Def. A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Def. Suppose that f is a one-to-one function on a domain D with range R. The inverse function is defined by

$$f^{-1}(b) = a \quad \text{if } f(a) = b.$$

The domain of f^{-1} is R and the range is D.

Logarithmic Functions

Def. The logarithm function with base a, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x \quad (a > 0,\ a \neq 1)$.

$$p = \log_b N \Longleftrightarrow b^p = N \qquad p = \ln N \Longleftrightarrow e^p = N$$

Thm 1 Algebraic Properties of the Natural Logarithm
For any numbers $A>0$ and $B>0$, the natural logarithm satisfies the following rules:

* 1) [Product Rule] $\ln(AB) = \ln A + \ln B$ $\qquad \log_c(AB) = \log_c A + \log_c B$

* 2) [Quotient Rule] $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$ $\qquad \log_c\left(\frac{A}{B}\right) = \log_c A - \log_c B$

3) [Reciprocal Rule] $\ln\left(\frac{1}{B}\right) = \ln 1 - \ln B = -\ln B$ $\qquad \log_c\left(\frac{1}{B}\right) = -\log_c B$

* 4) [Power Rule] $\ln(A^B) = B \ln A$ $\qquad \log_c(A^B) = B \log_c A$

Inverse Trigonometric Functions

Study "Domain restrictions that make the trigonometric functions one-to-one" on Pg 45. [especially, $\sin x$, $\cos x$, $\tan x$]

$y = \sin^{-1} x = \arcsin x \qquad y = \cos^{-1} x = \arccos x$

$y = \tan^{-1} x = \arctan x \qquad y = \sec^{-1} x = \operatorname{arcsec} x$

$y = \cot^{-1} x = \operatorname{arccot} x \qquad y = \csc^{-1} x = \operatorname{arccsc} x$

If θ is an radiant angle and R is a trigonometric ratio [note: both θ, R is real numbers]

$\sin\theta = R \iff \theta = \sin^{-1} R \qquad \cos\theta = R \iff \theta = \cos^{-1} R$

$\tan\theta = R \iff \theta = \tan^{-1} R$

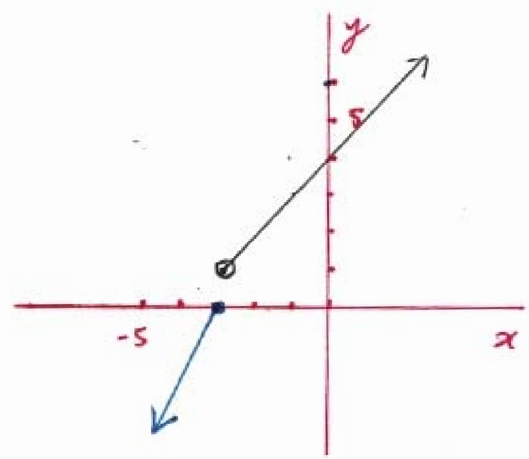
2) not 1-1 (one-to-one) because it fails the horizontal line test.

4) not 1-1 because it fails the horizontal line test

6) yes 1-1 because it satisfies the horizontal line test

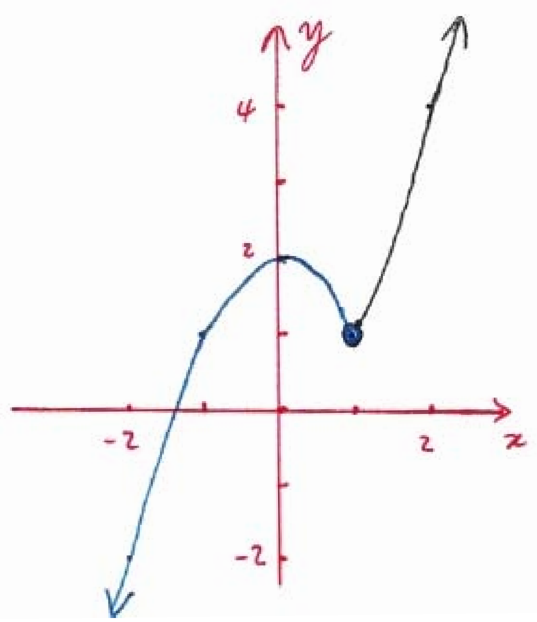
8) $f(x) = \begin{cases} 2x+6 & x \le -3 \\ x+4 & x > -3 \text{ or } -3 < x \end{cases}$

yes 1-1 because it satisfies the horizontal line test (see graph)

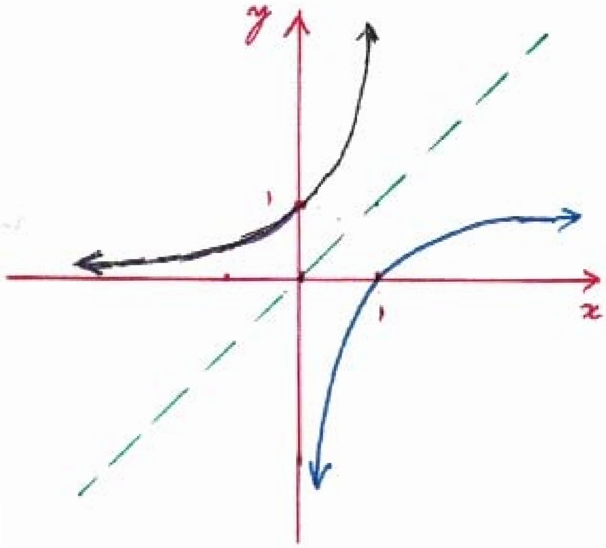


10) $f(x) = \begin{cases} 2-x^2 & x \le 1 \\ x^2 & x > 1 \text{ or } 1 < x \end{cases}$

not 1-1 because it fails the horizontal line test (see graph)



12) $y = f(x) = 1 - \frac{1}{x}$, $x > 0$

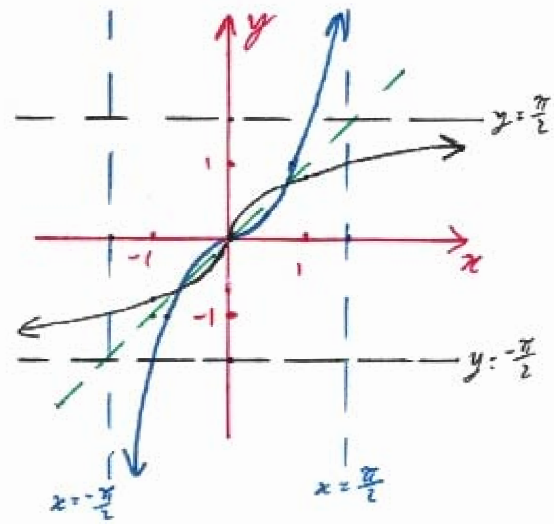


black: $f^{-1}(x)$

domain: $(-\infty, 1)$

range: $(0, \infty)$

14) $y = f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$



black: $f^{-1}(x)$

domain: $(-\infty, \infty)$

range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

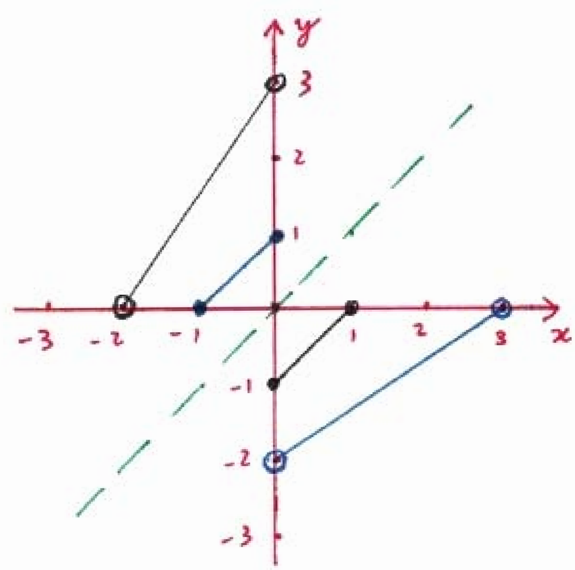
16)

$$f(x) = \begin{cases} x+1 & -1 \le x \le 0 \\ -2 + \frac{2}{3}x & 0 < x < 3 \end{cases}$$

black: $f^{-1}(x)$

domain: $(-2, 1]$

range: $[-1, 3)$



26) $f(x) = x^4$, $x \geq 0$

$y = x^4$ since $x \geq 0$

$\pm\sqrt[4]{y} = x$ take $x = +\sqrt[4]{y}$ and $f^{-1}(x) = \sqrt[4]{x}$

$f(f^{-1}(x)) = f(\sqrt[4]{x}) = (\sqrt[4]{x})^4 = x$

domain: $[0, \infty)$
range: $[0, \infty)$

$f^{-1}(f(x)) = f^{-1}(x^4) = \sqrt[4]{(x^4)} = x$

---

28) $f(x) = \frac{1}{2}x - \frac{7}{2}$ $\quad f^{-1}(x) = 2x + 7$

domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$

$y = \frac{1}{2}x - \frac{7}{2}$

$f(f^{-1}(x)) = f(2x+7) = \frac{1}{2}(2x+7) - \frac{7}{2} = x + \frac{7}{2} - \frac{7}{2} = x$

$y = \frac{x-7}{2}$

$f^{-1}(f(x)) = f^{-1}(\frac{1}{2}x - \frac{7}{2}) = 2(\frac{1}{2}x - \frac{7}{2}) + 7 = x - 7 + 7 = x$

$2y = x - 7$

$2y + 7 = x$

---

30) $f(x) = \frac{1}{x^3}$, $x \neq 0$ $\quad f^{-1}(x) = \frac{1}{\sqrt[3]{x}}$

domain: $(-\infty, 0) \cup (0, \infty)$
range: $(-\infty, 0) \cup (0, \infty)$

$y = \frac{1}{x^3}$

$f(f^{-1}(x)) = f\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{1}{\left(\frac{1}{\sqrt[3]{x}}\right)^3} = \frac{1}{\frac{(1)^3}{(\sqrt[3]{x})^3}} = \frac{1}{\frac{1}{x}} = x$

$x^3 = \frac{1}{y}$

$x = \sqrt[3]{\frac{1}{y}}$ $\quad f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x^3}\right) = \frac{1}{\sqrt[3]{\left(\frac{1}{x^3}\right)}} = \frac{1}{\frac{\sqrt[3]{1}}{\sqrt[3]{x^3}}} = \frac{1}{\frac{1}{x}} = x$

$x = \frac{1}{\sqrt[3]{y}}$

32) $f(x)=\frac{\sqrt{x}}{\sqrt{x}-3}$ domain of $f(x)$: $[0,9)\cup(9,\infty)$

$y=\frac{\sqrt{x}}{\sqrt{x}-3}$

$y(\sqrt{x}-3)=\sqrt{x}$

$\sqrt{x}\,y-3y=\sqrt{x}$

$\sqrt{x}\,y-\sqrt{x}=3y$

$\sqrt{x}(y-1)=3y$

$\sqrt{x}=\frac{3y}{y-1}$

$x=\left(\frac{3y}{y-1}\right)^2$

$f^{-1}(x)=\left(\frac{3x}{x-1}\right)^2 \Rightarrow \left(\sqrt{f^{-1}(x)}\right)^2=\frac{3x}{x-1}$

domain: since $f(x)$ contains $\sqrt{x}$, we must check $\left(\sqrt{f^{-1}(x)}\right)^2\geq 0$ to get our domain of $f^{-1}(x)$.

$\frac{3x}{x-1}\geq 0$

$3x=0 \quad x-1=0$

$x=\boxed{0} \quad x=1$

	$(-\infty,\boxed{0})$	$(0,1)$	$(1,\infty)$
x	neg	POS	POS
$x-1$	neg	neg	POS
$\frac{3x}{x-1}$	POS	neg	POS

so the domain of $f^{-1}(x)$ is $(-\infty,0]\cup(1,\infty)$

range: using the fact that domain of $f(x)$ is the range of $f^{-1}(x)$. $[0,9)\cup(9,\infty)$

$f(f^{-1}(x))=f\left(\left(\frac{3x}{x-1}\right)^2\right)=\frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}=\left(\frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-\frac{3}{1}}\right)\left(\frac{\frac{x-1}{1}}{\frac{x-1}{1}}\right)$

$=\frac{3x}{3x-3(x-1)}=\frac{3x}{3x-3x+3}=\frac{3x}{3}=x$

$f^{-1}(f(x))=f^{-1}\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)=\left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)-1}\right)^2=\left(\frac{\frac{3\sqrt{x}}{\sqrt{x}-3}}{\frac{\sqrt{x}}{\sqrt{x}-3}-1}\right)^2=\left(\left(\frac{\frac{3\sqrt{x}}{\sqrt{x}-3}}{\frac{\sqrt{x}}{\sqrt{x}-3}-\frac{1}{1}}\right)\left(\frac{\frac{\sqrt{x}-3}{1}}{\frac{\sqrt{x}-3}{1}}\right)\right)^2$

$=\left(\frac{3\sqrt{x}}{\sqrt{x}-(\sqrt{x}-3)}\right)^2=\left(\frac{3\sqrt{x}}{\sqrt{x}-\sqrt{x}+3}\right)^2=\left(\frac{3\sqrt{x}}{3}\right)^2=(\sqrt{x})^2=x$

34) $f(x) = (2x^3+1)^{\frac{1}{5}} = \sqrt[5]{2x^3+1}$

$y = \sqrt[5]{2x^3+1}$

$y^5 = 2x^3+1$

$y^5 - 1 = 2x^3$

$\frac{y^5-1}{2} = x^3$

$\sqrt[3]{\frac{y^5-1}{2}} = x$

$f^{-1}(x) = \sqrt[3]{\frac{x^5-1}{2}}$

domain: $(-\infty, \infty)$

range: $(-\infty, \infty)$

$$f(f^{-1}(x)) = f\left(\sqrt[3]{\frac{x^5-1}{2}}\right) = \sqrt[5]{2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3+1} = \sqrt[5]{2\left(\frac{x^5-1}{2}\right)+1}$$

$$= \sqrt[5]{x^5-1+1} = \sqrt[5]{x^5} = x$$

$$f^{-1}(f(x)) = f^{-1}\left(\sqrt[5]{2x^3+1}\right) = \sqrt[3]{\frac{\left(\sqrt[5]{2x^3+1}\right)^5-1}{2}} = \sqrt[3]{\frac{(2x^3+1)-1}{2}}$$

$$= \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$$

36) $f(x) = x^2 - 2bx$, $b > 0$ and constant, $x \leq b$

$y = x^2 - 2bx$ $\quad \frac{b_x}{2} = \frac{(-2b)}{2} = -b$ $\quad$ domain of $f(x)$: $(-\infty, b]$

$\left(\frac{b_x}{2}\right)^2 = (-b)^2 = b^2$

$y + b^2 = x^2 - 2bx + b^2$

$y + b^2 = (x - b)^2$

$\pm\sqrt{y + b^2} = x - b$

$b \pm \sqrt{y + b^2} = x$

vertex: $(b, -b^2)$

parabola opening upwards

since $x \leq b$ on $f(x)$, we take the minus part for $f^{-1}(x)$.

$f^{-1}(x) = b - \sqrt{x + b^2}$

domain: $x + b^2 \geq 0$

$x \geq -b^2$ $\quad [-b^2, \infty)$

$-b^2 \leq x$

range: using the fact that the domain of $f(x)$ is the range of $f^{-1}(x)$.

$(-\infty, b]$

$$f(f^{-1}(x)) = f\left(b - \sqrt{x + b^2}\right) = \left(b - \sqrt{x+b^2}\right)^2 - 2b\left(b - \sqrt{x+b^2}\right)$$

$$= \left(b^2 - 2b\sqrt{x+b^2} + (x + b^2)\right) - 2b^2 + 2b\sqrt{x+b^2} = x$$

$$f^{-1}(f(x)) = f^{-1}(x^2 - 2bx) = b - \sqrt{(x^2 - 2bx) + b^2} = b - \sqrt{(x-b)^2}$$

$$= b - |x - b| = b - (b - x) = b - b + x = x$$

since $x \leq b$

42-a) $\ln\left(\frac{1}{125}\right) = \ln\left(\frac{1}{5^3}\right) = \ln 1 - \ln(5^3) = \underline{\underline{-3\ln 5}}$

42-b) $\ln 9.8 = \ln\left(\frac{49}{5}\right) = \ln\left(\frac{7^2}{5}\right) = \ln(7^2) - \ln 5 = \underline{\underline{2\ln 7 - \ln 5}}$

42-c) $\ln 7\sqrt{7} = \ln(7^{\frac{3}{2}}) = \underline{\underline{\frac{3}{2}\ln 7}}$

42-d) $\ln 1225 = \ln(35^2) = 2\ln 35 = 2\ln((5)(7)) = 2\{\ln 5 + \ln 7\}$
$= \underline{\underline{2\ln 5 + 2\ln 7}}$

42-e) $\ln 0.056 = \ln\left(\frac{7}{125}\right) = \ln\left(\frac{7}{5^3}\right) = \ln 7 - \ln(5^3) = \underline{\underline{\ln 7 - 3\ln 5}}$

42-f) $\frac{\ln 35 + \ln\left[\frac{1}{7}\right]}{\ln 25} = \frac{\ln\{(5)(7)\} + \ln\left[\frac{1}{7}\right]}{\ln(5^2)} = \frac{\{\ln 5 + \ln 7\} + [\ln 1 - \ln 7]}{2\ln 5}$

$= \frac{\ln 5 + \ln 1}{2\ln 5} = \frac{\ln 5 + 0}{2\ln 5} = \frac{\ln 5}{2\ln 5} = \underline{\underline{\frac{1}{2}}}$

44-a) $\ln\sec\theta + \ln\cos\theta = \ln\{(\sec\theta)(\cos\theta)\} = \ln\left\{\left(\frac{1}{\cos\theta}\right)(\cos\theta)\right\}$
$= \ln\{1\} = \underline{\underline{0}}$

44-b) $\ln(8x+4) - 2\ln c = \ln(8x+4) - \ln(c^2) = \underline{\underline{\ln\left(\frac{8x+4}{c^2}\right)}}$

44-c) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = \ln\{(\sqrt[3]{t^2-1})^3\} - \ln(t+1)$
$= \ln(t^2-1) - \ln(t+1) = \ln\left(\frac{t^2-1}{t+1}\right) = \ln\left(\frac{(t+1)(t-1)}{t+1}\right)$
$= \underline{\underline{\ln(t-1)}}$

46-a) $e^{\ln(x^2+y^2)} = \underline{\underline{x^2+y^2}}$

46-b) $e^{-\ln 0.3} = e^{\ln(0.3)^{-1}} = (0.3)^{-1}$

$= \dfrac{1}{0.3} = \underline{\underline{\dfrac{10}{3}}}$

46-c) $e^{\ln \pi x - \ln 2} = e^{\ln\left(\frac{\pi x}{2}\right)} = \underline{\underline{\dfrac{\pi x}{2}}}$

48-a) $\ln(e^{\sec\theta}) = (\sec\theta)[\ln e] = \underline{\underline{\sec\theta}}$

48-b) $\ln(e^{(e^x)}) = (e^x)[\ln e] = \underline{\underline{e^x}}$

48-c) $\ln(e^{2\ln x}) = \ln\left(e^{\ln(x^2)}\right) = \underline{\underline{\ln(x^2) = 2\ln x}}$

50) $\ln y = -t+5$

$\Downarrow$

$\underline{\underline{y = e^{(-t+5)}}}$

$y = (e^{-t})(e^5)$

$\underline{\underline{y = \dfrac{e^5}{e^t}}}$

52) $\ln(c-2y) = t$

$\Downarrow$

$c - 2y = e^t$

$c - e^t = 2y$

$\underline{\underline{\dfrac{c-e^t}{2} = y}}$

54) $\ln(y^2-1) - \ln(y+1) = \ln(\sin x)$

$$\ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x)$$

$$\ln\left(\frac{(y+1)(y-1)}{y+1}\right) = \ln(\sin x)$$

because we have single ln on both sides of equal sign

$$y-1 = \sin x$$

$$\underline{\underline{y = 1+\sin x}}$$

56-a) $e^{5k} = \frac{1}{4}$

$\Downarrow$

$5k = \ln\left(\frac{1}{4}\right)$

$5k = \ln 1 - \ln 4$

$5k = -\ln 4$

$\underline{\underline{k = \frac{-\ln 4}{5}}}$

56-b) $80e^{k} = 1$

$e^{k} = \frac{1}{80}$

$\Downarrow$

$k = \ln\left(\frac{1}{80}\right)$

$k = \ln 1 - \ln 80$

$\underline{\underline{k = -\ln 80}}$

56-c) $e^{(\ln 0.8)k} = 0.8$

$\Downarrow$

$(\ln 0.8)k = \ln 0.8$

$k = \frac{\ln 0.8}{(\ln 0.8)}$

$k = \underline{\underline{1}}$

58-a) $e^{-0.01t} = 1000$

$\Downarrow$

$-0.01t = \ln 1000$

$t = \frac{\ln 1000}{-0.01}$

$\underline{\underline{t = \frac{-\ln 1000}{0.01} = -100\ln 1000}}$

58-b) $e^{kt} = \frac{1}{10}$

$\Downarrow$

$kt = \ln\left(\frac{1}{10}\right)$

$kt = \ln 1 - \ln 10$

$kt = -\ln 10$

$\underline{\underline{t = \frac{-\ln 10}{k}}}$

58-c) $e^{(\ln 2)t} = \frac{1}{2}$

$\Downarrow$

$(\ln 2)t = \ln\left(\frac{1}{2}\right)$

$(\ln 2)t = \ln 1 - \ln 2$

$(\ln 2)t = -\ln 2$

$t = \frac{-\ln 2}{(\ln 2)} = \underline{\underline{-1}}$

60) $e^{(x^2)} e^{(2x+1)} = e^t$

$e^{\{(x^2)+(2x+1)\}} = e^t$

$e^{\{x^2+2x+1\}} = e^t$

since base is identical, the exponents are equal

$\underline{x^2+2x+1 = t}$

$\underline{(x+1)^2 = t}$

62) $e^{-2t} + 6 = 5e^{-t}$

$e^{-2t} - 5e^{-t} + 6 = 0$

$(e^{-t})^2 - 5(e^{-t}) + 6 = 0$

$[(e^{-t}) - 2][(e^{-t}) - 3] = 0$

$(e^{-t}) - 2 = 0$	$(e^{-t}) - 3 = 0$
$e^{-t} = 2$	$e^{-t} = 3$
$\Downarrow$	$\Downarrow$
$-t = \ln 2$	$-t = \ln 3$
$\underline{t = -\ln 2}$	$\underline{t = -\ln 3}$

64) $\ln(t-2) = \ln 8 - \ln t$

$\ln(t-2) = \ln\left(\frac{8}{t}\right)$

$t - 2 = \frac{8}{t}$ LCD$=t$

$(t)(t-2) = \left(\frac{8}{t}\right)(t)$

$t^2 - 2t = 8$

$t^2 - 2t - 8 = 0$

$(t+2)(t-4) = 0$

$t+2=0$	$t-4=0$
$t=-2$	$\underline{t=4}$
discard	

66-a) $2^{\log_2 3} = \underline{3}$

66-b) $10^{\log_{10}\left(\frac{1}{2}\right)} = \underline{\frac{1}{2}}$

66-c) $\pi^{\log_\pi 7} = \underline{7}$

66-d) $\log_{11} 121 = \log_{11}(11)^2 = 2\log_{11}(11)$

$= 2(1) = \underline{2}$

66-e) $\log_{121} 11 = \log_{121}(\sqrt{121}) = \log_{121}(121)^{\frac{1}{2}}$

$= \frac{1}{2}\log_{121} 121 = \frac{1}{2}(1) = \underline{\frac{1}{2}}$

66-f) $\log_3\left(\frac{1}{9}\right) = \log_3\left(\frac{1}{3^2}\right) = \log_3(3)^{-2}$

$= (-2)\log_3 3 = (-2)(1)$

$= \underline{-2}$

68-a) $25^{\log_5(3x^2)} = (5^2)^{\log_5(3x^2)} = 5^{2\log_5(3x^2)}$

$= 5^{\log_5(3x^2)^2} = (3x^2)^2 = \underline{\underline{9x^4}}$

68-b) $\log_e(e^x) = (x)\log_e e = (x)(1) = \underline{\underline{x}}$

68-c) $\log_4\left(2^{e^x \sin x}\right) = \log_4\left((\sqrt{4})^{e^x \sin x}\right)$

$= \log_4\left(4^{\frac{1}{2}(e^x \sin x)}\right) = \left(\frac{1}{2}(e^x \sin x)\right)\log_4 4$

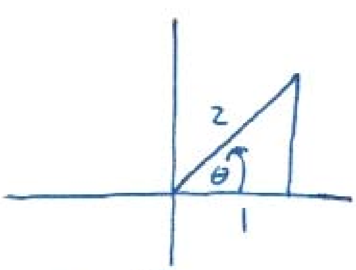
$= \left(\frac{1}{2}(e^x \sin x)\right)(1) = \underline{\underline{\frac{1}{2}(e^x \sin x)}}$

72-a) $\cos^{-1}\left(\frac{1}{2}\right)$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\Downarrow$

$\cos\theta = \frac{1}{2}$



$\theta = \frac{\pi}{3}$

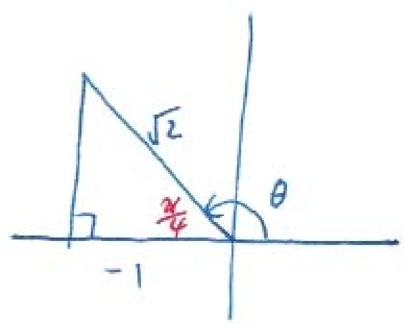
$\underline{\underline{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}}$

72-b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\Downarrow$

$\cos\theta = \frac{-1}{\sqrt{2}}$



$\theta = \frac{3\pi}{4}$

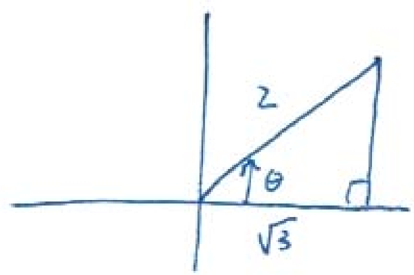
$\underline{\underline{\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}}}$

72-c) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Downarrow$

$\cos\theta = \frac{\sqrt{3}}{2}$



$\theta = \frac{\pi}{6}$

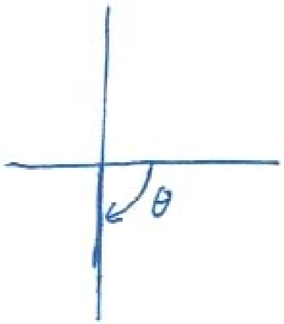
$\underline{\underline{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}}}$

74-a) $\arcsin(-1)$

$$\theta = \sin^{-1}(-1)$$

$$\Downarrow$$

$$\sin\theta = -1$$



$$\theta = \frac{-\pi}{2}$$

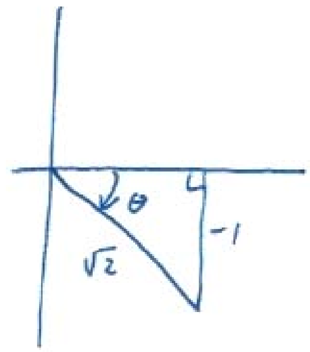
$$\underline{\underline{\arcsin(-1) = \frac{-\pi}{2}}}$$

74-b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

$$\theta = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$\Downarrow$$

$$\sin\theta = \frac{-1}{\sqrt{2}}$$



$$\theta = \frac{-\pi}{4}$$

$$\underline{\underline{\arcsin\left(\frac{-1}{\sqrt{2}}\right) = \frac{-\pi}{4}}}$$

---

88) $y = y_0 e^{-0.18t}$

$$y(t) = y_0 e^{-0.18t}$$

$$(0.90\,y_0) = y_0 e^{-0.18t}$$

$$0.90 = e^{-0.18t}$$

$$\Downarrow$$

$$\ln 0.90 = -0.18t$$

$90\% = 0.90$

$y(t) = 0.90\,y_0\ ,\ t = ?$

$$\underline{\underline{t = \frac{\ln 0.90}{-0.18} = \frac{-\ln 0.90}{0.18} \text{ days}}}$$

$t \approx 0.585336198$ days