MATH 20100 section 2.4

Thm 6: Suppose that a function f is defined on an open interval containing c, except at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L$$

Precise Definition of One-Sided Limits

Def.: a) Assume the domain of f contains an interval (c,d) to the right of c. We say that $f(x)$ has right-hand limit L at c, and write

$$\lim_{x\to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } c < x < c + \delta.$$

b) Assume the domain of f contains an interval (b,c) to the left of c. We say that f has left-hand limit L at c, and write

$$\lim_{x\to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } c - \delta < x < c.$$

Thm 7: Limit of the Ratio $\frac{\sin\theta}{\theta}$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad (\theta \text{ in radians})$$

also: $$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0 \quad (\theta \text{ in radians})$$

2-a) true   2-b) false because $\lim_{x\to 2} f(x) = 1$

2-c) false because $\lim_{x\to 2} f(x) = 1$ 2-d) true

2-e) true   2-f) true because $2 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 1$

2-g) true   2-h) true   2-i) true

2-j) false because $f(x)$ does not exist for $x < -1$

2-k) true because $f(x)$ does not exist for $3 < x$

4-a) $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(\frac{x}{2}\right) = \frac{(2)}{2} = \underline{\underline{1}}$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (3-x) = 3-(2) = \underline{\underline{1}}$,

$f(2) = \underline{\underline{2}}$

4-b) <u>yes</u>, from part a $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 1 = \lim_{x\to 2} f(x)$

4-c) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (3-x) = 3-(-1) = \underline{\underline{4}}$ $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (3-x) = 3-(-1) = \underline{\underline{4}}$

4-d) <u>yes</u>, from part c $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = 4 = \lim_{x\to -1} f(x)$

6-a) yes, because for $0 < x$ $\quad -\sqrt{x} \le g(x) \le \sqrt{x}$ and the sandwitch (squeze) thm $\lim_{x\to 0^+}(-\sqrt{x}) \le \lim_{x\to 0^+} g(x) \le \lim_{x\to 0^+}\sqrt{x}$

$$\underset{0}{\overset{\shortparallel}{}} \le \lim_{x\to 0^+} g(x) \le \underset{0}{\overset{\shortparallel}{}}$$

6-b) no, because $g(x)$ does not exist for $x<0$ and $\sqrt{x}$ and $-\sqrt{x}$ is not defined for $x<0$

6-c) yes, because the domains of $-\sqrt{x}$, $g(x)$, $\sqrt{x}$ is $[0,\infty)$ and $x=0$ is the boundary point of the domain so $\lim_{x\to 0} g(x) = \lim_{x\to 0^+} g(x) = 0$

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12) $$\lim_{x\to 1^+}\sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{(1^+)-1}{(1^+)+2}} = \sqrt{\frac{0^+}{3}} = \sqrt{0} = \underline{\underline{0}}$$

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14) $$\lim_{x\to 1^-}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right) = \left(\frac{1}{(1^-)+1}\right)\left(\frac{(1^-)+6}{(1^-)}\right)\left(\frac{3-(1^-)}{7}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right) = \underline{\underline{1}}$$

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16) $$\lim_{h\to 0}\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} = \lim_{h\to 0}\left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$$

$$= \lim_{h\to 0}\frac{6-(5h^2+11h+6)}{h(\sqrt{6}+\sqrt{5h^2+11h+6})} = \lim_{h\to 0}\frac{-5h^2-11h}{h(\sqrt{6}+\sqrt{5h^2+11h+6})}$$

$$= \lim_{h\to 0}\frac{h(-5h-11)}{h(\sqrt{6}+\sqrt{5h^2+11h+6})} = \lim_{h\to 0}\frac{-5h-11}{\sqrt{6}+\sqrt{5h^2+11h+6}} = \frac{-5(0)-11}{\sqrt{6}+\sqrt{5(0)^2+11(0)+6}} = \frac{-11}{\sqrt{6}+\sqrt{6}}$$

$$= \underline{\underline{\frac{-11}{2\sqrt{6}}}}$$

18-a) $|x-1| = +(x-1)$ for $1 < x$

$$\lim_{x \to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x \to 1^+} \frac{\sqrt{2x}\,(x-1)}{+(x-1)} = \lim_{x \to 1^+} \sqrt{2x} = \sqrt{2(1^+)} = \underline{\underline{\sqrt{2}}}$$

18-b) $|x-1| = -(x-1)$ for $x < 1$

$$\lim_{x \to 1^-} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x \to 1^-} \frac{\sqrt{2x}\,(x-1)}{-(x-1)} = \lim_{x \to 1^-} \left(-\sqrt{2x}\right) = -\sqrt{2(1^-)} = \underline{\underline{-\sqrt{2}}}$$

20-a) for $0 < x < \frac{\pi}{2}$, $0 < \cos x < 1$ and $|\cos x - 1| = -(\cos x - 1)$

$$\lim_{x \to 0^+} \frac{1-\cos x}{|\cos x - 1|} = \lim_{x \to 0^+} \frac{1-\cos x}{-(\cos x - 1)} = \lim_{x \to 0^+} \frac{1-\cos x}{1-\cos x} = \lim_{x \to 0^+} 1 = \underline{\underline{1}}$$

20-b) for $-\frac{\pi}{2} < x < 0$, $0 < \cos x < 1$ and $|\cos x - 1| = -(\cos x - 1)$

$$\lim_{x \to 0^-} \frac{\cos x - 1}{|\cos x - 1|} = \lim_{x \to 0^-} \frac{\cos x - 1}{-(\cos x - 1)} = \lim_{x \to 0^-} \frac{1}{-1} = \lim_{x \to 0^-} -1 = \underline{\underline{-1}}$$

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24) $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

$$= \lim_{t \to 0} \left(\frac{\sin(kt)}{t}\right)\left(\frac{k}{k}\right) = \lim_{t \to 0} \frac{k \sin(kt)}{(kt)} = k \lim_{t \to 0} \frac{\sin(kt)}{(kt)}$$

let $\theta = (kt)$ $\quad = k \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = k(1) = \underline{\underline{k}}$

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26) $$\lim_{h \to 0^-} \frac{h}{\sin 3h} = \lim_{h \to 0^-} \left(\frac{h}{\sin(3h)}\right)\left(\frac{3}{3}\right) = \lim_{h \to 0^-} \left(\frac{1}{3}\right)\left(\frac{(3h)}{\sin(3h)}\right)$$

$$= \lim_{h \to 0^-} \left(\frac{1}{3}\right)\left(\frac{1}{\frac{\sin(3h)}{(3h)}}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{\lim_{h \to 0^-} \frac{\sin(3h)}{(3h)}}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{(1)}\right) = \underline{\underline{\frac{1}{3}}}$$

28) $\lim_{t \to 0} \frac{2t}{\tan t} = \lim_{t \to 0} \frac{2t}{\frac{\sin t}{\cos t}} = \lim_{t \to 0} \frac{2t \cos t}{\sin t}$

$= \lim_{t \to 0} (2 \cos t)\left(\frac{t}{\sin t}\right) = \lim_{t \to 0} (2 \cos t)\left(\frac{1}{\frac{\sin t}{t}}\right)$

$= \left(2 \lim_{t \to 0} \cos t\right)\left(\frac{1}{\lim_{t \to 0} \frac{\sin t}{t}}\right) = (2(1))\left(\frac{1}{(1)}\right) = \underline{\underline{2}}$

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30) $\lim_{x \to 0} 6x^2 (\cot x)(\csc 2x) = \lim_{x \to 0} 6x^2 \left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin(2x)}\right)$

$= \lim_{x \to 0} (3 \cos x)\left(\frac{x}{\sin x}\right)\left(\frac{(2x)}{\sin(2x)}\right) = \lim_{x \to 0} (3 \cos x)\left(\frac{1}{\frac{\sin x}{x}}\right)\left(\frac{1}{\frac{\sin(2x)}{(2x)}}\right)$

$= \left(3 \lim_{x \to 0} \cos x\right)\left(\frac{1}{\lim_{x \to 0} \frac{\sin x}{x}}\right)\left(\frac{1}{\lim_{x \to 0} \frac{\sin(2x)}{(2x)}}\right) = (3(1))\left(\frac{1}{(1)}\right)\left(\frac{1}{(1)}\right) = \underline{\underline{3}}$

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32) $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x} = \lim_{x \to 0} \left(\frac{x^2}{2x} - \frac{x}{2x} + \frac{\sin x}{2x}\right)$

$= \lim_{x \to 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{\sin x}{2x}\right) = \lim_{x \to 0} \left(\frac{x}{2}\right) - \lim_{x \to 0} \left(\frac{1}{2}\right) + \frac{1}{2} \lim_{x \to 0} \left(\frac{\sin x}{x}\right)$

$= (0) - \frac{1}{2} + \frac{1}{2}(1) = \underline{\underline{0}}$

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34) $\lim_{x \to 0} \frac{x - x \cos x}{\sin^2 3x} = \lim_{x \to 0} \frac{x(1 - \cos x)}{\sin^2(3x)} = \lim_{x \to 0} \left(\frac{\frac{x(1-\cos x)}{1}}{\frac{\sin^2(3x)}{1}}\right)\left(\frac{\frac{1}{(3x)^2}}{\frac{1}{(3x)^2}}\right)$

$= \lim_{x \to 0} \frac{\frac{x(1-\cos x)}{(3x)^2}}{\frac{\sin^2(3x)}{(3x)^2}} = \lim_{x \to 0} \frac{\frac{1 - \cos x}{9x}}{\left(\frac{\sin(3x)}{(3x)}\right)^2} = \frac{\frac{1}{9} \lim_{x \to 0} \left(\frac{1 - \cos x}{x}\right)}{\left(\lim_{x \to 0} \frac{\sin(3x)}{(3x)}\right)^2} = \frac{\frac{1}{9}(0)}{(1)^2} = \underline{\underline{0}}$

36) $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$ since $\sin h \to 0$ as $h \to 0$, let $\theta = \sin h$

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = \underline{\underline{1}}$

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38) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 4x} = \lim\limits_{x\to 0} \left(\dfrac{\sin(5x)}{\sin(4x)}\right)\left(\dfrac{x}{x}\right)\left(\dfrac{4}{4}\right)\left(\dfrac{5}{5}\right)$

$= \lim\limits_{x\to 0} \left(\dfrac{\sin(5x)}{(5x)}\right)\left(\dfrac{(4x)}{\sin(4x)}\right)\left(\dfrac{5}{4}\right) = \lim\limits_{x\to 0} \left(\dfrac{\sin(5x)}{(5x)}\right)\left(\dfrac{1}{\frac{\sin(4x)}{(4x)}}\right)\left(\dfrac{5}{4}\right)$

$= \left(\lim\limits_{x\to 0} \dfrac{\sin(5x)}{(5x)}\right)\left(\dfrac{1}{\lim\limits_{x\to 0} \frac{\sin(4x)}{(4x)}}\right)\left(\dfrac{5}{4}\right) = (1)\left(\dfrac{1}{(1)}\right)\left(\dfrac{5}{4}\right) = \underline{\underline{\dfrac{5}{4}}}$

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40) $\lim\limits_{\theta\to 0} \sin\theta \cot 2\theta = \lim\limits_{\theta\to 0} (\sin\theta)\left(\dfrac{\cos(2\theta)}{\sin(2\theta)}\right)$

$= \lim\limits_{\theta\to 0} (\sin\theta)\left(\dfrac{\cos(2\theta)}{2\sin\theta\cos\theta}\right) = \lim\limits_{\theta\to 0} \dfrac{\cos(2\theta)}{2\cos\theta} = \dfrac{(1)}{2(1)} = \underline{\underline{\dfrac{1}{2}}}$

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42) $\lim\limits_{y\to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y} = \lim\limits_{y\to 0} \dfrac{\sin(3y)\left(\frac{\cos(5y)}{\sin(5y)}\right)}{y\left(\frac{\cos(4y)}{\sin(4y)}\right)}$

$= \lim\limits_{y\to 0} \left(\dfrac{\sin(3y)\cos(5y)\sin(4y)}{y\cos(4y)\sin(5y)}\right)\left(\dfrac{y}{y}\right)\left(\dfrac{3}{3}\right)\left(\dfrac{4}{4}\right)\left(\dfrac{5}{5}\right)$

$= \lim\limits_{y\to 0} \left(\dfrac{\sin(3y)}{(3y)}\right)\left(\dfrac{\sin(4y)}{(4y)}\right)\left(\dfrac{(5y)}{\sin(5y)}\right)\left(\dfrac{(3)(4)}{5}\right)$

$= \left(\dfrac{12}{5}\right)\left(\lim\limits_{y\to 0} \dfrac{\sin(3y)}{(3y)}\right)\left(\lim\limits_{y\to 0} \dfrac{\sin(4y)}{(4y)}\right)\left(\dfrac{1}{\lim\limits_{y\to 0} \frac{\sin(5y)}{(5y)}}\right) = \left(\dfrac{12}{5}\right)(1)(1)\left(\dfrac{1}{(1)}\right) = \underline{\underline{\dfrac{12}{5}}}$

44) $$\lim_{\theta\to0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}=\lim_{\theta\to0}\frac{\theta\left(\frac{\cos(4\theta)}{\sin(4\theta)}\right)}{\sin^2\theta\left(\frac{\cos^2(2\theta)}{\sin^2(2\theta)}\right)}=\lim_{\theta\to0}\frac{\theta\cos(4\theta)\sin^2(2\theta)}{\sin^2\theta\cos^2(2\theta)\sin(4\theta)}$$

$$=\lim_{\theta\to0}\frac{\theta\cos(4\theta)(2\sin\theta\cos\theta)^2}{\sin^2\theta\cos^2(2\theta)\sin(4\theta)}=\lim_{\theta\to0}\frac{\theta\cos(4\theta)(4\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2(2\theta)\sin(4\theta)}$$

$$=\lim_{\theta\to0}\frac{4\theta\cos(4\theta)\cos^2\theta}{\cos^2(2\theta)\sin(4\theta)}=\lim_{\theta\to0}\left(\frac{\cos(4\theta)\cos^2\theta}{\cos^2(2\theta)}\right)\left(\frac{(4\theta)}{\sin(4\theta)}\right)$$

$$=\left(\lim_{\theta\to0}\frac{\cos(4\theta)\cos^2\theta}{\cos^2(2\theta)}\right)\left(\frac{1}{\lim_{\theta\to0}\frac{\sin(4\theta)}{(4\theta)}}\right)=\left(\frac{(1)(1)^2}{(1)^2}\right)\left(\frac{1}{(1)}\right)=\underline{\underline{1}}$$

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46) $$\lim_{x\to0}\frac{\cos^2x-\cos x}{x^2}$$

to fix this expression we must fix the denominator x^2. We have to make $\frac{\sin x}{x}$ (actually $\frac{\sin^2x}{x^2}=\left(\frac{\sin x}{x}\right)^2$) in our expression on the left.

$$=\lim_{x\to0}\frac{\cos x(\cos x-1)}{x^2}$$

$$=\lim_{x\to0}\left(\frac{\cos x(\cos x-1)}{x^2}\right)\left(\frac{\cos x+1}{\cos x+1}\right)=\lim_{x\to0}\frac{\cos x(\cos^2x-1)}{x^2(\cos x+1)}$$

$$=\lim_{x\to0}\frac{\cos x(-\sin^2x)}{x^2(\cos x+1)}=\lim_{x\to0}\left(\frac{-\cos x}{\cos x+1}\right)\left(\frac{\sin^2x}{x^2}\right)$$

$$=\lim_{x\to0}\left(\frac{-\cos x}{\cos x+1}\right)\left(\frac{\sin x}{x}\right)^2=\left(\lim_{x\to0}\frac{-\cos x}{\cos x+1}\right)\left(\lim_{x\to0}\frac{\sin x}{x}\right)^2$$

$$=\left(\frac{-(1)}{(1)+1}\right)(1)^2=\underline{\underline{\frac{-1}{2}}}$$

48) Since $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$, then $\lim_{x \to c} f(x)$ can be found by calculating $\lim_{x \to c^+} f(x)$.

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50) If f is an even function of x, $f(-x) = f(x)$. We can conclude that $\lim_{x \to 2^-} f(x) = 7$ yields that $\lim_{x \to -2^+} f(x) = 7$. Unfortunately, nothing can be said about $\lim_{x \to -2^-} f(x)$ because we don't know $\lim_{x \to 2^+} f(x)$.