Math 39100 K (21937) - Lectures 03

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Power Series

The Taylor series expansion, centered at $x = a$, for a real-valued function f of a real variable x is given by:

$$\begin{aligned} f(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots \\ &= \Sigma_{n=0}^{\infty}\, a_n(x-a)^n. \end{aligned}$$

This "Sigma notation" will be used throughout what follows. Notice that the powers of $(x - a)$ that occur are the $n = 0$ power (because $(x - a)^0 = 1$) and the $n = 1, 2, 3, \dots$ powers so that a_n is defined only for these values of n.

However, we are going to introduce an IMPORTANT CONVENTION: $a_n = 0$, whenever n is negative.
This will mean that we can omit the range of the sum and write $f(x) \quad = \quad \Sigma\, a_n(x-a)^n$.
When the center $a = 0$ the series $f(x) = \Sigma\, a_n x^n$ is called the MacLaurin series. We will concentrate mainly on this case.

Series Solutions Centered at an Ordinary Point

We are going to look at second order, linear, homogeneous equations of the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with the coefficients P, Q, R polynomials in x. The center $x = a$ is an *ordinary point* when $P(a)$ is not zero so that Q/P and R/P are defined at $x = a$.

We will look for a series solution centered at $x = 0$. Thus,

$$\begin{aligned} y &= \Sigma\, a_n x^n. \\ y' &= \Sigma\, n a_n x^{n-1}. \\ y'' &= \Sigma\, n(n-1) a_n x^{n-2}. \end{aligned}$$

We use three steps

Step 1: Write a separate series for each term of the polynomials P, Q and R.

Step 2: Shift indices so that each series is represented as powers of x^k.

Step 3: Collect terms to obtain the *recursion formula* for the a_k's with $a_0 = y(0)$ and $a_1 = y'(0)$ the arbitrary constants.

Example: $(2+3x^2)y'' + (x-5x^3)y' + (1-x^2+x^4)y = 0$. So there will be seven series.

$$
\begin{aligned}
2y'' &= \Sigma\ 2n(n-1)a_n x^{n-2} [k=n-2] = \Sigma 2(k+2)(k+1)a_{k+2}x^k. \\
3x^2y'' &= \Sigma\ 3n(n-1)a_n x^n\ [k=n] = \Sigma\ 3k(k-1)a_k x^k. \\
xy' &= \Sigma\ na_n x^n\ [k=n] = \Sigma\ ka_k x^k. \\
-5x^3y' &= \Sigma - 5na_n x^{n+2} [k=n+2] = \Sigma - 5(k-2)a_{k-2}x^k. \\
1y &= \Sigma\ a_n x^n\ [k=n] = \Sigma\ a_k x^k. \\
-x^2y &= \Sigma - a_n x^{n+2} [k=n+2] = \Sigma\ - a_{k-2}x^k. \\
x^4y &= \Sigma\ a_n x^{n+4}\ [k=n+4] = \Sigma\ a_{k-4}x^k.
\end{aligned}
$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{2(k+2)(k+1)}[-3k(k-1)a_k$$
$$-ka_k + 5(k-2)a_{k-2} - a_k + a_{k-2} - a_{k-4}] =$$
$$\frac{1}{2(k+2)(k+1)}[(-k(3k-2)-1)a_k + (5k-9)a_{k-2} - a_{k-4}].$$

Substitute:

$k = 0, \quad a_2 = \frac{1}{4}[-a_0] = -\frac{1}{4}a_0.$

$k = 1, \quad a_3 = \frac{1}{12}[-2a_1] = -\frac{1}{6}a_1.$

$k = 2, \quad a_4 = \frac{1}{24}[-9a_2 + 1a_0] = \frac{1}{24}[\frac{13}{4}a_0] = \frac{13}{96}a_0.$

$k = 3, \quad a_5 = \frac{1}{40}[-22a_3 + 6a_1] = \frac{1}{40}[\frac{29}{3}a_1] = \frac{29}{120}a_1.$

$k = 4, \quad a_6 = \frac{1}{60}[-41a_4 + 11a_2 - a_0] =$
$\frac{1}{60}[-\frac{41 \cdot 13}{96}a_0 - \frac{11}{4}a_0 - a_0] = -\frac{893}{5760}a_0.$

Example: $(1-x)y'' + (x-2x^2)y' + (1-x+x^2)y = 0. y(0) = -12, y'(0) = 12$ So there will again be seven series. Step 1: and Step 2:

$$\begin{aligned}
1y'' &= \Sigma\ 1n(n-1)a_n x^{n-2}[k=n-2] = \Sigma 1(k+2)(k+1)a_{k+2}x \\
-xy'' &= \Sigma\ -n(n-1)a_n x^{n-1}\ [k=n-1] = \Sigma\ -(k+1)(k)a_{k+1}x^k. \\
xy' &= \Sigma\ na_n x^n \quad [k=n] = \Sigma\ ka_k x^k. \\
-2x^2y' &= \Sigma - 2na_n x^{n+1}[k=n+1] = \Sigma - 2(k-1)a_{k-1}x^k. \\
1y &= \Sigma\ a_n x^n \quad [k=n] = \Sigma\ a_k x^k. \\
-xy &= \Sigma - a_n x^{n+1}[k=n+1] = \Sigma\ -a_{k-1}x^k. \\
x^2y &= \Sigma\ a_n x^{n+2} \quad [k=n+2] = \Sigma\ a_{k-2}x^k.
\end{aligned}$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(k+1)(k)a_{k+1}$$
$$-ka_k + 2(k-1)a_{k-1} - a_k + a_{k-1} - a_{k-2}] =$$
$$\frac{1}{(k+2)(k+1)}[((k+1)(k)a_{k+1} - (k+1)a_k + (2k-1)a_{k-1} - a_{k-2}].$$

Substitute: $a_0 = -12, a_1 = 12$

$k = 0, \quad a_2 = \frac{1}{2}[0 - a_0 + 0 + 0] = 6.$

$k = 1, \quad a_3 = \frac{1}{6}[2a_2 - 2a_1 + a_0 + 0] = -4.$

$k = 2, \quad a_4 = \frac{1}{12}[6a_3 - 3a_2 + 3a_1 - a_0] = \frac{1}{2}.$

$k = 3, \quad a_5 = \frac{1}{20}[12a_4 - 4a_3 + 5a_2 - a_1] = 2.$

$k = 4, \quad a_6 = \frac{1}{30}[20a_5 - 5a_4 + 7a_3 - a_2] = \frac{7}{60}$

$$y = -12 + 12x + 6x^2 - 4x^3 + \frac{1}{2}x^4 + 2x^5 + \frac{7}{60}x^6 + \dots$$

Example: $y'' + [2 + (x-1)^2]y' + (x^2-1)y = 0$ centered at $x = 1$ So we want a solution of the form $y = \Sigma a_n(x-1)^n$. Let $X = x - 1$ so that $x = X + 1$. Then $x^2 - 1 = X^2 + 2X$ and $y = \Sigma a_n X^n$ with $y'' + [2 + X^2]y' + (X^2 + 2X)y = 0$. Notice that $\frac{dy}{dX} = \frac{dy}{dx}$ because $\frac{dX}{dx} = 1$.

$$
\begin{aligned}
y'' &= \Sigma\, n(n-1)a_n X^{n-2} [k = n-2] = \Sigma (k+2)(k+1)a_{k+2}X^k. \\
2y' &= \Sigma\, 2na_nX^{n-1} \quad [k = n-1] = \Sigma\, 2(k+1)a_{k+1}X^k. \\
X^2y' &= \Sigma\, na_nX^{n+1} \quad [k = n+1] = \Sigma\, (k-1)a_{k-1}X^k. \\
X^2y &= \Sigma\, a_nX^{n+2} \quad [k = n+2] = \Sigma\, a_{k-2}X^k. \\
2Xy &= \Sigma\, 2a_nX^{n+1} \quad [k = n+1] = \Sigma\, 2a_{k-1}X^k.
\end{aligned}
$$

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}].$$

From the recursion formula
$a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get $k = 0, \quad a_2 = \frac{1}{2}[-2a_1 - 0 - 0] = -a_1$.

$$k = 1, \quad a_3 = \frac{1}{6}[-4a_2 - 2a_0 - 0] = \frac{2}{3}a_1 - \frac{1}{3}a_0.$$
$$k = 2, \quad a_4 = \frac{1}{12}[-6a_3 - 3a_1 - a_0] = -\frac{7}{12}a_1 + \frac{1}{12}a_0$$
$$k = 3, \quad a_5 = \frac{1}{20}[-8a_4 - 4a_2 - a_1] = \frac{23}{60}a_1 - \frac{1}{30}a_0.$$
$$k = 4, \quad a_6 = \frac{1}{30}[-10a_5 - 5a_3 - a_2] = -\frac{37}{180}a_1 + \frac{1}{15}a_0.$$

$a_0 = y(1)$ and $a_1 = y'(1)$.

If the initial conditions $y(1) = 18, y'(1) = -24$ are given, it is best to substitute immediately. From the recursion formula $a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get

$$\begin{aligned}
k = 0, &\quad a_2 = \tfrac{1}{2}[48 - 0 - 0] = 24. \\
k = 1, &\quad a_3 = \tfrac{1}{6}[-96 - 36 - 0] = -22. \\
k = 2, &\quad a_4 = \tfrac{1}{12}[132 + 72 - 18] = \tfrac{31}{2}. \\
k = 3, &\quad a_5 = \tfrac{1}{20}[-124 - 96 + 24] = -\tfrac{49}{5}. \\
k = 4, &\quad a_6 = \tfrac{1}{30}[98 + 110 - 24] = \tfrac{92}{15}.
\end{aligned}$$

So that

$$y = 18 - 24(x-1) + 24(x-1)^2 - 22(x-1)^3 + \frac{31}{2}(x-1)^4 - \frac{49}{5}(x-1)^5 + \frac{92}{15}(x-1)^6 + \dots .$$

When the coefficients are not polynomials we cannot use this procedure, but we can still obtain the first few terms of the series expansion, or, equivalently, the first few derivatives at the center point.

Example: Section 5.3 / 2:
$y'' + (\sin x)y' + (\cos x)y = 0; /y(0) = 0, y'(0) = 1.$

Substitute to get $y''(0) + 0 \cdot 1 + 1 \cdot 0 = 0$ and so $y''(0) = 0$.

Differentiate to get
$y^{(3)} + (\sin x)y'' + (2\cos x)y' - (\sin x)y = 0$. Substitute to get $y^{(3)}(0) + 0 + 2 + 0 = 0$ and so $y^{(3)}(0) = -2$.

Differentiate to get
$y^{(4)} + (\sin x)y^{(3)} + (3\cos x)y'' - (3\sin x)y' - (\cos x)y = 0$. So $y^{(4)}(0) + 0 + 0 + 0 + 0$ and $y^{(4)}(0) = 0$.

Euler Equations

An Euler equation is of the form $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$. Because $x = 0$ is a singular point we will only look at solutions where $x > 0$ or where $x < 0$.

For constant coefficients we looked for a test solution of the form $y = e^{rt}$. Notice that Ce^{rt} is the solution of $\frac{dy}{dt} = ry$. This is the reason e^{rt} factored out when we substituted.

However, $y = Cx^r$ is the solution of the equation $x\frac{dy}{dx} = ry$. So we will look for a test solution of the form $y = x^r$. Substituting we get

$$0 = ar(r-1)x^r + brx^r + cx^r = (ar(r-1) + br + c)x^r.$$

Since we are avoiding $x = 0$ we can divide away the x^r and are left the the quadratic equation $ar^2 + (b-a)r + c = 0$, whose two roots give us the two solutions we need. As before Case 1: two distinct real roots r_1, r_2 yield the general solution $C_1x^{r_1} + C_2x^{r_2}$.

In order to see how to deal with the other cases, we use the change of variable $x = e^t, \ln x = t$ so that $\frac{dx}{dt} = e^t = x$. From this we see that for any Q

$$\frac{dQ}{dt} = \frac{dx}{dt}\frac{dQ}{dx} = x\frac{dQ}{dx}.$$

Apply this to $Q = \frac{dy}{dt}$ to get

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}[\frac{dy}{dt}] = x\frac{d}{dx}[x\frac{dy}{dx}] \\ &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^2\frac{d^2y}{dx^2} + \frac{dy}{dt}.\end{aligned}$$

So $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ and $x\frac{dy}{dx} = \frac{dy}{dt}$.
This transforms the equation:
$0 = ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = a[\frac{d^2y}{dt^2} - \frac{dy}{dt}] + b\frac{dy}{dt} + cy$. And so to

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0.$$

So for the equation $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$, we use the test solution $y = x^r$ which in t coordinates is $y = (e^t)^r = e^{rt}$. The characteristic equation is the quadratic equation $ar(r-1) + br + c = 0$.

Case 1: Two distinct real roots r_1, r_2, leading to the fundamental pair of solutions $e^{r_1 t} = x^{r_1}$ and $e^{r_2 t} = x^{r_2}$. The general solution of the homogeneous equation is thus

$$y = C_1 x^{r_1} + C_2 x^{r_2}.$$

Case 2: A complex conjugate pair $a \pm \mathbf{i}b$, leading to the fundamental pair $e^{at}\cos bt, e^{at}\sin bt$. Since $t = \ln x$, these are $x^a\cos(b\ln x), x^a\sin(b\ln x)$. The general solution of the homogeneous equation is thus

$$y = C_1 x^a\cos(b\ln x) + C_2 x^a\sin(b\ln x).$$

Case 3: Repeated real roots r, r, leading to the fundamental pair e^{rt}, te^{rt} which in x coordinates are $x^r, \ln(x)x^r$. The general solution of the homogeneous equation is thus

$$y = C_1 x^r + C_2\ln(x)x^r = x^r[C_1 + C_2\ln(x)].$$

Example: $x^2y'' + 3xy' + 4y = 0, x > 0$. Use $y = x^r$ to get
$0 = r(r-1) + 3r + 4 = r^2 + 2r + 4$.
Roots $r = -1 \pm \mathbf{i}\sqrt{3}$ So the general solution is

$$y = C_1 x^{-1}\cos(\sqrt{3}\ln x) + C_2 x^{-1}\sin(\sqrt{3}\ln x).$$

Example: $4x^2y'' + y (= 4x^2y'' + 0y' + y) = 0, x > 0$.
The indicial equation is
$0 = 4r(r-1) + 1 = 4r^2 - 4r + 1 = (2r-1)^2$ with roots $\frac{1}{2}, \frac{1}{2}$.
The general solution is

$$y = C_1 x^{\frac{1}{2}} + C_2 x^{\frac{1}{2}}\ln(x).$$

Regular Singular Points

A second order, linear, homogeneous equation $P(x)y'' + Q(x)y' + S(x)y = 0$ has 0 as a *singular point* if $P(0) = 0$. It is a *regular singular point* if it is of the form $x^2 p(x)y'' + xq(x)y' + s(x)y = 0$ with $p(0) \neq 0$.

While we will only consider 0 as a singular point, in general, $x = a$ is a regular singular point if the equation can be written $(x-a)^2 p(x)y'' + (x-a)q(x)y' + s(x)y = 0$ and so when the limits

$$Limit_{x \to a} \frac{P(x)}{(x-a)^2}, \qquad Limit_{x \to a} \frac{Q(x)}{(x-a)}$$

exist and the first one is not zero.

If the equation is $x^2p(x)y'' + xq(x)y' + s(x)y = 0$ then the *associated Euler equation* is given by

$$x^2p(0)y'' + xq(0)y' + s(0)y = 0.$$

We will look at such equations with $p(x), q(x), s(x)$ polynomials. We must keep track of the roots of indicial equation of the associated Euler equation.

$$p(0)r(r-1) + q(0)r + s(0) = 0.$$

We will look for series solutions of the form

$$y = x^r\Sigma a_nx^n = \Sigma a_nx^{n+r}$$

with r as well as the a_n's to be determined.

$$y' = \Sigma(n+r)a_nx^{n+r-1} \quad \text{and} \quad y'' = \Sigma(n+r)(n+r-1)a_nx^{n+r-2}.$$

Example: $(2x^2+5x^3)y''+(7x-x^3)y'+(x^2-3)y=0$.
Step1: Rewrite the equation:
$x^2(2+5x)y''+x(7-x^2)y'+(-3+x^2)y=0$ with associated Euler equation: $2x^2y''+7xy'-3y=0$.
Step2: The indicial equation is
$0=2r(r-1)+7r-3=2r^2+5r-3=(2r-1)(r+3)$.
Keep track of that factoring. The roots are $\frac{1}{2},-3$.
Step3: Separate series for each term of the coefficients of the original problem.

$$\begin{aligned}
2x^2y'' &= \quad \Sigma\, 2(n+r)(n+r-1)a_nx^{n+r} \quad [k=n] \\
5x^3y'' &= \quad \Sigma\, 5(n+r)(n+r-1)a_nx^{n+r+1} \quad [k=n+1] \\
7xy' &= \quad \Sigma\, 7(n+r)a_nx^{n+r} \quad [k=n] \\
-x^3y' &= \quad \Sigma-(n+r)a_nx^{n+r+2} \quad [k=n+2] \\
-3y &= \quad \Sigma\, -3a_nx^{n+r} \quad [k=n] \\
x^2y &= \quad \Sigma a_nx^{n+r+2} \quad [k=n+2]
\end{aligned}$$

IMPORTANT: When we shift indices we just carry along the r so we end up with a series with terms x^{k+r}.

$$
\begin{aligned}
\Sigma\, 2(n+r)(n+r-1)a_n x^{n+r} &= \Sigma 2(k+r)(k+r-1)a_k x^{k+r}.\\
\Sigma 5(n+r)(n+r-1)a_n x^{n+r+1} &= \Sigma 5(k+r-1)(k+r-2)a_{k-1} x^{k+r}\\
\Sigma\, 7(n+r)a_n x^{n+r} &= \Sigma\, 7(k+r)a_k x^{k+r}.\\
\Sigma - (n+r)a_n x^{n+r+2} &= \Sigma - (k+r-2)a_{k-2} x^{k+r}.\\
\Sigma\, -3a_n x^{n+r} &= \Sigma\, -3a_k x^{k+r}.\\
\Sigma\, a_n x^{n+r+2} &= \Sigma\, a_{k-2} x^{k+r}.
\end{aligned}
$$

Step3: The sum of the coefficients is zero. Collect the a_k terms on one side of the equation

$$
\begin{aligned}
[2(k+r)(k+r-1)+7(k+r)-3]a_k &= \\
&(-5(k+r-1)(k+r-2))a_{k-1} + ((k+r-2)-1)a_{k-2}.
\end{aligned}
$$

NOTICE $[2(k+r)(k+r-1)+7(k+r)-3]$ is the left side of the indicial equation with r replaced by $k+r$.
So
$[2(k+r)(k+r-1)+7(k+r)-3] = [(2(k+r)-1)((k+r)+3)]$.

For ordinary points, the first two coefficients a_0 and a_1 are the arbitrary constants. This time the only arbitrary constant is a_0. The two different solutions instead come from the two roots

With $r = -3$, $[2(k+r)(k+r-1) + 7(k+r) - 3] = [(2(k+r)-1)((k+r)+3)] = k(2k-7)$ and the recursion formula is given by:

$$a_k = \frac{1}{k(2k-7)}[-5(k-4)(k-5)a_{k-1} + (k-6)a_{k-2}].$$

$k = 1: \quad a_1 = -\frac{1}{5}[-5(-3)(-4)a_0 + 0] = 12a_0.$

$k = 2: \quad a_2 = -\frac{1}{6}[-5(-2)(-3)a_1 + (2-6)a_0] = \frac{182}{3}a_0.$

With $r = \frac{1}{2}$, $[2(k+r)(k+r-1)+7(k+r)-3] = [(2(k+r)-1)((k+r)+3)] = 2k(k+\frac{7}{2}) = k(2k+7)$ and the recursion formula is given by:

$$a_k = \frac{1}{4k(2k+7)}[-5(2k-1)(2k-3)a_{k-1} + (4k-10)a_{k-2}].$$

$k = 1:\quad a_1 = \frac{1}{36}[-5(1)(-1)a_0 + 0] = \frac{5}{36}a_0.$

$k = 2:\quad a_2 = \frac{1}{88}[-5(3)(1)a_1 - 2a_0 = \frac{-25-24}{88\cdot 12}a_0 = \frac{-49}{1056}a_0.$

So the two solutions are (with a_0 chosen as 1 in each case):

$$y_1 = x^{-3}[1+12x + \frac{182}{3}x^2 + \dots]$$
$$y_2 = x^{1/2}[1+\frac{5}{36}x - \frac{49}{1056}x^2 + \dots.$$

We will only look at the case where the indicial equation has real roots. There are problems with the smaller root when the two roots differ by an integer. For this reason we will look only at the solutions associated with the larger root.

Example: $(2x + x^2)y'' + (1 - x)y' + 3y = 0$. Rewrite by multiplying by x and replace the original equation by

$$(2x^2+x^3)y''+(x-x^2)y'+3xy = x^2(2+x)y''+x(1-x)y'+(3x)y = 0.$$

The associated Euler equation is $2x^2y'' + xy' + 0y = 0$ with indicial equation $0 = 2r(r-1) + r = r(2r-1)$ with roots $r = \frac{1}{2}, 0$.
We use the rewritten equation:

$$\begin{aligned}
\Sigma\, 2(n+r)(n+r-1)a_n x^{n+r} &= \Sigma 2(k+r)(k+r-1)a_k x^{k+r}.\\
\Sigma(n+r)(n+r-1)a_n x^{n+r+1} &= \Sigma(k+r-1)(k+r-2)a_{k-1}x^{k+r}.\\
\Sigma\,(n+r)a_n x^{n+r} &= \Sigma\,(k+r)a_k x^{k+r}.\\
\Sigma-(n+r)a_n x^{n+r+1} &= \Sigma-(k+r-1)a_{k-1}x^{k+r}.\\
\Sigma\, 3a_n x^{n+r+1} &= \Sigma\, 3a_{k-1}x^{k+r}.
\end{aligned}$$

Sum the terms:

$$[2(k+r)(k+r-1)+(k+r)]a_k \;=\; (-(k+r-1)(k+r-2)+(k+r-1)-3)a_{k-1}.$$

$[2(k+r)(k+r-1)+(k+r)] = (k+r)(2(k+r)-1)$ from the factoring of the indicial equation.

Substitute $r=\frac{1}{2}$ to get the recursion formula

$$a_k = \frac{1}{k(2k+1)}[-(k-\frac{1}{2})(k-\frac{3}{2})+(k-\frac{1}{2})-3)]a_{k-1}$$
$$-\frac{4k^2-12k+17}{4k(2k+1)}a_{k-1}$$

With $k=1,\quad a_1=-\frac{9}{12}a_0=-\frac{3}{4}a_0.$
With $k=2,\quad a_2=-\frac{9}{40}a_1=\frac{27}{160}a_0.$
So with $a_0=1,\; y_1=x^{1/2}[1-\frac{3}{4}x+\frac{27}{160}x^2+\ldots].$

Laplace Transforms

We have looked before at the linear differential operator given by $\mathcal{L}(y) = y'' + py' + qy$. The *Laplace Transform* is a linear integral operator which transforms a function y of t to a function $\mathcal{L}(y)$ of s. It is given by the improper integral:

$$\mathcal{L}(y) = \int_0^\infty e^{-st} y(t) dt.$$

Examples: If $y = 1$ then

$$\mathcal{L}(y) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{t=0}^{t=\infty} = \frac{1}{s}.$$

The integral only converges if s is positive and so the Laplace transform is a function defined when $s > 0$.

If $y = e^{at}$ then

$$\mathcal{L}(y) = \int_0^\infty e^{-st} e^{at} dt = -\frac{1}{s-a} e^{-(s-a)t} \Big|_{t=0}^{t=\infty} = \frac{1}{s-a}.$$

Again this is only defined when $s > a$.

We can apply the definition of the Laplace transform to more complicated functions. Suppose that

$$y(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ t & \text{for } 1 \leq t \leq 5, \\ 0 & \text{for } 5 < t. \end{cases}$$

$$\int_0^\infty e^{-st} y(t) dt = \int_0^1 e^{-st} dt + \int_1^5 te^{-st} dt =$$

$$-\frac{1}{s} e^{-st} \Big|_{t=0}^{t=1} + \left(-\frac{1}{s} te^{-st} - \frac{1}{s^2} e^{-st}\right)\Big|_1^5 =$$

$$-\frac{1}{s}e^{-st}\Big|_{t=0}^{t=1} + (-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st})\Big|_1^5 =$$

$$\frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-s} - 5e^{-5s}) + \frac{1}{s^2}(e^{-s} - e^{-5s}).$$

So the Laplace transform $L(y)(s) =$

$$\frac{1}{s}(1 - 5e^{-5s}) + \frac{1}{s^2}(e^{-s} - e^{-5s}).$$

One of the most useful properties of the Laplace transform is what it does to the derivative. We integrate by parts with $u = e^{-st}$ and $dv = y'dt$ so that $du = -se^{-st}dt$ and $v = y$:

$$\mathcal{L}(y') = \int_0^\infty e^{-st} y' dt = e^{-st} y(t)|_{t=0}^{t=\infty} + s \int_0^\infty e^{-st} y dt.$$

That is,

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0).$$

and so the Laplace transform converts differentiation to multiplication by the independent variable.
For the second derivative we have:
$\mathcal{L}(y'') = s\mathcal{L}(y') - y'(0) = s[s\mathcal{L}(y) - y(0)] - y'(0)$, and so we have

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) \quad \text{and} \quad \mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0).$$

We can apply this to solve initial value problems directly:
Example: $3y'' - 2y' - y = 1, y(0) = -3, y'(0) = -2$. Of course, we can solve this directly using Undetermined Coefficients.

Apply the Laplace transform operator:
$3\mathcal{L}(y'') - 2\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(1)$ and so

$3[s^2\mathcal{L}(y) + 3s + 2] - 2[s\mathcal{L}(y) + 3] - \mathcal{L}(y) = \frac{1}{s}$.

$(3s^2 - 2s - 1)\mathcal{L}(y) = -9s + \frac{1}{s}$.

$$\mathcal{L}(y) = \frac{1 - 9s^2}{s(3s^2 - 2s - 1)}.$$

This is as far as you will be required to go, but I will illustrate the next steps which use partial fractions and the inverse Laplace transform.

$\frac{1-9s^2}{s(3s^2-2s-1)} = \frac{1-9s^2}{s(3s+1)(s-1)} = \frac{A}{s} + \frac{B}{3s+1} + \frac{C}{s-1}$.

$A(3s+1)(s-1) + Bs(s-1) + Cs(3s+1) = 1 - 9s^2$.

$s = 0$ gives $-A = 1$ and so $A = -1$.

$s = 1$ gives $4C = -8$ and so $C = -2$.

$s = -\frac{1}{3}$ gives $\frac{4}{9}B = 1 - 1 = 0$ and so $B = 0$.

$\mathcal{L}(y) = -\frac{1}{s} - 2\frac{1}{s-1}$.

$y = -1 - 2e^t$.

Fourier Series

A function f defined on the real line is periodic with period $2L$ if $f(x+2L) = f(x)$ for every x. Thus, if we look at the graph of f the block over any interval of length $2L$ just repeats. It will be convenient to use the interval $[-L, L]$ centered on 0 (which is why we wrote $2L$ for the period).

A constant function is periodic with any period, but our main examples are sin and cos. These have period 2π. If n is a positive integer then each repeats n times in an interval of length $2n\pi$. If we look at $\cos(\omega x)$ with frequency ω, the period is inversely related to the frequency:

$$\cos(\omega x) = \cos(\omega x + 2n\pi) = \cos(\omega(x + \frac{2n\pi}{\omega})).$$

If we are using $2L$ as the period then $2L = \frac{2n\pi}{\omega}$ or $\omega = \frac{n\pi}{L}$. With this frequency, $\cos(\omega x)$ and $\sin(\omega x)$ each repeat n times in the interval $[-L, L]$.

The Fourier series of a periodic function f of period $2L$ writes the function as an infinite sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + \Sigma_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x).$$

Notice for future reference that the a's are the coefficients of the even functions and the b's are the coefficients of the odd functions.

To determine the coefficients, recall that for a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, we get the coefficients using the dot product:

$$a =< \mathbf{v} \cdot \mathbf{i} >, \; b =< \mathbf{v} \cdot \mathbf{j} >, \; c =< \mathbf{v} \cdot \mathbf{k} >,$$

because the three vectors are mutually perpendicular and with length 1

$$< \mathbf{i} \cdot \mathbf{j} >= < \mathbf{i} \cdot \mathbf{k} >=< \mathbf{k} \cdot \mathbf{j} >= 0,$$
$$\text{and} \qquad < \mathbf{i} \cdot \mathbf{i} >=< \mathbf{j} \cdot \mathbf{j} >=< \mathbf{k} \cdot \mathbf{k} >= 1.$$

So we define a dot product between periodic functions f and g of period $2L$:

$$< f \cdot g >= \frac{1}{L} \int_{-L}^{L} f(x) g(x) \, dx.$$

The dot product is symmetric and linear in each variable separately. The dot product of f with itself is positive unless $f = 0$.

To compute with it recall that if f is odd then $\int_{-L}^{L} f(x) dx = 0$ and if f is even $\int_{-L}^{L} f(x) dx = 2 \int_{0}^{L} f(x) dx$. Furthermore,

$$odd \cdot odd, \; even \cdot even \;\; \text{are} \;\; even, \qquad odd \cdot even \;\; \text{is} \;\; odd.$$

Since $\sin \cdot \cos$ is odd: $\int_{-L}^{L} \cos(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)\ dx = 0$ for any positive integers m, n

Since sin is odd $\int_{-L}^{L} \sin(\frac{m\pi}{L}x)\ dx = 0$. Now cos is even but nonetheless the integral over an entire period is zero. That is, $\int_{-L}^{L} \cos(\frac{n\pi}{L}x)\ dx = 0$.

Now we need the sum formula for cos: $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$. So

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)],$$
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)].$$

Thus, $\displaystyle\int_{-L}^{L} \cos(\frac{n\pi}{L}x)\cos(\frac{m\pi}{L}x)dx =$

$$\frac{1}{2}[\int_{-L}^{L} \cos(\frac{(m-n)\pi}{L}x)dx + \int_{-L}^{L} \cos(\frac{(m+n)\pi}{L}x)dx].$$

The second integral is always zero and the first is when $m \neq n$. For sin we take the difference of these same two integrals.

When $m = n$ we have:

$$\frac{1}{L}\int_{-L}^{L} \cos^2(\frac{n\pi}{L}x)\,dx = \frac{1}{L}\int_{-L}^{L} \sin^2(\frac{n\pi}{L}x)\,dx = \frac{1}{2L}\int_{-L}^{L} 1\,dx = 1.$$

Finally, $\frac{1}{L}\int_{-L}^{L} \frac{1}{2}\cdot 1\,dx = 1.$

So we get the dot product formula for the Fourier coefficients for the $2L$ periodic function f.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\ dx \qquad \text{for } n = 0, 1, 2, \dots,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\ dx \qquad \text{for } n = 1, 2, \dots.$$

Notice in the $n = 0$ case, $< f \cdot 1 >=< a_0\frac{1}{2} \cdot 1 >= a_0$. That is, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$. This is just the above formula with $n = 0$, but it is usually computed differently because of the absence of trig functions.

If f is an even function then all of the coefficients of the sin's, the b_n's are zero, while if f is odd then all of the coefficients of the cos's, the a_n's are zero. An even function is a sum of cosines (including constants) and an odd function is a sum of sines (with no constant term).

In doing the computations, we use that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$.

Many of the integrals use integration by parts and there is a short-cut, sometimes called the *tabular method*. We use $D(F)$ for the derivative and $I(F)$ for the integral. To integrate $F \cdot G$ with F a polynomial, we use

$$I(FG) \ = \ FI(G) - D(F)I^2(G) + D^2(F)I^3(G) - D^3(F)I^4(G) + \ldots.$$

That is, we begin by pulling the F out and then at each step we differentiate the previous F factor and integrate the previous G factor, using alternating signs. Because F is a polynomial, this terminates because $D^{n+1}(F) = 0$ if F is a polynomial of degree n.

Example: $f(x) = x^2$, for $-L < x \leq L$ and with $f(x+2L) = f(x)$.

f is even and so we need only compute the cosine terms.

$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2\ dx = \frac{2}{L}\int_{0}^{L} x^2\ dx = \frac{2}{3L}x^3|_0^L = \frac{2L^2}{3}.$$

$$\begin{aligned} a_n = \frac{2}{L}\int_0^L x^2\cos(\frac{n\pi}{L}x)\ dx \ &= \ \frac{2}{L}[(x^2)(\frac{L}{n\pi}\sin(\frac{n\pi}{L}x)) \\ -(2x)(-(\frac{L}{n\pi})^2\cos(\frac{n\pi}{L}x)) \quad &+ \quad (2)(-(\frac{L}{n\pi})^3\sin(\frac{n\pi}{L}x))]|_0^L \\ = \ \frac{2}{L}[+2L(\frac{L}{n\pi})^2\cos(n\pi)] \quad &= \quad (-1)^n\frac{4L^2}{n^2\pi^2}. \end{aligned}$$

So $f(x) = \frac{L^2}{3} \ + \ \Sigma_{n=1}^{\infty}\ (-1)^n\frac{4L^2}{n^2\pi^2}\cos(\frac{n\pi}{L}x)$.

Example, 10.2/ BD15; BDM15 :

$$f(x) = \begin{cases} x, & -\pi \le x < 0, \\ 0, & 0 \le x < \pi; \end{cases} \qquad f(x+2\pi) = f(x). \text{ So } L = \pi.$$

The function is neither even nor odd.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x \; dx = \frac{1}{2\pi} x^2|_{-\pi}^{0} = -\frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx) \; dx =$$

$$\frac{1}{\pi}[(x)(\frac{1}{n}\sin(nx)) \quad - \quad (1)(-\frac{1}{n^2}\cos(nx))]|_{-\pi}^{0}$$

$$= \frac{1}{n^2\pi}[1-\cos(n\pi)] \;=\; \frac{1}{n^2\pi}[1-(-1)^n].$$

Notice that $1-(-1)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$ but you may leave it in the form $1-(-1)^n$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\ dx =$$

$$\frac{1}{\pi}[(x)(-\frac{1}{n}\cos(nx)) \quad - \quad (1)(-\frac{1}{n^2}\sin(nx))]|_{-\pi}^{0}$$

$$= -\frac{1}{n}[\cos(n\pi)] \ = \ \frac{1}{n}[(-1)^{n+1}]$$

Therefore,

$$f(x) = -\frac{\pi}{4} + \Sigma_{n=1}^{\infty}\ \frac{1}{n^2\pi}[1-(-1)^n]\cos(nx)$$

$$+\ \Sigma_{n=1}^{\infty}\ \frac{1}{n}[(-1)^{n+1}]\sin(nx).$$

The only functions we will consider are continuous and with a continuous derivative except for a finite number of possible jumps (*piecewise continuously differentiable*). The Fourier series then converges to $f(x)$ at points x where f is continuous and to the midpoint $\frac{1}{2}(f(x-) + f(x+))$ at an jump. Notice that the actual value of f at the jump makes no difference.

If we start with any function f defined for $-L \leq x < L$ we can extend it to get a periodic function by insisting that $f(x + 2nL) = f(x)$ for every x between $-L$ and L and every integer n. We then get a Fourier series which converges to the original function.

So if we start with any function defined for $0 < x < L$ and extend it any way to the interval $[-L, 0]$ and then make it periodic we still get a Fourier series which converges to the original function for $0 < x < L$. There are infinitely many such Fourier series.

A $2L$ periodic function f has a unique Fourier series:
$\frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x) + \Sigma_{n=1}^{\infty} \; b_n \sin(\frac{n\pi}{L}x)$,
but if we look just at $(0, L)$ there are infinitely many ways of extending to $[-L, 0]$ leading to infinitely many Fourier series.

Of special importance, are the *cosine series* and the *sine series* for a function on $(0, L)$.

For the cosine series we extend the function f to get an even function on $[-L, L]$ and then make it $2L$ periodic. To be precise, we choose $f(0)$ to be anything (usually, we use $f(0+)$ to get continuity at zero) and then let $f(-x) = f(x)$ for $-L \leq x \leq 0$. For an even function all the b_n's equal zero and we get a series with just cosine terms (and a possible constant term) and for $x \in (0, L)$

$$f(x) \;=\; \frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x).$$

For the sine series we extend the function f to get an odd function on $[-L, L]$ and then make it $2L$ periodic. There is a slight quibble here: If f is an odd $2L$ periodic function then:
$f(0) = f(-0) = -f(0)$ and
$f(L) = f(-L + 2L) = f(-L) = -f(L)$.
This needs $f(-L) = f(0) = f(L) = 0$.

This is a quibble because the actual value at these three points doesn't affect the Fourier series.

Starting with any f defined on $(-0, L)$ we set $f(0) = f(L) = 0$ and then extend to $[-L, 0]$ by setting $f(-x) = -f(x)$ for $-L \leq x \leq 0$. The associated Fourier series has only sine terms.

$$f(x) \;=\; \Sigma_{n=1}^{\infty}\; b_n \sin(\frac{n\pi}{L}x).$$

In each of these cases we only need the formula for f on $(0, L)$ we do not need the extension to $[-L, 0]$

For the cosine series:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)\ dx \qquad \text{for } n = 0, 1, 2, \ldots.$$

For the sine series:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\ dx.$$

Example: 10.4/ BD25; BDM25 : $f(x) = 2 - x^2$ for $0 < x < 2$. Thus, $L = 2$.
For the sine series:

$$b_n = \frac{2}{2}\int_0^2 (2 - x^2)\sin(\frac{n\pi}{2}x)\ dx =$$

$$[(2 - x^2)(-\frac{2}{n\pi}\cos\frac{n\pi}{2}x)) \ - \ (-2x)(-\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}x)$$

$$+(-2)(\frac{8}{n^3\pi^3}\cos\frac{n\pi}{2}x)]_0^2 \ = \ [(-2)(-\frac{2}{n\pi}(-1)^n) - (2)(-\frac{2}{n\pi})]$$

$$+(-2)(\frac{8}{n^3\pi^3}(-1)^n) - (-2)(\frac{8}{n^3\pi^3})]$$

$$= \frac{4}{n\pi}[(-1)^n + 1] - \frac{16}{n^3\pi^3}[(-1)^n - 1]$$

Thus, $b_n = \begin{cases} \frac{8}{n\pi} & \text{if } n \text{ is even,} \\ \frac{32}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}$

Separation of Variables

A partial differential equation (pde) describes a real-valued function of at least two independent variables. We will consider only linear, homogeneous pde's so that if u_1 and u_2 are solutions then $C_1 u_1 + C_2 u_2$ is a solution. Just as with linear ordinary differential equations we look for special solutions and then mix them to obtain the general solution. However, in these cases we often require an infinite sum of the special solutions.

If u is a function of the two variables x and t we look for special solutions of the form $u = X(x) \cdot T(t)$ where X is a function of x alone and T is a function of t alone. The procedure, when it works, is called *separation of variables*.

We write u_x for $\frac{\partial u}{\partial x}$, u_{xt} for $\frac{\partial^2 u}{\partial x \partial t}$ etc. If $u = XT$ then $u_x = X'T$ $u_{xt} = X'T'$ etc.

Example 10.5/ BD3; BDM3 : $u_{xx} + u_{xt} + u_t = 0$. If $u = XT$ this becomes $X''T + X'T' + XT' = 0$, or $X''T + (X' + X)T' = 0$ and we can separate the variables and write this as

$$\frac{X''}{X' + X} = -\frac{T'}{T}.$$

The only way this can happen is if both sides are constant, the same constant, which we will call λ.

Setting each side equal to λ we cross multiply to get rid of the fractions and so obtain two ordinary differential equations: one for X and one for T.

$$X'' - \lambda X' - \lambda X = 0, \qquad \text{and} \qquad T' = -\lambda T.$$

The process is reversible. If we start with any solutions X and T of these two equations, their product is a solution of the original pde.

The procedure does not always work. Suppose the equation had been $u_{xx} + u_{xt} + u_{tt} = 0$ instead.
With $u = XT$ we get $X''T + X'T' + XT'' = 0$. This time you can't separate the variables.

As you might expect, it will work with the equation upon which we will focus our attention: The *heat equation* which is given by $u_t = \alpha^2 u_{xx}$. It is a function of $t > 0$ but with $0 < x < L$.

Imagine a thin bar of length L. The coordinate x describes the position of a point on the bar. Then $u(x,t)$ describes the temperature at position x, at time t. The equation is obtained from the intuitive idea that if the bar at a point x is hotter then the average temperature of the nearby points, then heat tends to diffuse away and so the temperate drops at x and so at that moment u_t is negative. The term u_{xx} measure, infinitesimally, the difference between the average temperature of nearby points and the temperature at x. The constant α^2 measures the rate at which heat diffuses through the rod. We write is a α^2 because it is always positive.

We use separation of variables. Substituting $u = XT$ into $u_t = \alpha^2 u_{xx}$ gives $XT' = \alpha^2 X'' T$

So $\frac{T'}{\alpha^2 T} = \frac{X''}{X}$ which leads to the two equations

$$T' = \lambda \alpha^2 T, \qquad \text{and} \qquad X'' - \lambda X = 0.$$

The equation in T is exponential growth or decay, and the equation in X is a second order, linear, homogeneous equation with constant coefficients and characteristic equation $r^2 - \lambda = 0$.

In addition to specifying the initial temperature distribution $u(x, 0) = f(x)$, we need to know what is happening at the ends of the rod. These are the *boundary conditions*. We will only consider the homogeneous boundary conditions which fixes the temperature at zero at each end for all time. That is, $u(0, t) = u(L, t) = 0$. We need special solutions which satisfy these boundary conditions.

The solution of the equation for T is given by constant multiples of $T = e^{\alpha^2 \lambda t}$. But the solution of the X equation depends on the sign of λ.

Case 1 ($\lambda > 0$): The characteristic equation has the distinct real roots $\pm\sqrt{\lambda}$ with the general solution $C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$. So we have $u(x,t) = [C_1 e^{\sqrt{\lambda}x} + C_2 e^{\sqrt{\lambda}x}] e^{\alpha^2 \lambda t}$.

Putting in the boundary conditions we get
$0 = u(0,t) = [C_1 + C_2] e^{\alpha^2 \lambda t}$ and
$0 = u(L,t) = [C_1 e^{\sqrt{\lambda}L} + C_2 e^{-\sqrt{\lambda}L}] e^{\alpha^2 \lambda t}$. The T part is always positive and so the only solution is $C_1 = C_2 = 0$. In other words, no nonzero Case 1 solution satisfies the boundary condition.

Case 2 ($\lambda = 0$): T is then constant and we choose it to be 1. The characteristic equation $r^2 - \lambda = 0$ has the repeated roots $0, 0$ with the general solution $C_1 + C_2 x$.

So we have $u(x, t) = C_1 + C_2 x$.

Putting in the boundary conditions we get $0 = u(0, t) = C_1$ and $0 = u(L, t) = C_1 + C_2 L$. Again the only solution is $C_1 = C_2 = 0$. In other words, no nonzero Case 2 solution satisfies the boundary condition either.

Case 3 ($\lambda < 0$): It will be convenient to write $\lambda = -\omega^2$. Then, up to constant multiple, $T = e^{-\alpha^2\omega^2 t}$.

The characteristic equation $r^2 + \omega^2 = 0$ has the imaginary roots $\pm\omega\mathbf{i}$ with the general solution $C_1\cos(\omega x) + C_2\sin(\omega x)$ and $u(x,t) = [C_1\cos(\omega x) + C_2\sin(\omega x)]e^{-\alpha^2\omega^2 t}$.

The boundary conditions give us $0 = u(0,t) = C_1 e^{-\alpha^2\omega^2 t}$ and so $C_1 = 0$. Then $0 = u(L,t) = C_2\sin(\omega L)e^{-\alpha^2\omega^2 t}$. To avoid $C_2 = 0$ we must have $\sin(\omega L) = 0$ which requires that $\omega L = n\pi$ for a positive integer n. (With $-n$ we just get the negative of the solution with n). So ω has to be $\frac{n\pi}{L}$.

Thus, we obtain special solutions which satisfy the boundary conditions only when $\lambda = -(\frac{n\pi}{L})^2$ and we get one special solution u_n for every positive integer n.

$$u_n(x,t) = e^{-(\frac{\alpha n\pi}{L})^2 t}\sin(\frac{n\pi}{L}x).$$

Thus, to solve the system: $u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$ for all $t > 0$ with initial distribution $u(x,0) = f(x), 0 < x < L$. We begin with the sine series for $f(x)$. That is, we extend f to an odd function on $[-L, L]$ and then make it periodic with period $2L$. The sine series is:

$$\begin{aligned} f(x) &= \Sigma_{n=1}^{\infty}\ b_n \sin(\frac{n\pi}{L}x), \\ \text{with} \qquad b_n &= \frac{2}{L}\int_0^L\ f(x)\sin(\frac{n\pi}{L}x)\ dx. \end{aligned}$$

For the solution we simply write

$$u(x,t) = \Sigma_{n=1}^{\infty}\ b_n e^{-(\frac{\alpha n\pi}{L})^2 t}\sin(\frac{n\pi}{L}x).$$

Notice that if $f(x)$ is just given as a sum of the appropriate sine terms then we need only insert the corresponding exponential decay factors.

Example: $u_t = \alpha^2 u_{xx}, u(0,t) = u(L,t) = 0$ with $f(x) = x^2, 0 < x < L$.

$$b_n = \frac{2}{L}\int_0^L x^2 \sin(\frac{n\pi}{L}x)\ dx \ = \ \frac{2}{L}[(x^2)(-\frac{L}{n\pi}\cos(\frac{n\pi}{L}x))$$

$$-(2x)(-(\frac{L}{n\pi})^2 \sin(\frac{n\pi}{L}x)) \quad + \quad (2)(+(\frac{L}{n\pi})^3 \cos(\frac{n\pi}{L}x))]|_0^L$$

$$= \ \frac{2}{L}[(L^2)(-\frac{L}{n\pi})\cos(n\pi)] \ + \ (2)(+(\frac{L}{n\pi})^3\{\cos(n\pi) - 1\}]$$

$$= \frac{2L^2}{n\pi}(-1)^{n+1} + \frac{4L^2}{n^3\pi^3}((-1)^n - 1).$$

The solution is

$$u(x,t) \ = \ \Sigma_{n=1}^{\infty}\,[\frac{2L^2}{n\pi}(-1)^{n+1} + \frac{4L^2}{n^3\pi^3}((-1)^n - 1)]e^{-(\frac{\alpha n\pi}{L})^2 t}\sin(\frac{n\pi}{L}x).$$

Example 10.5/ BD8; BDM8 : $u_{xx} = 4u_t, 0 < x < 2, t > 0$ with $u(0,t) = u(2,t) = 0, t > 0$ and with $u(x,0) = f(x) = 2\sin(\frac{\pi}{2}x) - \sin(\pi x) + 4\sin(2\pi x)$.

Notice first that $L = 2$, but that $\alpha^2 \neq 4$. Instead $\alpha^2 = \frac{1}{4}$. Also $f(x)$ is given as a sine series with just three nonzero terms. Since $L = 2$,

$$\begin{aligned} \sin(\frac{\pi}{2}x) &= \sin(\frac{1\pi}{L}x), \\ \sin(\pi x) &= \sin(\frac{2\pi}{L}x), \\ \sin(2\pi x) &= \sin(\frac{4\pi}{L}x), \end{aligned}$$

$$u(x,t) = 2e^{-(\frac{\pi}{4})^2 t}\sin(\frac{\pi}{2}x) - e^{-(\frac{\pi}{2})^2 t}\sin(\pi x) + 4e^{-\pi^2 t}\sin(2\pi x).$$

is the solution.