)$

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30)

$$f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & , x \neq 2, x \neq -2 \\ 3 & , x = 2 \\ 4 & , x = -2 \end{cases}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3-8}{x^2-4} = \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{(2)+2} = \frac{12}{4} = 3 = f(2)$$

so $f(x)$ is continuous at $x = 2$

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^3-8}{x^2-4} = \lim_{x \to -2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)} = \lim_{x \to -2} \frac{x^2+2x+4}{x+2}$$

$$= \text{D.N.E.} \neq 4 = f(-2)$$

so $f(x)$ is discontinuous at $x = -2$

continuous: $(-\infty, -2) \cup (-2, \infty)$

32) $f(x) = \dfrac{x+3}{2-e^x}$

V.A.: $2 - e^x = 0$

$2 = e^x$

$\Downarrow$

$\ln 2 = x$

domain: $(-\infty, \ln 2) \cup (\ln 2, \infty)$

continuous: $(-\infty, \ln 2) \cup (\ln 2, \infty)$

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34) $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right) = \sin\left(\frac{\pi}{2}\cos(\tan(0))\right)$

$= \sin\left(\frac{\pi}{2}\cos(0)\right) = \sin\left(\frac{\pi}{2}(1)\right) = \sin\left(\frac{\pi}{2}\right) = 1$

and function $\sin\left(\frac{\pi}{2}\cos(\tan t)\right)$ is continuous at $t=0$.

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36) $\lim\limits_{x \to 0} \tan\left(\frac{\pi}{4}\cos(\sin x^{\frac{1}{3}})\right) = \lim\limits_{x \to 0} \tan\left(\frac{\pi}{4}\cos(\sin \sqrt[3]{x})\right)$

$= \tan\left(\frac{\pi}{4}\cos(\sin \sqrt[3]{(0)})\right) = \tan\left(\frac{\pi}{4}\cos(\sin(0))\right)$

$= \tan\left(\frac{\pi}{4}\cos(0)\right) = \tan\left(\frac{\pi}{4}(1)\right) = \tan\left(\frac{\pi}{4}\right) = 1$

and function $\tan\left(\frac{\pi}{4}\cos(\sin x^{\frac{1}{3}})\right)$ is continuous at $x=0$

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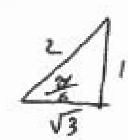
38) $\lim\limits_{x \to \frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3}\tan x} = \lim\limits_{x \to \frac{\pi}{6}} \sqrt{\frac{1}{\sin^2 x} + 5\sqrt{3}\tan x}$



$= \sqrt{\frac{1}{(\frac{1}{2})^2} + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{\frac{1}{\frac{1}{4}} + 5} = \sqrt{4+5} = \sqrt{9} = 3$

and function $\sqrt{\csc^2 x + 5\sqrt{3}\tan x}$ is continuous at $x = \frac{\pi}{6}$

40) $\lim_{x\to 1} \cos^{-1}(\ln\sqrt{x}) = \cos^{-1}(\ln\sqrt{(1)}) = \cos^{-1}(\ln 1)$
$= \cos^{-1}(0) = \frac{\pi}{2}$

and function $\cos^{-1}(\ln\sqrt{x})$ is continuous at $x=1$

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42) $h(t) = \dfrac{t^2+3t-10}{t-2}$, $t=2$

to make $h(t)$ continuous at $t=2$ we must make

$$h(2) = \lim_{t\to 2} \frac{t^2+3t-10}{t-2} = \lim_{t\to 2} \frac{(t+5)(t-2)}{t-2} = \lim_{t\to 2}(t+5) = (2)+5 = 7$$

so let $h(2)=7$

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44) $g(x) = \dfrac{x^2-16}{x^2-3x-4}$, $x=4$

to make $g(x)$ continuous at $x=4$ we must make

$$g(4) = \lim_{x\to 4} \frac{x^2-16}{x^2-3x-4} = \lim_{x\to 4} \frac{(x+4)(x-4)}{(x+1)(x-4)} = \lim_{x\to 4} \frac{x+4}{x+1} = \frac{(4)+4}{(4)+1} = \frac{8}{5}$$

so let $g(4) = \frac{8}{5}$

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46) $g(x) = \begin{cases} x, & x<-2 \\ bx^2, & x\geq -2 \text{ or } -2\leq x \end{cases}$

we need to make $\lim_{x\to -2^-} g(x) = g(-2) = \lim_{x\to -2^+} g(x)$

$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-}(x) = -2$

$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+}(bx^2) = b(-2)^2 = 4b$

$-2 = \lim_{x\to -2^-} g(x) = \lim_{x\to -2^+} g(x) = 4b$

$-2 = 4b$

$\underline{\underline{\frac{-1}{2} = b}}$

50) $$g(x)=\begin{cases} ax+2b, & x\le 0 \\ x^2+3a-b, & 0<x\le 2 \\ 3x-5, & x>2 \text{ or } 2<x \end{cases}$$

at $x=0$

$$\lim_{x\to 0^-} g(x)=\lim_{x\to 0^-}(ax+2b)=a(0)+2b=2b$$

$$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}(x^2+3a-b)=(0)^2+3a-b=3a-b$$

$$2b=\lim_{x\to 0^-} g(x)=\lim_{x\to 0^+} g(x)=3a-b$$

$$2b=3a-b$$

$$0=3a-3b$$

at $x=2$

$$\lim_{x\to 2^-} g(x)=\lim_{x\to 2^-}(x^2+3a-b)=(2)^2+3a-b=4+3a-b$$

$$\lim_{x\to 2^+} g(x)=\lim_{x\to 2^+}(3x-5)=3(2)-5=6-5=1$$

$$4+3a-b=\lim_{x\to 2^-} g(x)=\lim_{x\to 2^+} g(x)=1$$

$$4+3a-b=1$$

$$3a-b=-3$$

$0=3a-3b$

$3b=3a$

$b=a$

$b=\frac{-3}{2}$

$3a-b=-3$

$3a-(a)=-3$

$2a=-3$

$a=\frac{-3}{2}$

we must let $a=\frac{-3}{2}$ and $b=\frac{-3}{2}$ to make $g(x)$ continuous at every x.

58) $F(x) = (x-a)^2 \cdot (x-b)^2 + x$

Assume that $a < b$. Then $F(x)$ is continuous for all values of x because it is a polynomial. We can now conclade that $F(x)$ is continuous on the interval $[a, b]$.

$$F(a) = ((a)-a)^2 \cdot ((a)-b)^2 + (a) = a$$

$$F(b) = ((b)-a)^2 \cdot ((b)-b) + (b) = b$$

Since $a < \frac{a+b}{2} < b$, by the Intermediate Value Theorem, there is a number c between a and b such that $F(x) = \frac{a+b}{2}$ $\{F(c) = \frac{a+b}{2}\}$

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64) $f(x)$, $g(x)$ contnuous for $0 \le x \le 1$

$\frac{f(x)}{g(x)}$ be discontinuous at a point of $[0, 1]$?

<u>yes</u> here is one of many examples of why $\frac{f(x)}{g(x)}$ can be discontinuous

let $f(x) = x+2$ and $g(x) = 2x-1$ {both continuous on $[0,1]$}

But $\frac{f(x)}{g(x)} = \frac{x+2}{2x-1}$ is undefined at $x = \frac{1}{2}$ because $g(\frac{1}{2}) = 2(\frac{1}{2}) - 1 = 0$ making $\frac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$.

66) Let $f(x)=\frac{1}{x+1}$ and $g(x)=x-1$. Both functions are continuous at $x=0$. The composition $(f\circ g)(x)=f(g(x))=f(x-1)=\frac{1}{(x-1)+1}=\frac{1}{x}$ which is discontinuous at $x=0$ because it is not defined there. Thm 9 requires that $f(x)$ be continuous at $g(0)$, which is not the case here because $g(0)=(0)-1=-1$ and $f(x)$ is undefined at $x=-1$.

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70) Let $\varepsilon=\frac{|f(c)|}{2}>0$. Since f is continuous at $x=c$ there is a $\delta>0$ such that $|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$

$$\Downarrow$$

$$f(c)-\varepsilon<f(x)<f(c)+\varepsilon.$$

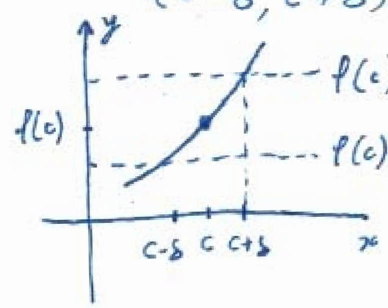
If $f(c)>0$, then

$$\varepsilon=\frac{f(c)}{2}$$

$$f(c)-\left(\frac{f(c)}{2}\right)<f(x)<f(c)+\left(\frac{f(c)}{2}\right)$$

$$\frac{f(c)}{2}<f(x)<\frac{3f(c)}{2}$$

$f(x)>0$ on the interval $(c-\delta, c+\delta)$



If $f(c)<0$, then

$$\varepsilon=\frac{-f(c)}{2}$$

$$f(c)-\left(\frac{-f(c)}{2}\right)<f(x)<f(c)+\left(\frac{-f(c)}{2}\right)$$

$$\frac{3f(c)}{2}<f(x)<\frac{f(c)}{2}$$

$f(x)<0$ on the interval $(c-\delta, c+\delta)$