MATH 20100 section 2.1

Average Speed [like slope formula of equation of a line]

When $f(t)$ measures the distance traveled at time t,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Def.: The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

if $x_2 = x_1 + h$, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$

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2) $g(x) = x^2 - 2x$

a) $g(1) = (1)^2 - 2(1) = -1$

$g(3) = (3)^2 - 2(3) = 3$

$$\frac{\Delta g}{\Delta x} = \frac{g(3) - g(1)}{(3) - (1)} = \frac{(3) - (-1)}{3 - 1} = \frac{4}{2} = \underline{\underline{2}}$$

b) $g(-2) = (-2)^2 - 2(-2) = 8$

$g(4) = (4)^2 - 2(4) = 8$

$$\frac{\Delta g}{\Delta x} = \frac{g(4) - g(-2)}{(4) - (-2)} = \frac{(8) - (8)}{4 + 2} = \frac{0}{6} = \underline{\underline{0}}$$

4) $g(t) = 2 + \cos t$

a) $g(0) = 2 + \cos(0) = 2 + 1 = 3$

$g(\pi) = 2 + \cos(\pi) = 2 + (-1) = 1$

$$\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(0)}{(\pi) - (0)} = \frac{(1) - (3)}{\pi} = \underline{\underline{\frac{-2}{\pi}}}$$

b) $g(-\pi) = 2 + \cos(-\pi) = 2 + (-1) = 1$

$g(\pi) = 1$

$$\frac{\Delta g}{\Delta t} = \frac{g(\pi) - g(-\pi)}{(\pi) - (-\pi)} = \frac{(1) - (1)}{2\pi} = \frac{0}{2\pi} = \underline{\underline{0}}$$

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6) $P(\theta) = \theta^3 - 4\theta^2 + 5\theta \; ; \; [1, 2]$

$P(1) = (1)^3 - 4(1)^2 + 5(1) = 2$

$P(2) = (2)^3 - 4(2)^2 + 5(2) = 2$

$$\frac{\Delta P}{\Delta \theta} = \frac{P(2) - P(1)}{(2) - (1)} = \frac{(2) - (2)}{1} = \frac{0}{1} = \underline{\underline{0}}$$

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8) $y = 7 - x^2 \; , \; P(2, 3)$

$y(x) = 7 - x^2$

$y(2) = 7 - (2)^2 = 3$

$y(2+h) = 7 - (2+h)^2 =$

$= 7 - (4 + 4h + h^2)$

$= 3 - 4h - h^2$

$$\frac{\Delta y}{\Delta x} = \frac{y(2+h) - y(2)}{h}$$

$$= \frac{(3 - 4h - h^2) - (3)}{h}$$

$$= \frac{-4h - h^2}{h} = \frac{h(-4 - h)}{h} = -4 - h$$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = -4$

$y - (3) = -4(x - (2))$

$y - 3 = -4x + 8$

$y = -4x + 11$

10) $y = x^2 - 4x$, $P = (1, -3)$

$y(x) = x^2 - 4x$

$y(1) = (1)^2 - 4(1) = -3$

$y(1+h) = (1+h)^2 - 4(1+h)$

$= (1 + 2h + h^2) - 4 - 4h$

$= h^2 - 2h - 3$

$\frac{\Delta y}{\Delta x} = \frac{y(1+h) - y(1)}{h} = \frac{(h^2 - 2h - 3) - (-3)}{h}$

$= \frac{h^2 - 2h}{h} = \frac{h(h-2)}{h} = h - 2$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = -2$

$y - (-3) = -2(x - (1))$

$y + 3 = -2x + 2$

$y = x - 1$

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12) $y = 2 - x^3$, $P(1, 1)$

$y(x) = 2 - x^3$

$y(1) = 2 - (1)^3 = 1$

$y(1+h) = 2 - (1+h)^3$

$= 2 - (1 + 3h + 3h^2 + h^3)$

$= 1 - 3h - 3h^2 - h^3$

$\frac{\Delta y}{\Delta x} = \frac{y(1+h) - y(1)}{h}$

$= \frac{(1 - 3h - 3h^2 - h^3) - (1)}{h}$

$= \frac{-3h - 3h^2 - h^3}{h}$

$= \frac{h(-3 - 3h - h^2)}{h}$

$= -3 - 3h - h^2$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = -3$

$y - (1) = -3(x - (1))$

$y - 1 = -3x + 3$

$y = -3x + 4$

14) $y = x^3 - 3x^2 + 4 \quad , \quad P(2,0)$

$$y(x) = x^3 - 3x^2 + 4$$

$$y(2) = (2)^3 - 3(2)^2 + 4 = 0$$

$$y(2+h) = (2+h)^3 - 3(2+h)^2 + 4$$

$$= (8 + 12h + 6h^2 + h^3) - 3(4 + 4h + h^2) + 4$$

$$= 8 + 12h + 6h^2 + h^3 - 12 - 12h - 3h^2 + 4 = 3h^2 + h^3$$

$$\frac{\Delta y}{\Delta x} = \frac{y(2+h) - y(2)}{h} = \frac{(3h^2 + h^3) - (0)}{h} = \frac{3h^2 + h^3}{h} = \frac{h(3h + h^2)}{h}$$

$$= 3h + h^2$$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = 0$

$$y - (0) = 0(x - (2))$$

$$y - 0 = 0$$

$$y = 0$$

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16) $y = \frac{x}{2-x} \quad , \quad P(4,-2)$

$$y(x) = \frac{x}{2-x}$$

$$y(4) = \frac{(4)}{2-(4)} = \frac{4}{-2} = -2$$

$$y(4+h) = \frac{(4+h)}{2-(4+h)} = \frac{4+h}{-2-h}$$

$$\frac{\Delta y}{\Delta x} = \frac{y(4+h) - y(4)}{h} = \frac{\left(\frac{4+h}{-2-h}\right) - (-2)}{h}$$

$$= \left(\frac{\frac{4+h}{-2-h} + \frac{2}{1}}{\frac{h}{1}}\right)\left(\frac{\frac{-2-h}{1}}{\frac{-2-h}{1}}\right) = \frac{(4+h) + 2(-2-h)}{h(-2-h)}$$

$$= \frac{4 + h - 4 - 2h}{h(-2-h)} = \frac{-h}{h(-2-h)} = \frac{-1}{-2-h}$$

$$= \frac{-1}{-(2+h)} = \frac{1}{2+h}$$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = \frac{1}{2}$

$$y - (-2) = \frac{1}{2}(x - (4))$$

$$y + 2 = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x - 4$$

18) $y = \sqrt{7-x}$, $P(-2, 3)$

$y(x) = \sqrt{7-x}$

$y(-2) = \sqrt{7-(-2)} = \sqrt{9}$

$y(-2+h) = \sqrt{7-(-2+h)} = \sqrt{9-h}$

$$\frac{\Delta y}{\Delta x} = \frac{y(-2+h) - y(-2)}{h}$$

$$= \left(\frac{\sqrt{9-h} - \sqrt{9}}{h}\right)\left(\frac{\sqrt{9-h} + \sqrt{9}}{\sqrt{9-h} + \sqrt{9}}\right)$$

$$= \frac{(9-h) - 9}{h(\sqrt{9-h} + \sqrt{9})} = \frac{-h}{h(\sqrt{9-h} + \sqrt{9})}$$

$$= \frac{-1}{\sqrt{9-h} + \sqrt{9}}$$

as $h \to 0$, $m = \frac{\Delta y}{\Delta x} = \frac{-1}{\sqrt{9} + \sqrt{9}} = \frac{-1}{3+3} = \frac{-1}{6}$

$y - (3) = \frac{-1}{6}(x - (-2))$

$y - 3 = \frac{-1}{6}x - \frac{2}{6}$

$y - 3 = \frac{-1}{6}x - \frac{1}{3}$

$y = \frac{-1}{6}x + \frac{8}{3}$

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24) $f(t) = \frac{1}{t}$ $t \neq 0$

a-i) $t = 2$ to $t = 3$

$f(2) = \frac{1}{(2)} = \frac{1}{2}$

$f(3) = \frac{1}{(3)} = \frac{1}{3}$

$$\frac{f(3) - f(2)}{(3)-(2)} = \frac{\left(\frac{1}{3}\right) - \left(\frac{1}{2}\right)}{1} = \frac{\frac{2}{6} - \frac{3}{6}}{1} = \underline{\underline{\frac{-1}{6}}}$$

a-ii) $t = 2$ to $t = T$

$f(2) = \frac{1}{(2)} = \frac{1}{2}$

$f(T) = \frac{1}{(T)} = \frac{1}{T}$

$$\frac{f(T) - f(2)}{(T) - (2)} = \frac{\left(\frac{1}{T}\right) - \left(\frac{1}{2}\right)}{T-2} = \left(\frac{\frac{1}{T} - \frac{1}{2}}{\frac{T-2}{1}}\right)\left(\frac{\frac{2T}{1}}{\frac{2T}{1}}\right)$$

$$= \frac{2-T}{2T(T-2)} = \frac{-T+2}{2T(T-2)} = \frac{-1(T-2)}{2T(T-2)} = \underline{\underline{\frac{-1}{2T}}}$$

24) continued

b)

T	$2.1 = \frac{21}{10}$	$2.01 = \frac{201}{100}$	$2.001 = \frac{2001}{1000}$	$2.0001 = \frac{20001}{10000}$	$2.00001 = \frac{200001}{100000}$	$2.000001 = \frac{2000001}{1000000}$
$\ell(T)$	$\frac{1}{\left(\frac{21}{10}\right)} = \frac{10}{21}$	$\frac{100}{201}$	$\frac{1000}{2001}$	$\frac{10000}{20001}$	$\frac{100000}{200001}$	$\frac{1000000}{2000001}$
$T-2$	$2.1-2 = 0.1 = \frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10000}$	$\frac{1}{100000}$	$\frac{1}{1000000}$
$\frac{\ell(T)-\ell(2)}{T-2}$	$\frac{\left(\frac{10}{21}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{10}\right)}$ $= \frac{\left(\frac{10}{21}\right)\left(\frac{2}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{21}{21}\right)}{\left(\frac{1}{10}\right)}$ $= 10\left(\frac{20-21}{42}\right)$ $= \frac{-10}{42}$	$\frac{\left(\frac{100}{201}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{100}\right)}$ ⋮ $= \frac{-100}{402}$	$\frac{\left(\frac{1000}{2001}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{1000}\right)}$ ⋮ $= \frac{-1000}{4002}$	$\frac{\left(\frac{10000}{20001}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{10000}\right)}$ ⋮ $= \frac{-10000}{40002}$	$\frac{\left(\frac{100000}{200001}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{100000}\right)}$ ⋮ $= \frac{-100000}{400002}$	$\frac{\left(\frac{1000000}{2000001}\right)-\left(\frac{1}{2}\right)}{\left(\frac{1}{1000000}\right)}$ ⋮ $= \frac{-1000000}{4000002}$

c) the table is showing that as $T \to 2$, the average rate of change $\frac{\ell(T)-\ell(2)}{T-2} \to \frac{-1}{4}$

d) $\lim_{T\to 2}\left(\frac{-1}{2T}\right) = \frac{-1}{2(2)} = \frac{-1}{4}$

"borrowing limit evaluation of section 2.2"