Def. Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta.$$

How to Find Algebraically a δ for a Given f, L, c, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - c| < \delta$$

can be accomplished in two steps.

1) Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.

2) Find a value of $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

For question 7 to 14, we need only need to do step 2 of "How to Find Algebraically a δ for Given f, L, c, and $\varepsilon > 0$"

8) $f(x) = \frac{-3}{2}x + 3$, $c = -3$

$|x - c| < \delta$

$|x - (-3)| < \delta$

$|x + 3| < \delta$

$-\delta < x + 3 < \delta$

$-\delta - 3 < x < \delta - 3$

from graph $-3.1 < x < -2.9$

$-\delta - 3 = -3.1$	$\delta - 3 = -2.9$
$0.1 = \delta$	$\delta = 0.1$

so $\underline{\underline{\delta = 0.1}}$

10) $f(x) = 2\sqrt{x+1}$, $c = 3$

$|x - c| < \delta$

$|x - (3)| < \delta$

$|x - 3| < \delta$

$-\delta < x - 3 < \delta$

$-\delta + 3 < x < \delta + 3$

from graph $2.61 < x < 3.41$

$-\delta + 3 = 2.61$	$\delta + 3 = 3.41$
$0.39 = \delta$	$\delta = 0.41$

since we must ensure that $|f(x) - 4| < \varepsilon$ whenever $0 < |x - 3| < \delta$ we must choose the smaller value of δ

so $\underline{\underline{\delta = 0.39}}$

12) $f(x)=4-x^2$, $c=-1$

$|x-c|<\delta$

$|x-(-1)|<\delta$

$|x+1|<\delta$

$-\delta<x+1<\delta$

$-\delta-1<x<\delta-1$

from the graph $\frac{-\sqrt{5}}{2}<x<\frac{-\sqrt{3}}{2}$

$-\delta-1=\frac{-\sqrt{5}}{2}$	$\delta-1=\frac{-\sqrt{3}}{2}$
$\frac{\sqrt{5}}{2}-1=\delta$	$\delta=1-\frac{\sqrt{3}}{2}$
$\frac{\sqrt{5}-2}{2}=\delta$	$\delta=\frac{2-\sqrt{3}}{2}$

smaller value of δ

$\delta=\frac{\sqrt{5}-2}{2}$

14) $f(x)=\frac{1}{x}$, $c=\frac{1}{2}$

$|x-c|<\delta$

$|x-(\frac{1}{2})|<\delta$

$|x-\frac{1}{2}|<\delta$

$-\delta<x-\frac{1}{2}<\delta$

$-\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$

from the graph $\frac{1}{2.01}<x<\frac{1}{1.99}$

$-\delta+\frac{1}{2}=\frac{1}{2.01}$	$\delta+\frac{1}{2}=\frac{1}{1.99}$
$\frac{1}{2}-\frac{1}{2.01}=\delta$	$\delta=\frac{1}{1.99}-\frac{1}{2}$
$\frac{1}{402}=\delta$	$\delta=\frac{1}{398}$

smaller value of δ

$\delta=\frac{1}{402}$

---

16) $f(x)=2x-2$, $L=-6$, $c=-2$, $\varepsilon=0.02$

Given $\varepsilon>0$, we need to find a $\delta>0$ such that

$\lim_{x\to c} f(x)=L$ if $0<|x-c|<\delta$ then $|f(x)-L|<\varepsilon$

1. $|(2x-2)-(-6)|<0.02$

$|2x+4|<0.02$

$-0.02<2x+4<0.02$

$-0.02-4<2x<0.02-4$

$-4.02<2x<-3.98$

$-2.01<x<-1.99$

2. $|x-c|<\delta$

$|x-(-2)|<\delta$

$|x+2|<\delta$

$-\delta<x+2<\delta$

$-\delta-2<x<\delta-2$

$-\delta-2=-2.01$	$\delta-2=-1.99$
$0.01=\delta$	$\delta=0.01$

so $\delta=0.01$

18) $f(x) = \sqrt{x}$, $L = \frac{1}{2}$, $c = \frac{1}{4}$, $\varepsilon = 0.1$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$ if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

1 $|\sqrt{x} - (\frac{1}{2})| < 0.1$

$|\sqrt{x} - 0.5| < 0.1$

$-0.1 < \sqrt{x} - 0.5 < 0.1$

$0.4 < \sqrt{x} < 0.6$

$0.16 < x < 0.36$

2 $|x - (\frac{1}{4})| < \delta$

$|x - 0.25| < \delta$

$-\delta < x - 0.25 < \delta$

$-\delta + 0.25 < x < \delta + 0.25$

$-\delta + 0.25 = 0.16$	$\delta + 0.25 = 0.36$
$0.09 = \delta$	$\delta = 0.11$

so $\delta = 0.09$

---

20) $f(x) = \sqrt{x - 7}$, $L = 4$, $c = 23$, $\varepsilon = 1$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$ if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

1 $|\sqrt{x-7} - (4)| < 1$

$|\sqrt{x-7} - 4| < 1$

$-1 < \sqrt{x-7} - 4 < 1$

$3 < \sqrt{x-7} < 5$

$9 < x - 7 < 25$

$16 < x < 32$

2 $|x - (23)| < \delta$

$|x - 23| < \delta$

$-\delta < x - 23 < \delta$

$-\delta + 23 < x < \delta + 23$

$-\delta + 23 = 16$	$\delta + 23 = 32$
$7 = \delta$	$\delta = 9$

so $\delta = 7$

22) $f(x)=x^2$, $L=3$, $c=\sqrt{3}$, $\varepsilon=0.1$

Given $\varepsilon>0$, we need to find a $\delta>0$ such that $\lim_{x\to c} f(x)=L$
if $0<|x-c|<\delta$ then $|f(x)-L|<\varepsilon$

① $|x^2-(3)|<0.1$

$|x^2-3|<0.1$

$-0.1<x^2-3<0.1$

$-0.1+3<x^2<0.1+3$

$2.9<x^2<3.1$

$\sqrt{2.9}<x<\sqrt{3.1}$

② $|x-(\sqrt{3})|<\delta$

$|x-\sqrt{3}|<\delta$

$-\delta<x-\sqrt{3}<\delta$

$-\delta+\sqrt{3}<x<\delta+\sqrt{3}$

$-\delta+\sqrt{3}=\sqrt{2.9}$	$\delta+\sqrt{3}=\sqrt{3.1}$
$\sqrt{3}-\sqrt{2.9}=\delta$	$\delta=\sqrt{3.1}-\sqrt{3}$
	smaller

so $\delta=\sqrt{3.1}-\sqrt{3}$

---

24) $f(x)=\frac{1}{x}$, $L=-1$, $c=-1$, $\varepsilon=0.1$

Given $\varepsilon>0$, we need to find a $\delta>0$ such that $\lim_{x\to c} f(x)=L$
if $0<|x-c|<\delta$ then $|f(x)-L|<\varepsilon$

① $\left|\frac{1}{x}-(-1)\right|<0.1$

$\left|\frac{1}{x}+1\right|<0.1$

$-0.1<\frac{1}{x}+1<0.1$

$-0.1-1<\frac{1}{x}<0.1-1$

$-1.1<\frac{1}{x}<-0.9$

$\frac{-11}{10}<\frac{1}{x}<\frac{-9}{10}$

$\frac{-10}{11}>x>\frac{-10}{9}$

or

$\frac{-10}{9}<x<\frac{-10}{1}$

② $|x-(-1)|<\delta$

$|x+1|<\delta$

$-\delta<x+1<\delta$

$-\delta-1<x<\delta-1$

$-\delta-1=\frac{-10}{9}$	$\delta-1=\frac{-10}{11}$
$-1+\frac{10}{9}=\delta$	$\delta=\frac{-10}{11}+1$
$\frac{-9}{9}+\frac{10}{9}=\delta$	$\delta=\frac{-10}{11}+\frac{11}{11}$
$\frac{1}{9}=\delta$	$\delta=\frac{1}{11}$

so $\delta=\frac{1}{11}$

26) $f(x) = \frac{120}{x}$, $L = 5$, $c = 24$, $\varepsilon = 1$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$

if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

① $\left|\frac{120}{x} - (5)\right| < 1$

$\left|\frac{120}{x} - 5\right| < 1$

$-1 < \frac{120}{x} - 5 < 1$

$4 < \frac{120}{x} < 6$

$\frac{1}{4} > \frac{x}{120} > \frac{1}{6}$

or

$\frac{1}{6} < \frac{x}{120} < \frac{1}{4}$

$\frac{120}{6} < x < \frac{120}{4}$

$20 < x < 30$

② $|x - (24)| < \delta$

$|x - 24| < \delta$

$-\delta < x - 24 < \delta$

$-\delta + 24 < x < \delta + 24$

$-\delta + 24 = 20$	$\delta + 24 = 30$
$4 = \delta$	$\delta = 6$

so $\delta = 4$

---

28) $f(x) = mx$, $m > 0$, $L = 3m$, $c = 3$, $\varepsilon = c > 0$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$

if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

① $|mx - (3m)| < c$

$|mx - 3m| < c$

$-c < mx - 3m < c$

$-c < m(x - 3) < c$

$\frac{-c}{m} < x - 3 < \frac{c}{m}$

$\frac{-c}{m} + 3 < x < \frac{c}{m} + 3$

② $|x - (3)| < \delta$

$|x - 3| < \delta$

$-\delta < x - 3 < \delta$

$-\delta + 3 < x < \delta + 3$

$-\delta + 3 = \frac{-c}{m} + 3$	$\delta + 3 = \frac{c}{m} + 3$
$-\delta = \frac{-c}{m}$	$\delta = \frac{c}{m}$
$\frac{c}{m} = \delta$	

so $\delta = \frac{c}{m}$

30) $f(x) = mx + b$, $m > 0$, $L = m + b$, $c = 1$, $\epsilon = 0.05$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

1 $|mx + b - (m + b)| < 0.05$

$|mx + b - m - b| < 0.05$

$|mx - m| < 0.05$

$-0.05 < mx - m < 0.05$

$-0.05 < m(x-1) < 0.05$

$\frac{-0.05}{m} < x - 1 < \frac{0.05}{m}$

$\frac{-0.05}{m} + 1 < x < \frac{0.05}{m} + 1$

2 $|x - (1)| < \delta$

$|x - 1| < \delta$

$-\delta < x - 1 < \delta$

$-\delta + 1 < x < \delta + 1$

$-\delta + 1 = \frac{-0.05}{m} + 1$	$\delta + 1 = \frac{0.05}{m} + 1$
$-\delta = \frac{-0.05}{m}$	$\delta = \frac{0.05}{m}$
$\frac{0.05}{m} = \delta$	

so $\delta = \frac{0.05}{m}$

---

38) $\lim_{x \to 3} (3x - 7) = 2$: $f(x) = 3x - 7$, $L = 2$, $c = 3$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

1 $|3x - 7 - (2)| < \varepsilon$

$|3x - 9| < \varepsilon$

$-\varepsilon < 3x - 9 < \varepsilon$

$-\varepsilon + 9 < 3x < \varepsilon + 9$

$\frac{-\varepsilon}{3} + 3 < x < \frac{\varepsilon}{3} + 3$

2 $|x - (3)| < \delta$

$|x - 3| < \delta$

$-\delta < x - 3 < \delta$

$-\delta + 3 < x < \delta + 3$

$-\delta + 3 = \frac{-\varepsilon}{3} + 3$	$\delta + 3 = \frac{\varepsilon}{3} + 3$
$-\delta = \frac{-\varepsilon}{3}$	$\delta = \frac{\varepsilon}{3}$
$\frac{\varepsilon}{3} = \delta$	

so $\delta = \frac{\varepsilon}{3}$

40) $\lim_{x\to 0} \sqrt{4-x} = 2$: $f(x)=\sqrt{4-x}$, $L=2$, $c=0$

Given $\varepsilon>0$, we need to find a $\delta>0$ such that $\lim_{x\to c} f(x)=L$
if $0<|x-c|<\delta$ then $|f(x)-L|<\varepsilon$

① $|\sqrt{4-x}-(2)|<\varepsilon$

$|\sqrt{4-x}-2|<\varepsilon$

$-\varepsilon<\sqrt{4-x}-2<\varepsilon$

$-\varepsilon+2<\sqrt{4-x}<\varepsilon+2$

$2-\varepsilon<\sqrt{4-x}<\varepsilon+2$

$(2-\varepsilon)^2<4-x<(\varepsilon+2)^2$

$(2-\varepsilon)^2<-1(x-4)<(\varepsilon+2)^2$

$-(2-\varepsilon)^2>x-4>-(\varepsilon+2)^2$

$-(\varepsilon+2)^2<x-4<-(2-\varepsilon)^2$

$4-(\varepsilon+2)^2<x<4-(2-\varepsilon)^2$

② $|x-(0)|<\delta$

$|x|<\delta$

$-\delta<x<\delta$

$-\delta=4-(\varepsilon+2)^2$	$\delta=4-(2-\varepsilon)^2$
$(\varepsilon+2)^2-4=\delta$	$\delta=4-(4-4\varepsilon+\varepsilon^2)$
$(\varepsilon^2+4\varepsilon+4)-4=\delta$	$\delta=4\varepsilon-\varepsilon^2$
$\varepsilon^2+4\varepsilon=\delta$	smaller

so $\delta=4\varepsilon-\varepsilon^2$

---

42) $\lim_{x\to -2} f(x)=4$: $f(x)=\begin{cases} x^2, & x\neq -2 \\ 1, & x=-2 \end{cases}$, $L=4$, $c=-2$

Given $\varepsilon>0$, we need to find a $\delta>0$ such that $\lim_{x\to c} f(x)=L$
if $0<|x-c|<\delta$ then $|f(x)-L|<\varepsilon$

① for $x\neq -2$ $|x^2-(4)|<\varepsilon$

$|x^2-4|<\varepsilon$

$-\varepsilon<x^2-4<\varepsilon$

$-\varepsilon+4<x^2<\varepsilon+4$

$4-\varepsilon<x^2<\varepsilon+4$

$\sqrt{4-\varepsilon}<x<\sqrt{\varepsilon+4}$

② $|x-(-2)|<\delta$

$|x+2|<\delta$

$-\delta<x+2<\delta$

$-\delta-2<x<\delta-2$

$-\delta-2=\sqrt{4-\varepsilon}$	$\delta-2=\sqrt{\varepsilon+4}$
$\sqrt{4-\varepsilon}-2=\delta$	$\delta=2+\sqrt{\varepsilon+4}$

so $\delta=\min\{\sqrt{4-\varepsilon}-2, 2+\sqrt{\varepsilon+4}\}$

44) $\lim_{x \to \sqrt{3}} \frac{1}{x^2} = \frac{1}{3}$: $f(x) = \frac{1}{x^2}$, $L = \frac{1}{3}$, $c = \sqrt{3}$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

① $\left|\frac{1}{x^2} - \left(\frac{1}{3}\right)\right| < \varepsilon$

$\left|\frac{1}{x^2} - \frac{1}{3}\right| < \varepsilon$

$-\varepsilon < \frac{1}{x^2} - \frac{1}{3} < \varepsilon$

$-\varepsilon + \frac{1}{3} < \frac{1}{x^2} < \varepsilon + \frac{1}{3}$

$\frac{1}{3} - \varepsilon < \frac{1}{x^2} < \varepsilon + \frac{1}{3}$

$\frac{1 - 3\varepsilon}{3} < \frac{1}{x^2} < \frac{3\varepsilon + 1}{3}$

$\frac{3}{1 - 3\varepsilon} > x^2 > \frac{3}{3\varepsilon + 1}$

or

$\frac{3}{3\varepsilon + 1} < x^2 < \frac{3}{1 - 3\varepsilon}$

$\sqrt{\frac{3}{3\varepsilon + 1}} < x < \sqrt{\frac{3}{1 - 3\varepsilon}}$

② $|x - (\sqrt{3})| < \delta$

$|x - \sqrt{3}| < \delta$

$-\delta < x - \sqrt{3} < \delta$

$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$

$-\delta + \sqrt{3} = \sqrt{\frac{3}{3\varepsilon + 1}}$ | $\delta + \sqrt{3} = \sqrt{\frac{3}{1 - 3\varepsilon}}$

$\sqrt{3} - \sqrt{\frac{3}{3\varepsilon + 1}} = \delta$ | $\delta = \sqrt{\frac{3}{1 - 3\varepsilon}} - \sqrt{3}$

so $\delta = \min\left\{\sqrt{3} - \sqrt{\frac{3}{3\varepsilon + 1}}, \sqrt{\frac{3}{1 - 3\varepsilon}} - \sqrt{3}\right\}$

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46) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$; $f(x) = \frac{x^2 - 1}{x - 1}$, $L = 2$, $c = 1$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x \to c} f(x) = L$
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

① $\left|\frac{x^2 - 1}{x - 1} - (2)\right| < \varepsilon$

$\left|\frac{(x+1)(x-1)}{x - 1} - 2\right| < \varepsilon$

$|x + 1 - 2| < \varepsilon$

$|x - 1| < \varepsilon$

$-\varepsilon < x - 1 < \varepsilon$

$-\varepsilon + 1 < x < \varepsilon + 1$

② $|x - (1)| < \delta$

$|x - 1| < \delta$

$-\delta < x - 1 < \delta$

$-\delta + 1 < x < \delta + 1$

$-\delta + 1 = -\varepsilon + 1$ | $\delta + 1 = \varepsilon + 1$

$-\delta = -\varepsilon$ | $\delta = \varepsilon$

$\varepsilon = \delta$

so $\delta = \varepsilon$

48) $\lim_{x\to 0} f(x) = 0$: $f(x) = \begin{cases} 2x & , x<0 \\ \frac{x}{2} & , x \geq 0 \end{cases}$, $L=0$, $c=0$

Given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $\lim_{x\to c} f(x) = L$
if $0 < |x-c| < \delta$ then $|f(x) - L| < \varepsilon$

Since $x=0$ is the border of evaluating pieces of a piecewise function, we need to investigate separately from left and right of the border value of 0 in step 1.

$f(0) = 0$

1 $x<0$: $-\varepsilon < (2x - (0)) < 0$

$-\varepsilon < 2x < 0$

$-\frac{\varepsilon}{2} < x < 0$

$x \geq 0$: $0 < \left(\frac{x}{2} - (0)\right) < \varepsilon$

$0 < \frac{x}{2} < \varepsilon$

$0 < x < 2\varepsilon$

2 $|x - (0)| < \delta$

$|x| < \delta$

$-\delta < x < \delta$ $\Longrightarrow$ $-\delta = -\frac{\varepsilon}{2}$, $\frac{\varepsilon}{2} = \delta$ | $\delta = 2\varepsilon$

so $\underline{\underline{\delta = \frac{\varepsilon}{2}}}$

---

50) $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

by the figure, we can see that

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2 \quad \text{for all } x$$

also, $\lim_{x\to 0}(-x^2) \leq \lim_{x\to 0} x^2 \sin\frac{1}{x} \leq \lim_{x\to 0} x^2$

$\lim_{x\to 0}(-x^2) = -(0)^2 = 0$ and $\lim_{x\to 0} x^2 = (0)^2 = 0$

by the sandwitch (squeeze) theorem,

$0 \leq \lim_{x\to 0} x^2 \sin\frac{1}{x} \leq 0$ so $\underline{\underline{\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0}}$