(1)(18 points) Solve the initial value problem: $y'' - 2y' - 3y = 12e^{3t} + 9$ with $y(0) = 3$ and $y'(0) = 0$.

The characteristic polynomial for the homogeneous equation is $r^2 - 2r - 3 = (r-3)(r+1)$ with roots $-1, 3$. So $y_h = C_1 e^{-t} + C_2 e^{3t}$.

The associated root for $12e^{3t}$ is 3 and the associated root for 9 is 0 so the first version of the test function is $Y_p^1 = Ae^{3t} + B$. Since 3 is a root of the homogeneous the test function is $Y_p = Ate^{3t} \quad + B$. So

$$\begin{array}{rrcllll} -3\times & Y_p & = & Ate^{3t} + & & & +B \\ -2\times & Y_p' & = & 3Ate^{3t} & + & Ae^{3t} & \\ 1\times & Y_p'' & = & 9Ate^{3t} & + & 6Ae^{3t} & \end{array}$$

$$12e^{3t} + 9 \ = \ 0 \quad + \quad 4Ae^{3t} \ - \ 3B$$

So $A = 3, B = -3$. The general solution is $y_g = C_1 e^t + C_2 e^{3t} + 3te^{3t} - 3$.

$$\begin{aligned} y_g &= C_1 e^{-t} \ + \ C_2 e^{3t} \ + \ 3te^{3t} \ - \ 3 \\ y_g' &= -C_1 e^t + (3C_2 + 3)e^{3t} + 15te^{3t}. \end{aligned}$$

So $3 = C_1 + C_2 - 3$ and $0 = -C_1 + 3C_2 + 3$. So $C_1 = 21/4, C_2 = 3/4$.

$$y = (21/4)e^{-t} + (3/4)e^{3t} + 3te^{3t} - 3.$$

(2)(20 points) Consider the seventh order differential equation $y^{(7)} - 2y^{(4)} + y' = g(t)$.

(a) Compute the general solution of the homogeneous equation with $g(t) = 0$.

(b) For each of the following forcing functions $g(t)$, write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

$$(i) \quad g(t) = t^2 + 2. \qquad (ii) \quad g(t) = 2te^{-t/2}\sin(\sqrt{3}t/2) + e^{-t/2}$$

(a) The characteristic polynomial is $r^7 - 2r^4 + r = r(r^3 - 1)^2 = r(r-1)(r^2 + r = 1)$. The roots are $0, 1, 1, \frac{-1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}, \frac{-1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}$.

$$\begin{gathered} y_h = C_1 + C_2 e^t + C_3 te^t + C_4 e^{-t/2}\cos(\sqrt{3}t/2) \\ +C_5 te^{-t/2}\cos(\sqrt{3}t/2) + C_6 e^{-t/2}\sin(\sqrt{3}t/2) + C_6 te^{-t/2}\sin(\sqrt{3}t/2). \end{gathered}$$

(b) (i) Associated roots are both 0. So $Y_p^1 = At^2 + Bt + C$ and as 0 is a root of the homogeneous $Y_p = t[At^2 + Bt + C]$.

(ii) Associated roots for $2te^{-t/2}\sin(\sqrt{3}t/2)$ are $\frac{-1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}$, and for $e^{-t/2}$ is 1/2

So $Y_p^1 = [(At+B)e^{-t/2}\cos(\sqrt{3}t/2) + (Ct+D)e^{-t/2}\sin(\sqrt{3}t/2)] \ + \ [Ee^{-t/2}]$.

Because $\frac{-1}{2} \pm \mathbf{i}\frac{\sqrt{3}}{2}$ are roots of the homogeneous, repeated twice we have

$$Y_p = t^2[(At+B)e^{-t/2}\cos(\sqrt{3}t/2) + (Ct+D)e^{-t/2}\sin(\sqrt{3}t/2)] \ + \ [Ee^{-t/2}].$$

(3)(16 points) Compute the general solution of $y'' + 2y' + y = t^{-1}e^{-t}$.

The characteristic polynomial for the homogeneous equation is $r^2 + 2r + 1 = (r+1)(r+1)$ with roots $-1, -1$. The solution of the homogeneous is $y_h = C_1e^{-t} + C_2te^{-t}$.

We look for a particular solution $y_p = u_1e^{-t} + u_2te^{-t}$.

The equations are:

$$\begin{aligned} u_1'e^{-t} + u_2'te^{-t} &= 0, \\ u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) &= t^{-1}e^{-t} \end{aligned}$$

The Wronskian is e^{-2t}.

$u_1' = -e^{-2t}/e^{-2t} = -1, \quad u_2' = t^{-1}e^{-2t}/e^{-2t} = t^{-1}$.

So $u_1 = -t, u_2 = \ln t$ and $y_p = -te^{-t} + te^{-t}\ln t$.

$$y_g = C_1e^{-t} + C_2te^{-t} - te^{-t} + te^{-t}\ln t \ \text{ or } \ C_1e^{-t} + C_2te^{-t} + te^{-t}\ln t.$$

(4)(14 points) For the differential equation, $(x-1)y'' - xy' + y = 0$, $y_1 = e^x$ is a solution. Use the method of Reduction of Order to compute a second solution y_2 which is independent of the first one.

Look for $y_2 = ue^x$ So that

$$\begin{aligned} 1 \times \quad y_2 &= ue^x \\ -x \times \quad y_2' &= ue^x \ + \ ue^x \\ (x-1) \times \quad y_2'' &= ue^x \ + \ 2u'e^x \ + \ u''e^x \end{aligned}$$

$$0 \ = \ 0 \ + \ (x-2)u'e^x \ + \ (x-1)u''e^x.$$

With $v = u'$, $\frac{dv}{v} = -\frac{(x-2)dx}{(x-1)} = (-1 + \frac{1}{x-1})dx$

Therefore, $\ln v = -x + \ln(x-1)$ and so $\frac{du}{dx} = (x-1)e^{-x}$ and $u = -xe^{-x}$

$$y_2 = ue^x = -x.$$

(5)(14 points) A hanging spring is stretched 6 inches (= .5 feet) by a weight of 64 pounds.

(a) Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight is pulled down an additional foot and is then released and is subjected to an external force of $6\cos(\omega t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when $\omega = 4$.

(c) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when ω is the *resonance frequency* that is, the natural frequency of the spring.

(a) $mg = w = 64, \quad m = 2\ \Delta L = 1/2$ and so $w = k\Delta L$ implies $k = 128$. The equation is

$$2y'' + 0y' + 128y = 6\cos(\omega t) \quad \text{or} \quad y'' + 64y = 3\cos(\omega t)$$

with initial conditions $y(0) = -1, y'(0) = 0$.

The characteristic polynomial for the homogeneous equation is $r^2 + 64$ with roots $\pm 8\mathbf{i}$. Thus, the natural frequency of the spring is 8.

(b) $Y_p = A\cos(4t) + B\sin(4t)$.

(c) $Y_p = t[A\cos(8t) + B\sin(8t)]$.

(6) (18 points) For the differential equation:

$$(1 - x^2)y'' + (1 + x^2)y' - 2y \quad = \quad 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$. Use it to compute the first four nonzero terms of the series for the solution with $y(0) = -3$ and $y'(0) = -2$.

$$\begin{aligned}
1y'' &= \sum n(n-1)a_n x^{n-2} &&= \sum (k+2)(k+1)a_{k+2}x^k, \\
-x^2y'' &= \sum n(n-1) - a_n x^n &&= \sum -k(k-1)a_k x^k, \\
1y' &= \sum n a_n x^{n-1} &&= \sum (k+1)a_{k+1}x^k, \\
x^2y' &= \sum n a_n x^{n+1} &&= \sum (k-1)a_{k-1}x^k, \\
-2y &= \sum -2a_n x^n &&= \sum -2a_k x^k
\end{aligned}$$

The recursion formula is

$$a_{k+2} \quad = \quad \frac{1}{(k+2)(k+1)}[-(k+1)a_{k+1} + (k(k-1)+2)a_k - (k-1)a_{k-1}].$$

$a_0 = -3.$
$a_1 = -2$
$k = 0 : a_2 = \frac{1}{2}[-a_1 + 2a_0 + a_{-1}] = \frac{1}{2}[2 - 6 + 0] = -2.$
$k = 1 : a_3 = \frac{1}{6}[-2a_2 + 2a_1 - 0a_0] = \frac{1}{6}[4 - 4 - 0] = 0.$
$k = 2 : a_4 = \frac{1}{12}[-3a_3 + 4a_2 - a_1] = \frac{1}{12}[0 - 8 + 2] = -\frac{1}{2}.$

$$y = -3 - 2x - 2x^2 - \frac{1}{2}x^4 + \dots.$$