

**Definition:** If  $x$  is a real number,  $m$  is a positive odd integer, and  $n$  is a positive even integer, then

$$\text{Positive } 2^{\text{nd}} \text{ root (square root) of } x, \sqrt{x}, \text{ is such that } (\sqrt{x})^2 = x \quad x \geq 0$$

$$\text{3}^{\text{rd}} \text{ root (cube root) of } x, \sqrt[3]{x}, \text{ is such that } (\sqrt[3]{x})^3 = x$$

$$\text{Positive } 4^{\text{th}} \text{ root of } x, \sqrt[4]{x}, \text{ is such that } (\sqrt[4]{x})^4 = x \quad x \geq 0$$

$$\text{5}^{\text{th}} \text{ root of } x, \sqrt[5]{x}, \text{ is such that } (\sqrt[5]{x})^5 = x$$

⋮

$$\text{m}^{\text{th}} \text{ root of } x, \sqrt[m]{x}, \text{ is such that } (\sqrt[m]{x})^m = x \quad (m \text{ is odd integer})$$

$$\text{Positive } n^{\text{th}} \text{ root of } x, \sqrt[n]{x}, \text{ is such that } (\sqrt[n]{x})^n = x \quad x \geq 0 \quad (n \text{ is positive even integer})$$

**Definition:** If  $x$  is a real number and  $n$  is a positive integer greater than 1, then

$$x^{\frac{1}{n}} = x^{\frac{1}{n}} = \sqrt[n]{x} \quad (x \geq 0 \text{ when } n \text{ is even})$$

In words: The quantity  $x^{\frac{1}{n}} = \sqrt[n]{x}$  is the  $n^{\text{th}}$  root of  $x$ .

### Properties of Exponents

If  $a$  and  $b$  are real numbers and  $r$  and  $s$  are rational numbers, and  $a$  and  $b$  are nonnegative whenever  $r$  and  $s$  indicate even roots, then

- |                                |  |
|--------------------------------|--|
| 1. $a^r \cdot a^s = a^{(r+s)}$ | 4. $a^{-r} = \frac{1}{a^r} \quad (a \neq 0)$                       |
| 2. $(a^r)^s = a^{rs}$          | 5. $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \quad (b \neq 0)$ |
| 3. $(ab)^r = a^r b^r$          | 6. $\frac{a^r}{a^s} = a^{(r-s)} \quad (a \neq 0)$                  |

### Theorem 5.1:

If  $a$  is a nonnegative real number,  $m$  is an integer, and  $n$  is a positive integer, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$$

Sometimes it is easier to transform the equation on theorem 5.1 as the one shown below:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Additional examples

2)  $-\sqrt{144}$

$$-\sqrt{144} = -\sqrt{(12)^2} = -12$$

4)  $\sqrt{-49}$

Not possible because it is not a real number.

6)  $\sqrt{49}$

$$\sqrt{49} = \sqrt{(7)^2} = 7$$

8)  $-\sqrt[3]{27}$

$$-\sqrt[3]{27} = -\sqrt[3]{(3)^3} = -3$$

10)  $-\sqrt[4]{16}$

$$-\sqrt[4]{16} = -\sqrt[4]{(2)^2} = -2$$

12)  $-\sqrt[4]{-16}$

Not possible because it is not a real number.

14)  $\sqrt{0.81}$

$$\sqrt{0.81} = \sqrt{(0.9)^2} = 0.9$$

16)  $\sqrt[3]{0.125}$

$$\sqrt[3]{0.125} = \sqrt[3]{(0.5)^3} = 0.5$$

18)  $\sqrt{49a^{10}}$

$$\sqrt{49a^{10}} = \sqrt{(7a^5)^2} = 7a^5$$

20)  $\sqrt[3]{8a^{15}}$

$$\sqrt[3]{8a^{15}} = \sqrt[3]{(2a^5)^3} = 2a^5$$

22)  $\sqrt[3]{x^6y^3}$

$$\sqrt[3]{x^6y^3} = \sqrt[3]{(x^2y^1)^3} = x^2y^1 = x^2y$$

24)  $\sqrt[5]{32x^5y^{10}}$

$$\sqrt[5]{32x^5y^{10}} = \sqrt[5]{(2x^1y^2)^5} = 2x^1y^2 = 2xy^2$$

26)  $\sqrt[4]{81a^{24}b^8}$

$$\sqrt[4]{81a^{24}b^8} = \sqrt[4]{(3a^6b^2)^4} = 3a^6b^2$$

28)  $49^{\frac{1}{2}}$

$$49^{\frac{1}{2}} = \sqrt{49} = 7$$

30)  $-16^{\frac{1}{2}}$   
 $-16^{\frac{1}{2}} = -\sqrt{16} = -4$

32)  $-8^{\frac{1}{3}}$   
 $-8^{\frac{1}{3}} = -\sqrt[3]{8} = -2$

34)  $-27^{\frac{1}{3}}$   
 $-27^{\frac{1}{3}} = -\sqrt[3]{27} = -3$

36)  $81^{\frac{1}{4}}$   
 $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

38)  $\left(\frac{9}{16}\right)^{\frac{1}{2}}$   
 $\left(\frac{9}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$

40)  $\left(\frac{8}{27}\right)^{\frac{1}{3}}$   
 $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$

42)  $8^{\frac{4}{3}}$   
 $8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = (2)^4 = 16$

44)  $9^{\frac{3}{2}}$   
 $9^{\frac{3}{2}} = (\sqrt{9})^3 = (3)^3 = 27$

46)  $81^{\frac{3}{4}}$   
 $81^{\frac{3}{4}} = (\sqrt[4]{81})^3 = (3)^3 = 27$

48)  $9^{-\frac{1}{2}}$   
 $9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$

50)  $4^{-\frac{3}{2}}$

$$4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{(2)^3} = \frac{1}{8}$$

52)  $\left(\frac{16}{49}\right)^{-\frac{1}{2}}$

$$\left(\frac{16}{49}\right)^{-\frac{1}{2}} = \left(\frac{49}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{49}{16}} = \frac{\sqrt{49}}{\sqrt{16}} = \frac{7}{4}$$

54)  $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

$$\left(\frac{27}{8}\right)^{-\frac{2}{3}} = \left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{8}{27}}\right)^2 = \left(\frac{\sqrt[3]{8}}{\sqrt[3]{27}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{(2)^2}{(3)^2} = \frac{4}{9}$$

56)  $25^{\frac{1}{2}} + 100^{\frac{1}{2}}$

$$25^{\frac{1}{2}} + 100^{\frac{1}{2}} = \sqrt{25} + \sqrt{100} = 5 + 10 = 15$$

58)  $49^{-\frac{1}{2}} + 25^{-\frac{1}{2}}$

$$LCD = (7)(5) = 35$$

$$49^{-\frac{1}{2}} + 25^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}} + \frac{1}{25^{\frac{1}{2}}} = \frac{1}{\sqrt{49}} + \frac{1}{\sqrt{25}} = \frac{1}{7} + \frac{1}{5} = \frac{1(5) + 1(7)}{35} = \frac{5+7}{35} = \frac{12}{35}$$

60)  $x^{\frac{3}{4}} \cdot x^{\frac{5}{4}}$

$$x^{\frac{3}{4}} \cdot x^{\frac{5}{4}} = x^{\left(\frac{3}{4} + \frac{5}{4}\right)} = x^{\left(\frac{8}{4}\right)} = x^2$$

62)  $\left(a^{\frac{2}{3}}\right)^{\frac{3}{4}}$

$$\left(a^{\frac{2}{3}}\right)^{\frac{3}{4}} = a^{\left(\frac{2}{3} \cdot \frac{3}{4}\right)} = a^{\left(\frac{(2)(3)}{(3)(4)}\right)} = a^{\left(\frac{1}{2}\right)} = a^{\frac{1}{2}} = \sqrt{a}$$

64)  $\frac{x^{\frac{2}{7}}}{x^{\frac{5}{7}}}$

$$\frac{x^{\frac{2}{7}}}{x^{\frac{5}{7}}} = x^{\left(\frac{2}{7} - \frac{5}{7}\right)} = x^{\left(\frac{-3}{7}\right)} = x^{-\frac{3}{7}} = \frac{1}{x^{\frac{3}{7}}} = \frac{1}{(\sqrt[7]{x})^3}$$

$$66) \quad \frac{x^{\frac{7}{8}}}{x^{\frac{8}{7}}}$$

$$LCD = (8)(7) = 56$$

$$\frac{x^{\frac{7}{8}}}{x^{\frac{8}{7}}} = x^{\left(\frac{7}{8} - \frac{8}{7}\right)} = x^{\left(\frac{7(7)-8(8)}{56}\right)} = x^{\left(\frac{49-64}{56}\right)} = x^{\left(\frac{-15}{56}\right)} = x^{-\frac{15}{56}} = \frac{1}{x^{\frac{15}{56}}} = \frac{1}{\left(\sqrt[56]{x}\right)^{15}}$$

$$68) \quad \left(x^{\frac{3}{4}}y^{\frac{1}{8}}z^{\frac{5}{6}}\right)^{\frac{4}{5}}$$

$$\begin{aligned} \left(x^{\frac{3}{4}}y^{\frac{1}{8}}z^{\frac{5}{6}}\right)^{\frac{4}{5}} &= x^{\left(\frac{(3)(4)}{4}(5)\right)}y^{\left(\frac{(1)(4)}{8}(5)\right)}z^{\left(\frac{(5)(4)}{6}(5)\right)} = x^{\left(\frac{(3)(4)}{(4)(5)}\right)}y^{\left(\frac{(1)(4)}{(8)(5)}\right)}z^{\left(\frac{(5)(4)}{(6)(5)}\right)} = x^{\left(\frac{(3)(1)}{(1)(5)}\right)}y^{\left(\frac{(1)(1)}{(2)(5)}\right)}z^{\left(\frac{(1)(2)}{(3)(1)}\right)} = x^{\left(\frac{3}{5}\right)}y^{\left(\frac{1}{10}\right)}z^{\left(\frac{2}{3}\right)} \\ &= x^{\frac{3}{5}}y^{\frac{1}{8}}z^{\frac{2}{3}} = \left(\sqrt[5]{x}\right)^3 \left(\sqrt[8]{y}\right) \left(\sqrt[3]{z}\right)^2 \end{aligned}$$

$$70) \quad \frac{a^{\frac{1}{3}}b^4}{a^{\frac{3}{5}}b^{\frac{1}{3}}}$$

$$LCD_a = (3)(5) = 15 \quad LCD_b = (1)(3) = 3$$

$$\frac{a^{\frac{1}{3}}b^4}{a^{\frac{3}{5}}b^{\frac{1}{3}}} = a^{\left(\frac{1}{3}-\frac{3}{5}\right)}b^{\left(\frac{4}{1}-\frac{1}{3}\right)} = a^{\left(\frac{1(5)-(3)(3)}{15}\right)}b^{\left(\frac{4(3)-1(1)}{3}\right)} = a^{\left(\frac{5-9}{15}\right)}b^{\left(\frac{12-1}{3}\right)} = a^{\left(\frac{-4}{15}\right)}b^{\left(\frac{11}{3}\right)} = a^{-\frac{4}{5}}b^{\frac{11}{3}} = \frac{b^{\frac{11}{3}}}{a^{\frac{4}{5}}} = \frac{\left(\sqrt[3]{b}\right)^{11}}{\left(\sqrt[5]{a}\right)^5}$$

$$72) \quad \frac{\left(y^{\frac{5}{4}}\right)^{\frac{2}{5}}}{\left(y^{\frac{1}{4}}\right)^{\frac{4}{3}}}$$

$$LCD = (2)(3) = 6$$

$$\frac{\left(y^{\frac{5}{4}}\right)^{\frac{2}{5}}}{\left(y^{\frac{1}{4}}\right)^{\frac{4}{3}}} = \frac{y^{\left(\frac{(5)(2)}{(4)(5)}\right)}}{y^{\left(\frac{(1)(4)}{(4)(3)}\right)}} = \frac{y^{\left(\frac{(5)(2)}{(4)(5)}\right)}}{y^{\left(\frac{(1)(4)}{(4)(3)}\right)}} = \frac{y^{\left(\frac{1}{2}\right)}}{y^{\left(\frac{1}{3}\right)}} = y^{\left(\frac{1-\frac{1}{2}}{6}\right)} = y^{\left(\frac{1(3)-1(2)}{6}\right)} = y^{\left(\frac{3-2}{6}\right)} = y^{\left(\frac{1}{6}\right)} = y^{\frac{1}{6}} = \sqrt[6]{y}$$

$$74) \quad \left(\frac{a^{-\frac{1}{5}}}{b^{\frac{1}{3}}}\right)^{15}$$

$$\left(\frac{a^{-\frac{1}{5}}}{b^{\frac{1}{3}}}\right)^{15} = \frac{\left(a^{-\frac{1}{5}}\right)^{15}}{\left(b^{\frac{1}{3}}\right)^{15}} = \frac{a^{\left(\frac{(-1)(15)}{5}\right)}}{b^{\left(\frac{(1)(15)}{3}\right)}} = \frac{a^{\left(\frac{(-1)(15)}{5}\right)}}{b^{\left(\frac{(1)(15)}{3}\right)}} = \frac{a^{-3}}{b^5} = \frac{1}{a^3 b^5}$$

$$76) \quad \frac{\left(r^{-5}s^{\frac{1}{2}}\right)^4}{r^{12}s^{\frac{5}{2}}}$$

$$LCD_s = 2$$

$$\begin{aligned} \frac{\left(r^{-5}s^{\frac{1}{2}}\right)^4}{r^{12}s^{\frac{5}{2}}} &= \frac{\left(r^{-5}\right)^4\left(s^{\frac{1}{2}}\right)^4}{r^{12}s^{\frac{5}{2}}} = \frac{\left(r^{(-5)(4)}\right)\left(s^{\left(\frac{1}{2}\right)\left(\frac{4}{1}\right)}\right)}{r^{12}s^{\frac{5}{2}}} = \frac{\left(r^{-20}\right)\left(s^2\right)}{r^{12}s^{\frac{5}{2}}} = r^{(-20)-12}s^{\left(2-\frac{5}{2}\right)} = r^{(-20)-12}s^{\left(\frac{2}{1}-\frac{5}{2}\right)} \\ &= r^{(-20)-12}s^{\left(\frac{2(2)-5}{2}\right)} = r^{(-32)}s^{\left(\frac{4-5}{2}\right)} = r^{(-32)}s^{\left(\frac{-1}{2}\right)} = r^{-32}s^{-\frac{1}{2}} = \frac{1}{r^{32}s^{\frac{1}{2}}} = \frac{1}{r^{32}\sqrt{s}} \end{aligned}$$

$$78) \quad \frac{(27a^3b^6)^{\frac{1}{3}}}{(81a^8b^{-4})^{\frac{1}{4}}}$$

$$\begin{aligned} \frac{(27a^3b^6)^{\frac{1}{3}}}{(81a^8b^{-4})^{\frac{1}{4}}} &= \frac{(27)^{\frac{1}{3}}(a^3)^{\frac{1}{3}}(b^6)^{\frac{1}{3}}}{(81)^{\frac{1}{4}}(a^8)^{\frac{1}{4}}(b^{-4})^{\frac{1}{4}}} = \frac{\left(\sqrt[3]{27}\right)\left(a^{\left(\frac{3}{1}\right)\left(\frac{1}{3}\right)}\right)\left(b^{\left(\frac{6}{1}\right)\left(\frac{1}{3}\right)}\right)}{\left(\sqrt[3]{81}\right)\left(a^{\left(\frac{8}{1}\right)\left(\frac{1}{4}\right)}\right)\left(b^{\left(\frac{-4}{1}\right)\left(\frac{1}{4}\right)}\right)} = \frac{(3)\left(a^{(1)}\right)\left(b^{(2)}\right)}{(3)\left(a^{(2)}\right)\left(b^{(-1)}\right)} = a^{(1-2)}b^{(2-(-1))} \\ &= a^{-1}b^3 = \frac{b^3}{a} \end{aligned}$$