MATH 20100 section 2.2

Thm 1: Limit laws

If $L, M, c,$ and k are real numbers and

$$\lim_{x\to c} f(x)=L \text{ and } \lim_{x\to c} g(x)=M, \text{ then}$$

1) [Sum Rule] $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x)+\lim_{x\to c} g(x) = L+M$

2) [Difference Rule] $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x)-\lim_{x\to c} g(x) = L-M$

3) [Constant Multiple Rule] $\lim_{x\to c}(k\cdot f(x)) = k\cdot \lim_{x\to c} f(x) = k\cdot L$

4) [Product Rule] $\lim_{x\to c}(f(x)\cdot g(x)) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right) = L\cdot M$

5) [Quotient Rule] $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}, \; M\neq 0$

6) [Power Rule] $\lim_{x\to c}\left[f(x)\right]^n = \left[\lim_{x\to c} f(x)\right]^n = L^n$

7) [Root Rule] $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$

n, n positive integer

(If n is even, we assume that $f(x)\geq 0$ for x in an interval containing c.)

Thm 2 : Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$$

Thm 3 : Limit of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

Thm 4: The Sandwitch Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

note: $\quad g(x) \leq f(x) \leq h(x)$

so $L = \lim_{x \to c} g(x) \leq \lim_{x \to c} f(x) \leq \lim_{x \to c} h(x) = L$

$$L \leq \lim_{x \to c} f(x) \leq L$$

Then $\lim_{x \to c} f(x) = L$.

2-a) $\lim_{t\to -2} f(t) = 0$ 2-b) $\lim_{t\to -1} f(t) = -1$

2-c) $\lim_{t\to 0} f(t) =$ D.N.E. because from left the value of $f(t)$ is approaching -1 and from right the value is approaching $+1$. Since both sides does not approach the same value, the general limit does not exist (D.N.E.).

2-d) $\lim_{t\to -0.5} f(t) = -1$

4-a) false, because $\lim_{x\to 2} f(x) = 1$

4-b) false, because $\lim_{x\to 2} f(x) = 1$

4-c) true 4-d) true 4-e) true

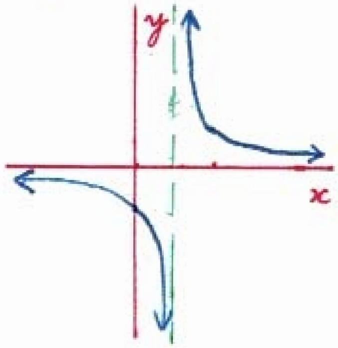
4-f) true

4-g) false, because $f(1) = 0$

4-h) true

4-i) false, because $f(2) = 0$

6) $\lim_{x \to 1} \frac{1}{x-1} = \text{D.N.E.}$ because (see graph) from left the function approaches $-\infty$ and from right the function approaches $+\infty$



8) Nothing can be said because in order for $\lim_{x \to 0} f(x)$ to exist, $f(x)$ must close to a single value for x near 0 regardless of the value $f(0)$ itself.

10) No, because the existence of a limit depends on the values of $f(x)$ when x is near 1, not on $f(1)$ itself. If $\lim_{x \to 1} f(x)$ exist, its value may be some other number other than $f(1)=5$. We can conclude nothing about $\lim_{x \to 1} f(x)$, whenever it exists or what its value is if it does exist, from knowing the value of $f(1)$ alone.

12) $\lim_{x \to 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = \underline{\underline{4}}$

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14) $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = \underline{\underline{-16}}$

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16) $\lim_{s \to \frac{2}{3}} (8 - 3s)(2s - 1) = \left(8 - 3\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right) - 1\right) = (8 - 2)\left(\frac{4}{3} - 1\right)$

$= (6)\left(\frac{4}{3} - \frac{3}{3}\right) = (6)\left(\frac{1}{3}\right) = \underline{\underline{2}}$

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18) $\lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6} = \frac{(2) + 2}{(2)^2 + 5(2) + 6} = \frac{4}{4 + 10 + 6} = \frac{4}{20} = \underline{\underline{\frac{1}{5}}}$

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20) $\lim_{z \to 4} \sqrt{z^2 - 10} = \sqrt{(4)^2 - 10} = \sqrt{16 - 10} = \sqrt{4} = \underline{\underline{2}}$

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22) $\lim_{h \to 0} \frac{\sqrt{5h + 4} - 2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{5h + 4} - 2}{h}\right)\left(\frac{\sqrt{5h + 4} + 2}{\sqrt{5h + 4} + 2}\right)$

$= \lim_{h \to 0} \frac{(5h + 4) - 4}{h(\sqrt{5h + 4} + 2)} = \lim_{h \to 0} \frac{5h}{h(\sqrt{5h + 4} + 2)} = \lim_{h \to 0} \frac{5}{\sqrt{5h + 4} + 2}$

$= \frac{5}{\sqrt{5(0) + 4} + 2} = \frac{5}{\sqrt{4} + 2} = \frac{5}{2 + 2} = \underline{\underline{\frac{5}{4}}}$

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24) $\lim_{x \to -3} \frac{\overset{0}{x + 3}}{\underset{0}{x^2 + 4x + 3}} = \lim_{x \to -3} \frac{x + 3}{(x + 3)(x + 1)} = \lim_{x \to -3} \frac{1}{x + 1} = \frac{1}{(-3) + 1} = \frac{1}{-2} = \underline{\underline{\frac{-1}{2}}}$

26) $\displaystyle \lim_{x \to 2} \frac{\overset{0}{x^2 - 7x + 10}}{\underset{0}{x - 2}} = \lim_{x \to 2} \frac{(x-2)(x-5)}{x-2} = \lim_{x \to 2} (x-5) = (2) - 5 = \underline{\underline{-3}}$

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28) $\displaystyle \lim_{t \to -1} \frac{\overset{0}{t^2 + 3t + 2}}{\underset{0}{t^2 - t - 2}} = \lim_{t \to -1} \frac{(t+2)(t+1)}{(t+1)(t-2)} = \lim_{t \to -1} \frac{t+2}{t-2} = \frac{(-1)+2}{(-1)-2} = \frac{1}{-3} = \underline{\underline{\frac{-1}{3}}}$

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30) $\displaystyle \lim_{y \to 0} \frac{\overset{0}{5y^3 + 8y^2}}{\underset{0}{3y^4 - 16y^2}} = \lim_{y \to 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \to 0} \frac{5y+8}{3y^2-16}$

$\displaystyle = \frac{5(0)+8}{3(0)^2-16} = \frac{8}{-16} = \underline{\underline{\frac{-1}{2}}}$

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32) $\displaystyle \lim_{x \to 0} \frac{\overset{0}{\frac{1}{x-1} + \frac{1}{x+1}}}{\underset{0}{x}} = \lim_{x \to 0} \left(\frac{\frac{1}{x-1} + \frac{1}{x+1}}{\frac{x}{1}} \right) \left(\frac{\frac{(x+1)(x-1)}{1}}{\frac{(x+1)(x-1)}{1}} \right)$

GLCD $= (x+1)(x-1)$

$\displaystyle = \lim_{x \to 0} \frac{(x+1) + (x-1)}{x(x+1)(x-1)} = \lim_{x \to 0} \frac{2x}{x(x+1)(x-1)}$

$\displaystyle = \lim_{x \to 0} \frac{2}{(x+1)(x-1)} = \frac{2}{((0)+1)((0)-1)} = \frac{2}{-1} = \underline{\underline{-2}}$

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34) $\displaystyle \lim_{v \to 2} \frac{\overset{0}{v^3 - 8}}{\underset{0}{v^4 - 16}} = \lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v^2+4)(v^2-4)}$

$\displaystyle = \lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v^2+4)(v+2)(v-2)} = \lim_{v \to 2} \frac{v^2+2v+4}{(v^2+4)(v+2)}$

$\displaystyle = \frac{(2)^2 + 2(2) + 4}{((2)^2+4)((2)+2)} = \frac{4+4+4}{(4+4)(4)} = \frac{4(1+1+1)}{(8)(4)} = \underline{\underline{\frac{3}{8}}}$

36) $$\lim_{x\to 4}\frac{4x-x^2}{2-\sqrt{x}}=\lim_{x\to 4}\frac{x(4-x)}{2-\sqrt{x}}=\lim_{x\to 4}\frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}}$$

$$=\lim_{x\to 4}x(2+\sqrt{x})=(4)(2+\sqrt{(4)})=(4)(2+2)=(4)(4)=\underline{\underline{16}}$$

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38) $$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}=\lim_{x\to -1}\left(\frac{\sqrt{x^2+8}-3}{x+1}\right)\left(\frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}\right)$$

$$=\lim_{x\to -1}\frac{(x^2+8)-9}{(x+1)(\sqrt{x^2+8}+3)}=\lim_{x\to -1}\frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$$

$$=\lim_{x\to -1}\frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)}=\lim_{x\to -1}\frac{x-1}{\sqrt{x^2+8}+3}=\frac{(-1)-1}{\sqrt{(-1)^2+8}+3}$$

$$=\frac{-2}{\sqrt{9}+3}=\frac{-2}{3+3}=\frac{-2}{6}=\underline{\underline{\frac{-1}{3}}}$$

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40) $$\lim_{x\to -2}\frac{x+2}{\sqrt{x^2+5}-3}=\lim_{x\to -2}\left(\frac{x+2}{\sqrt{x^2+5}-3}\right)\left(\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}\right)$$

$$=\lim_{x\to -2}\frac{(x+2)(\sqrt{x^2+5}+3)}{(x^2+5)-9}=\lim_{x\to -2}\frac{(x+2)(\sqrt{x^2+5}+3)}{x^2-4}$$

$$=\lim_{x\to -2}\frac{(x+2)(\sqrt{x^2+5}+3)}{(x+2)(x-2)}=\lim_{x\to -2}\frac{\sqrt{x^2+5}+3}{x-2}$$

$$=\frac{\sqrt{(-2)^2+5}+3}{(-2)-2}=\frac{\sqrt{4+5}+3}{-4}=\frac{\sqrt{9}+3}{-4}=\frac{3+3}{-4}=\frac{-6}{4}=\underline{\underline{\frac{-3}{2}}}$$

42) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x\to 4}\left(\frac{4-x}{5-\sqrt{x^2+9}}\right)\left(\frac{5+\sqrt{x^2+9}}{5+\sqrt{x^2+9}}\right)$

$= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{25-(x^2+9)} = \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{16-x^2}$

$= \lim_{x\to 4} \frac{(4-x)(5+\sqrt{x^2+9})}{(4+x)(4-x)} = \lim_{x\to 4} \frac{5+\sqrt{x^2+9}}{4+x}$

$= \frac{5+\sqrt{(4)^2+9}}{4+(4)} = \frac{5+\sqrt{16+9}}{8} = \frac{5+\sqrt{25}}{8} = \frac{5+5}{8} = \frac{10}{8} = \underline{\underline{\frac{5}{4}}}$

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44) $\lim_{x\to 0} \sin^2 x = \lim_{x\to 0}(\sin x)^2 = \left(\lim_{x\to 0} \sin x\right)^2 = (\sin(0))^2 = (0)^2 = \underline{\underline{0}}$

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46) $\lim_{x\to 0} \tan x = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{\sin(0)}{\cos(0)} = \frac{(0)}{(1)} = \underline{\underline{0}}$

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48) $\lim_{x\to 0} (x^2-1)(2-\cos x) = \left(\lim_{x\to 0}(x^2-1)\right)\left(\lim_{x\to 0}(2-\cos x)\right)$

$= \left((0)^2-1\right)\left(2-\cos(0)\right) = (-1)\left(2-(1)\right) = (-1)(1) = \underline{\underline{-1}}$

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50) $\lim_{x\to 0} \sqrt{7+\sec^2 x} = \sqrt{\lim_{x\to 0}(7+\sec^2 x)} = \sqrt{7+\lim_{x\to 0}\sec^2 x}$

$= \sqrt{7+\left(\lim_{x\to 0}\sec x\right)^2} = \sqrt{7+\left(\lim_{x\to 0}\frac{1}{\cos x}\right)^2} = \sqrt{7+\left(\frac{1}{\cos(0)}\right)^2}$

$= \sqrt{7+\left(\frac{1}{1}\right)^2} = \sqrt{7+1} = \underline{\underline{\sqrt{8} = 2\sqrt{2}}}$

52-a) Quotient Rule

52-b) Difference and Power Rules

52-c) Sum and Constant Multiple Rules

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54) $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$

a) $\lim_{x\to 4}\left(g(x)+3\right) = \lim_{x\to 4} g(x) + \lim_{x\to 4} 3 = (-3)+3 = \underline{\underline{0}}$

b) $\lim_{x\to 4} x f(x) = \left(\lim_{x\to 4} x\right)\left(\lim_{x\to 4} f(x)\right) = (4)(0) = \underline{\underline{0}}$

c) $\lim_{x\to 4}\left(g(x)\right)^2 = \left(\lim_{x\to 4} g(x)\right)^2 = (-3)^2 = \underline{\underline{9}}$

d) $\lim_{x\to 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x\to 4} g(x)}{\lim_{x\to 4}\left(f(x)-1\right)} = \frac{\lim_{x\to 4} g(x)}{\lim_{x\to 4} f(x) - \lim_{x\to 4} 1}$

$= \frac{(-3)}{(0)-1} = \frac{-3}{-1} = \underline{\underline{3}}$

56) $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, $\lim_{x\to -2} s(x) = -3$

a) $\lim_{x\to -2} \left(p(x) + r(x) + s(x)\right) = \lim_{x\to -2} p(x) + \lim_{x\to -2} r(x) + \lim_{x\to -2} s(x)$

$= (4) + (0) + (-3) = \underline{\underline{1}}$

b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x) = \left(\lim_{x\to -2} p(x)\right)\left(\lim_{x\to -2} r(x)\right)\left(\lim_{x\to -2} s(x)\right)$

$= (4)(0)(-3) = \underline{\underline{0}}$

c) $\lim_{x\to -2} \dfrac{-4p(x) + 5r(x)}{s(x)} = \dfrac{\lim_{x\to -2}(-4p(x)) + \lim_{x\to -2}(5r(x))}{\lim_{x\to -2} s(x)}$

$= \dfrac{-4\lim_{x\to -2} p(x) + 5\lim_{x\to -2} r(x)}{\lim_{x\to -2} s(x)} = \dfrac{-4(4) + 5(0)}{(-3)} = \dfrac{-16}{-3} = \underline{\underline{\dfrac{16}{3}}}$

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58) $f(x) = x^2$, $x = -2$

$f(-2) = (-2)^2 = 4$ $\quad f(-2+h) = (-2+h)^2 = (4 - 4h + h^2)$

$\lim_{h\to 0} \dfrac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \dfrac{(4 - 4h + h^2) - (4)}{h} = \lim_{h\to 0} \dfrac{h^2 - 4h}{h}$

$= \lim_{h\to 0} \dfrac{h(h-4)}{h} = \lim_{h\to 0} (h-4) = (0) - 4 = \underline{\underline{-4}}$

60) $f(x) = \dfrac{1}{x} \quad , x = -2$

$$f(-2) = \frac{1}{(-2)} = \frac{1}{-2} \qquad f(-2+h) = \frac{1}{(-2+h)} = \frac{1}{-2+h}$$

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \left(\frac{\left(\frac{1}{-2+h}\right) - \left(\frac{1}{-2}\right)}{\frac{h}{1}} \right) \left(\frac{\frac{-2(-2+h)}{1}}{\frac{-2(-2+h)}{1}} \right) \quad \text{GLCD} = -2(-2+h)$$

$$= \lim_{h \to 0} \frac{(-2) - (-2+h)}{h(-2)(-2+h)} = \lim_{h \to 0} \frac{-h}{h(-2)(-2+h)} = \lim_{h \to 0} \frac{-1}{-2(-2+h)} = \frac{-1}{-2(-2+(0))} = \underline{\underline{\frac{-1}{4}}}$$

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62) $f(x) = \sqrt{3x+1} \quad , x = 0$

$$f(0) = \sqrt{3(0)+1} = \sqrt{1} = 1 \qquad f(0+h) = \sqrt{3(0+h)+1} = \sqrt{3h+1}$$

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \left(\frac{\sqrt{3h+1} - 1}{h} \right) \left(\frac{\sqrt{3h+1} + 1}{\sqrt{3h+1} + 1} \right)$$

$$= \lim_{h \to 0} \frac{(3h+1) - 1}{h(\sqrt{3h+1} + 1)} = \lim_{h \to 0} \frac{3h}{h(\sqrt{3h+1} + 1)} = \lim_{h \to 0} \frac{3}{\sqrt{3h+1} + 1}$$

$$= \frac{3}{\sqrt{3(0)+1} + 1} = \frac{3}{\sqrt{1} + 1} = \frac{3}{1+1} = \underline{\underline{\frac{3}{2}}}$$

80) $\lim_{x\to -2} \frac{f(x)}{x^2} = 1$

a) $\lim_{x\to -2} f(x) = ?$

$$\lim_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} x^2} = \frac{\lim_{x\to -2} f(x)}{(-2)^2} = \frac{\lim_{x\to -2} f(x)}{4}$$

$$1 = \frac{\lim_{x\to -2} f(x)}{4}$$

$$\underline{\underline{4 = \lim_{x\to -2} f(x)}}$$

b) $\lim_{x\to -2} \frac{f(x)}{x} = ?$

$$\lim_{x\to -2} \frac{f(x)}{x^2} = \lim_{x\to -2} \left[\left(\frac{1}{x}\right)\left(\frac{f(x)}{x}\right)\right] = \left(\lim_{x\to -2} \frac{1}{x}\right)\left(\lim_{x\to -2} \frac{f(x)}{x}\right) = \left(\frac{1}{(-2)}\right) \lim_{x\to -2} \frac{f(x)}{x}$$

$$\left(\frac{1}{(-2)}\right) \lim_{x\to -2} \frac{f(x)}{x} = 1 \implies \underline{\underline{\lim_{x\to -2} \frac{f(x)}{x} = -2}}$$

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82) $\lim_{x\to 0} \frac{f(x)}{x^2} = 1$ since when $x\to 0$ the denominator is 0, we need to see when the product of $\left(\lim_{x\to 0} \frac{f(x)}{x^2} = 1\right)(0) = 0$?

a) $\lim_{x\to 0} f(x) = ?$

$$0 = 0\{1\} = (0)^2\{1\} = \left(\lim_{x\to 0} x\right)^2 \left\{\lim_{x\to 0} \frac{f(x)}{x^2}\right\} = \left(\lim_{x\to 0} x^2\right)\left\{\lim_{x\to 0} \frac{f(x)}{x^2}\right\} = \lim_{x\to 0} (x^2)\left\{\frac{f(x)}{x^2}\right\}$$

$$= \lim_{x\to 0} f(x) \qquad \text{so } \underline{\underline{\lim_{x\to 0} f(x) = 0}}$$

b) $\lim_{x\to 0} \frac{f(x)}{x}$

$$0 = (0)\{1\} = \left(\lim_{x\to 0} x\right)\left\{\lim_{x\to 0} \frac{f(x)}{x^2}\right\} = \lim_{x\to 0} (x)\left\{\frac{f(x)}{x^2}\right\} = \lim_{x\to 0} \frac{f(x)}{x}$$

$$\text{so } \underline{\underline{\lim_{x\to 0} \frac{f(x)}{x} = 0}}$$