

$$1) \quad (0,1), \quad \theta = 60^\circ \quad \hat{x} = x \cos \theta + y \sin \theta \quad \hat{y} = -x \sin \theta + y \cos \theta$$

$$\hat{x} = (0) \cos(60^\circ) + (1) \sin(60^\circ) = 0 + (1) \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2}$$

$$\hat{y} = -(0) \sin(60^\circ) + (1) \cos(60^\circ) = (0) + (1) \left(\frac{1}{2} \right) = \frac{1}{2}$$

$$(\hat{x}, \hat{y}) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

$$3) \quad (-3,1), \quad \theta = 30^\circ \quad \hat{x} = x \cos \theta + y \sin \theta \quad \hat{y} = -x \sin \theta + y \cos \theta$$

$$\hat{x} = (-3) \cos(30^\circ) + (1) \sin(30^\circ) = (-3) \left(\frac{\sqrt{3}}{2} \right) + (1) \left(\frac{1}{2} \right) = \frac{-3\sqrt{3}}{2} + \frac{1}{2} = \frac{1-3\sqrt{3}}{2}$$

$$\hat{y} = -(-3) \sin(30^\circ) + (1) \cos(30^\circ) = (-3) \left(\frac{1}{2} \right) + (1) \left(\frac{\sqrt{3}}{2} \right) = \frac{-3}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-3}{2}$$

$$(\hat{x}, \hat{y}) = \left(\frac{1-3\sqrt{3}}{2}, \frac{\sqrt{3}-3}{2} \right)$$

$$5) \quad x^2 - xy + y^2 = 2 \quad x = \hat{x} \cos \theta - \hat{y} \sin \theta \quad y = \hat{x} \sin \theta + \hat{y} \cos \theta$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{(1)-(1)}{(-1)} = \frac{0}{-1} = 0 \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$A=1 \quad B=-1 \quad C=1 \quad 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$x = \hat{x} \cos\left(\frac{\pi}{4}\right) - \hat{y} \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y} \quad y = \hat{x} \sin\left(\frac{\pi}{4}\right) + \hat{y} \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$$

$$x^2 - xy + y^2 = 2$$

$$\left(\frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y} \right)^2 - \left(\frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y} \right) \left(\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right) + \left(\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right)^2 = 2$$

$$\left(\frac{1}{2} \hat{x}^2 - \hat{x}\hat{y} + \frac{1}{2} \hat{y}^2 \right) - \left(\frac{1}{2} \hat{x}^2 - \frac{1}{2} \hat{y}^2 \right) + \left(\frac{1}{2} \hat{x}^2 + \hat{x}\hat{y} + \frac{1}{2} \hat{y}^2 \right) = 2$$

$$\frac{1}{2} \hat{x}^2 - \hat{x}\hat{y} + \frac{1}{2} \hat{y}^2 - \frac{1}{2} \hat{x}^2 + \frac{1}{2} \hat{y}^2 + \frac{1}{2} \hat{x}^2 + \hat{x}\hat{y} + \frac{1}{2} \hat{y}^2 = 2 \Rightarrow \frac{\hat{x}^2}{(2)^2} + \frac{\hat{y}^2}{\left(\frac{2}{\sqrt{3}}\right)^2} = 1$$

$$\frac{1}{2} \hat{x}^2 + \frac{3}{2} \hat{y}^2 = 2$$

$$\frac{\hat{x}^2}{4} + \frac{3\hat{y}^2}{4} = 1$$

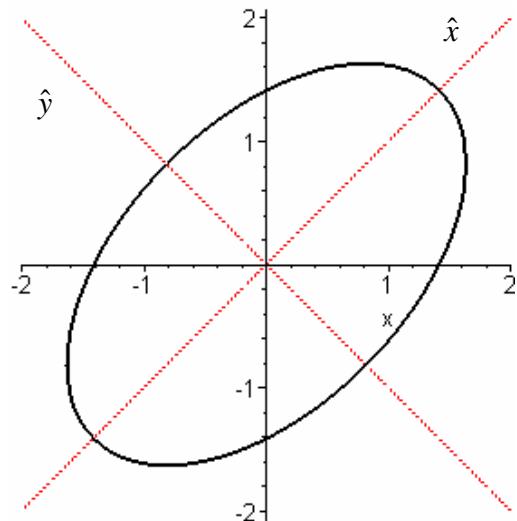
This is an ellipse.

All values here are in $\hat{x}\hat{y}$ -axis:

Center: $(0,0)$ Vertices: $(0 \pm 2, 0)$

length of Major axis: $2a = 2(2) = 4$

$$\text{length of minor axis: } 2b = 2\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$$



$$7) \quad 17x^2 - 6xy + 9y^2 = 0 \quad x = \hat{x}\cos\theta - \hat{y}\sin\theta \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta$$

$$\cot 2\theta = \frac{A-C}{B} = \frac{(17)-(9)}{(-6)} = \frac{8}{-6} = \frac{4}{-3}$$

$$A = 17 \quad B = -6 \quad C = 9$$

$$\tan 2\theta = \frac{1}{\cot 2\theta} = \frac{-3}{4}$$

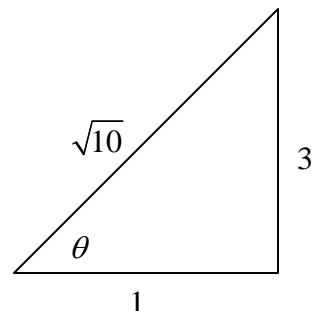
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{-3}{4}$$

$$3\tan^2\theta - 3 = 8\tan\theta$$

$$3\tan^2\theta - 8\tan\theta - 3 = 0$$

$$(3\tan\theta + 1)(\tan\theta - 3) = 0$$

$$\tan\theta = \frac{-1}{3} \quad \tan\theta = 3$$



We only have to pick one of the 2 values of $\tan\theta$ given above. The positive value is counterclockwise rotation and negative is clockwise rotation. For simplicity, let's pick the positive. Therefore, using the triangle:

$$\cos\theta = \frac{1}{\sqrt{10}} \quad \sin\theta = \frac{3}{\sqrt{10}}$$

$$x = \hat{x}\cos\theta - \hat{y}\sin\theta = \frac{1}{\sqrt{10}}\hat{x} - \frac{3}{\sqrt{10}}\hat{y} \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta = \frac{3}{\sqrt{10}}\hat{x} + \frac{1}{\sqrt{10}}\hat{y}$$

$$17x^2 - 6xy + 9y^2 = 0$$

$$17\left(\frac{1}{\sqrt{10}}\hat{x} - \frac{3}{\sqrt{10}}\hat{y}\right)^2 - 6\left(\frac{1}{\sqrt{10}}\hat{x} - \frac{3}{\sqrt{10}}\hat{y}\right)\left(\frac{3}{\sqrt{10}}\hat{x} + \frac{1}{\sqrt{10}}\hat{y}\right) + 9\left(\frac{3}{\sqrt{10}}\hat{x} + \frac{1}{\sqrt{10}}\hat{y}\right)^2 = 0$$

$$17\left(\frac{1}{10}\hat{x}^2 - \frac{3}{5}\hat{x}\hat{y} + \frac{9}{10}\hat{y}^2\right) - 6\left(\frac{3}{10}\hat{x}^2 - \frac{4}{5}\hat{x}\hat{y} - \frac{3}{10}\hat{y}^2\right) + 9\left(\frac{9}{10}\hat{x}^2 + \frac{3}{5}\hat{x}\hat{y} + \frac{1}{10}\hat{y}^2\right) = 0$$

$$\frac{17}{10}\hat{x}^2 - \frac{51}{5}\hat{x}\hat{y} + \frac{153}{10}\hat{y}^2 - \frac{18}{10}\hat{x}^2 + \frac{24}{5}\hat{x}\hat{y} + \frac{18}{10}\hat{y}^2 + \frac{81}{10}\hat{x}^2 + \frac{27}{5}\hat{x}\hat{y} + \frac{9}{10}\hat{y}^2 = 0$$

$$8\hat{x}^2 + 18\hat{y}^2 = 0$$

Since this equation is equal to zero, this is a point of center at origin.

9) $4x^2 - 4xy + 7y^2 = 24 \quad x = \hat{x}\cos\theta - \hat{y}\sin\theta \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta$

$$A = 4 \quad B = -4 \quad C = 7 \quad \cot 2\theta = \frac{A-C}{B} = \frac{(4)-(-7)}{(-4)} = \frac{-3}{-4} = \frac{3}{4}$$

$$\tan 2\theta = \frac{1}{\cot 2\theta} = \frac{4}{3}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{3}$$

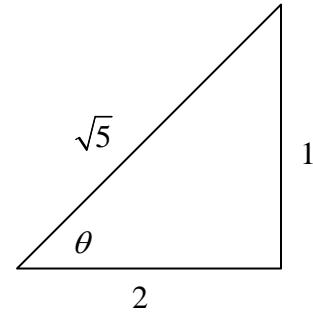
$$4 - 4\tan^2\theta = 6\tan\theta$$

$$0 = 4\tan^2\theta + 6\tan\theta - 4$$

$$0 = 2(2\tan^2\theta + 3\tan\theta - 2)$$

$$0 = 2(\tan\theta + 2)(2\tan\theta - 1)$$

$$\tan\theta = -2 \quad \tan\theta = \frac{1}{2}$$



We only have to pick one of the 2 values of $\tan\theta$ given above. The positive value is counterclockwise rotation and negative is clockwise rotation. For simplicity, let's pick the positive. Therefore, using the triangle:

$$\cos\theta = \frac{2}{\sqrt{5}} \quad \sin\theta = \frac{1}{\sqrt{5}}$$

$$x = \hat{x}\cos\theta - \hat{y}\sin\theta = \frac{2}{\sqrt{5}}\hat{x} - \frac{1}{\sqrt{5}}\hat{y} \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta = \frac{1}{\sqrt{5}}\hat{x} + \frac{2}{\sqrt{5}}\hat{y}$$

$$4x^2 - 4xy + 7y^2 = 24$$

$$4\left(\frac{2}{\sqrt{5}}\hat{x} - \frac{1}{\sqrt{5}}\hat{y}\right)^2 - 4\left(\frac{2}{\sqrt{5}}\hat{x} - \frac{1}{\sqrt{5}}\hat{y}\right)\left(\frac{1}{\sqrt{5}}\hat{x} + \frac{2}{\sqrt{5}}\hat{y}\right) + 7\left(\frac{1}{\sqrt{5}}\hat{x} + \frac{2}{\sqrt{5}}\hat{y}\right)^2 = 24$$

$$4\left(\frac{4}{5}\hat{x}^2 - \frac{4}{5}\hat{x}\hat{y} + \frac{1}{5}\hat{y}^2\right) - 4\left(\frac{2}{5}\hat{x}^2 + \frac{3}{5}\hat{x}\hat{y} - \frac{2}{5}\hat{y}^2\right) + 7\left(\frac{1}{5}\hat{x}^2 + \frac{4}{5}\hat{x}\hat{y} + \frac{4}{5}\hat{y}^2\right) = 24$$

$$\frac{16}{5}\hat{x}^2 - \frac{16}{5}\hat{x}\hat{y} + \frac{4}{5}\hat{y}^2 - \frac{8}{5}\hat{x}^2 - \frac{12}{5}\hat{x}\hat{y} + \frac{8}{5}\hat{y}^2 + \frac{7}{5}\hat{x}^2 + \frac{28}{5}\hat{x}\hat{y} + \frac{28}{5}\hat{y}^2 = 24$$

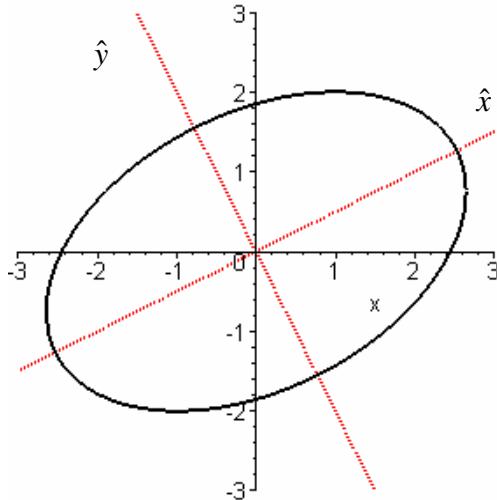
$$3\hat{x}^2 + 8\hat{y}^2 = 24$$

$$\frac{\hat{x}^2}{8} + \frac{\hat{y}^2}{3} = 1$$

an ellipse.

All values here are in $\hat{x}\hat{y}$ -axis:

Center: $(0,0)$ Vertices: $(0 \pm 2\sqrt{2}, 0)$
 length of Major axis: $2a = 2(2\sqrt{2}) = 4\sqrt{2}$
 length of minor axis: $2b = 2(\sqrt{3}) = 2\sqrt{3}$

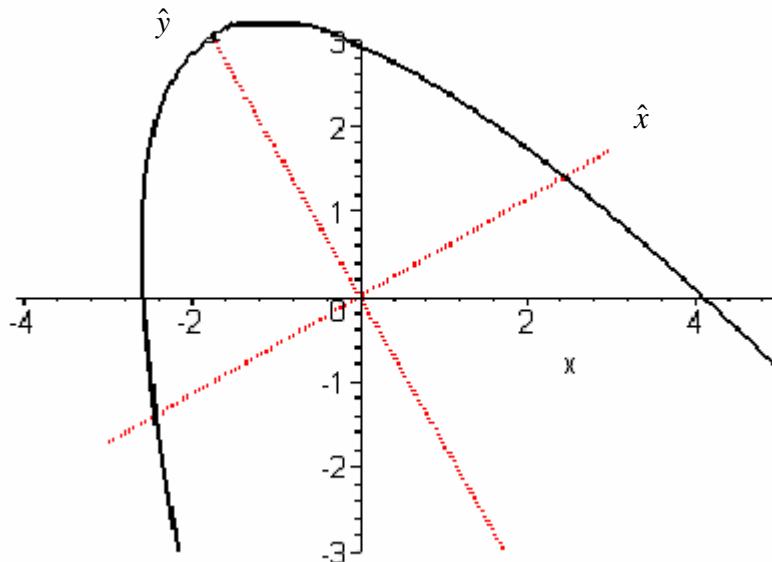


$$\begin{aligned}
 11) \quad & 6x^2 + 4\sqrt{3}xy + 2y^2 - 9x + 9\sqrt{3}y - 63 = 0 \quad x = \hat{x}\cos\theta - \hat{y}\sin\theta \quad y = \hat{x}\sin\theta + \hat{y}\cos\theta \\
 & \cot 2\theta = \frac{A-C}{B} = \frac{(6)-(2)}{(4\sqrt{3})} = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \\
 A = 6 \quad B = 4\sqrt{3} \quad C = 2 \quad & 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6} \\
 & \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
 & x = \hat{x}\cos\left(\frac{\pi}{6}\right) - \hat{y}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y} \quad y = \hat{x}\sin\left(\frac{\pi}{6}\right) + \hat{y}\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y} \\
 & 6x^2 + 4\sqrt{3}xy + 2y^2 - 9x + 9\sqrt{3}y - 63 = 0 \\
 & 6\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)^2 + 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) + 2\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)^2 - 9\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right) + 9\sqrt{3}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) - 63 = 0 \\
 & \frac{9}{2}\hat{x}^2 - 3\sqrt{3}\hat{x}\hat{y} + \frac{3}{2}\hat{y}^2 + 3\hat{x}^2 + 2\sqrt{3}\hat{x}\hat{y} - 3\hat{y}^2 + \frac{1}{2}\hat{x}^2 + \sqrt{3}\hat{x}\hat{y} + \frac{3}{2}\hat{y}^2 + 18\hat{y} - 63 = 0 \\
 & 8\hat{x}^2 + 18\hat{y} - 63 = 0 \\
 & 6\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)^2 = 6\left(\frac{3}{4}\hat{x}^2 - \frac{\sqrt{3}}{2}\hat{x}\hat{y} + \frac{1}{4}\hat{y}^2\right) = \frac{9}{2}\hat{x}^2 - 3\sqrt{3}\hat{x}\hat{y} + \frac{3}{2}\hat{y}^2 \\
 & 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right)\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) = 4\sqrt{3}\left(\frac{\sqrt{3}}{4}\hat{x}^2 + \frac{1}{2}\hat{x}\hat{y} - \frac{\sqrt{3}}{4}\hat{y}^2\right) = 3\hat{x}^2 + 2\sqrt{3}\hat{x}\hat{y} - 3\hat{y}^2 \\
 & 2\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right)^2 = 2\left(\frac{1}{4}\hat{x}^2 + \frac{\sqrt{3}}{2}\hat{x}\hat{y} + \frac{3}{4}\hat{y}^2\right) = \frac{1}{2}\hat{x}^2 + \sqrt{3}\hat{x}\hat{y} + \frac{3}{2}\hat{y}^2 \\
 & -9\left(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\right) + 9\sqrt{3}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) = \frac{-9\sqrt{3}}{2}\hat{x} + \frac{9}{2}\hat{y} + \frac{9\sqrt{3}}{2}\hat{x} + \frac{27}{2}\hat{y} = 18\hat{y}
 \end{aligned}$$

$$\begin{aligned} 8\hat{x}^2 + 18\hat{y} - 63 = 0 &\Rightarrow \hat{x}^2 = \frac{-9}{4}\hat{y} + \frac{63}{8} \Rightarrow \hat{x}^2 = \frac{-9}{4}\left(\hat{y} - \frac{7}{2}\right) \Rightarrow \hat{x}^2 = 4\left(\frac{-9}{16}\right)\left(\hat{y} - \frac{7}{2}\right) \\ 8\hat{x}^2 = -18\hat{y} + 63 \end{aligned}$$

This is an equation of a parabola:

$$\text{Vertex: } (0, \frac{7}{2}) \quad p = \frac{-9}{16} \quad \text{Focus: } (0, \frac{7}{2} - \frac{9}{16}) \quad \text{Directrix: } \hat{y} = \frac{7}{2} - \left(\frac{-9}{16}\right) = \frac{37}{16}$$



- 13) Find an equation of the parabola with axis $y = x$ passing through the points $(1, 0)$, $(0, 1)$, $(1, 1)$ (a) in $\hat{x}\hat{y}$ -coordinates, and (b) xy -coordinates.

- (a) Since the axis is $y = x$, the rotation angle is $\theta = 45^\circ = \frac{\pi}{4}$.

Now use the conversion formulas: $\hat{x} = x \cos \theta + y \sin \theta$ $\hat{y} = -x \sin \theta + y \cos \theta$

First, the conversion formulas must be used to find out what are the coordinates of the points given:

	$x = 1$	$y = 0$		$x = 0$	$y = 1$		$x = 1$	$y = 1$
\hat{x}	$\frac{\sqrt{2}}{2}$		\hat{x}	$\frac{\sqrt{2}}{2}$		\hat{x}		$\sqrt{2}$
\hat{y}	$\frac{-\sqrt{2}}{2}$		\hat{y}	$\frac{\sqrt{2}}{2}$		\hat{y}		0
$(1, 0) \Rightarrow \left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$			$(0, 1) \Rightarrow \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$			$(1, 1) \Rightarrow (\sqrt{2}, 0)$		

By observation of these 3 points in $\hat{x}\hat{y}$ -coordinates, this parabola has opens to the left, and its vertex is $(\sqrt{2}, 0)$.

Now let's use the equation of parabola that opens sideways in general form with vertex (h, k) :

$$(y - k)^2 = 4p(x - h)$$

Modifying to fit $\hat{x}\hat{y}$ -coordinates we get: $(\hat{y} - k)^2 = 4p(\hat{x} - h)$

After using the vertex of $(\sqrt{2}, 0)$, the equation of the parabola is $(\hat{y} - 0)^2 = 4p(\hat{x} - \sqrt{2}) \Rightarrow (\hat{y})^2 = 4p(\hat{x} - \sqrt{2})$

To find the value of p , use one of the points on the graph; say, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ of the equation obtained above:

$$\left(\frac{\sqrt{2}}{2}\right)^2 = 4p\left(\frac{\sqrt{2}}{2} - \sqrt{2}\right) \Rightarrow p = \frac{-\sqrt{2}}{8}$$

Therefore, the equation of the parabola in $\hat{x}\hat{y}$ -coordinates is:

$$(\hat{y})^2 = 4\left(\frac{-\sqrt{2}}{8}\right)(\hat{x} - \sqrt{2}) \Rightarrow (\hat{y})^2 = \frac{-\sqrt{2}}{2}(\hat{x} - \sqrt{2})$$

(b) To find the equation of parabola in xy -coordinates, substitute $\hat{x} = x \cos \theta + y \sin \theta$ $\hat{y} = -x \sin \theta + y \cos \theta$ in the equation obtained in part (a).

$$\hat{x} = x \cos \theta + y \sin \theta = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = \frac{\sqrt{2}}{2}(x + y) \quad \hat{y} = -x \sin \theta + y \cos \theta = \frac{-\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = \frac{\sqrt{2}}{2}(-x + y)$$

$$\begin{aligned} \left(\frac{\sqrt{2}}{2}(-x + y)\right)^2 &= \frac{-\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}(x + y) - \sqrt{2}\right) & (-x + y)^2 &= -(x + y) + 2 \\ \frac{1}{2}(-x + y)^2 &= \frac{-1}{2}(x + y) + 1 & x^2 - 2xy + y^2 &= -x - y + 2 \end{aligned}$$

Therefore the equation of the parabola in xy -coordinates is:

$$x^2 - 2xy + y^2 + x + y = 2$$