9.1.1b coordinate vector of $v = ax^2 + bx + c$

basis $B = \{x^2, x+1, x+2\}$.

$$r(x^2) + s(x+1) + t(x+2) = ax^2 + bx + c$$

convert to column

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

with $D = \{x^2, x, 1\}$ $\quad [v]_D = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$[v]_B = [I]_{BD} [v]_D \text{ or } [I]_{DB} [v]_B = [v]_D$$

Solve $\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 1 & 2 & c \end{array}\right)$ $\quad R_3 \to R_3 - R_2$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c-b \end{array}\right)$ $\quad R_2 \to R_2 - R_3$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 2b-c \\ 0 & 0 & 1 & c-b \end{array}\right)$

Solution: $r = a \quad s = 2b - c \quad t = c - b$

$$ax^2 + (2b-c)(x+1) + (c-b)(x+2)$$

$$= ax^2 + bx + c.$$

9.1.1d $\quad v = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$r \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + s \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$[v]_B = \begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix}$$

convert to columns

$$\left(\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

$R_2 \leftrightarrow R_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$R_4 \to \frac{1}{2} R_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Finally $\quad R_3 \to R_3 + R_4$

$R_2 \to R_2 - R_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} r \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$v = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} =$$

$$2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ 0 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9.1.3c $\quad T: P_2 \to P_3 \quad T[p(x)] = xp(x)$

$$B = \{1, x, x^2\} \qquad D = \{1, x, x^2, x^3\}$$

$$[T]_{DB} = \left([T(1)]_D \;\; [T(x)]_D \;\; [T(x^2)]_D \right)$$

$$= \left([x]_D \;\; [x^2]_D \;\; [x^3]_D \right) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.1.3d $\quad T: P_2 \to P_2 \quad T[p(x)] = p(x+1)$

$$B = D = \{1, x, x^2\}$$

$$[T]_{DD} = \left([T(1)]_D \;\; [T(x)]_D \;\; [T(x^2)]_D \right)$$

$$= \left([1]_D \;\; [x+1]_D \;\; [x^2+2x+1]_D \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

9.1.4c $\quad T: \mathcal{P}_2 \to \mathbb{R}^2 \quad T(a+bx+cx^2) = (a+b, c)$

$B = \{1, x, x^2\}, \quad D = \{(1,0), (1,-1)\}$

$$[T]_{DB} = \left([T(1)]_D \quad [T(x)]_D \quad [T(x^2)]_D \right)$$

$$= \left([(1,0)]_D \quad [(1,0)]_D \quad [(0,1)]_D \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

because $(0,1) = +(1,0) - (1,-1)$

$$[T(v)]_D = [T]_{DB} [v]_B$$

$v = a + bx + cx^2 \qquad [v]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b+c \\ -c \end{pmatrix}$$

$T(v) = (a+b, c) =$

$(a+b+c)(1,0) - c(1,-1)$

3.5.1 h $\quad xI - A = \begin{bmatrix} x-2 & -1 & -1 \\ 0 & x-1 & 0 \\ -1 & -1 & x-2 \end{bmatrix}$

For the determinant expand along the second row

$$\det(xI - A) = (x-1)\det\begin{pmatrix} x-2 & -1 \\ -1 & x-2 \end{pmatrix}$$

$$= (x-1)(x^2 - 4x + 4 - 1) = (x-1)(x^2 - 4x + 3)$$

$$= (x-1)(x-1)(x-3)$$ roots 1, 3

$x = 1$ $\quad I - A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = r$, $x_2 = s$ $\quad x_1 = -r - s$ $\quad e_r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\quad e_s = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$x = 3$ $\quad 3I - A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3+R_1]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

$\xrightarrow[R_3+R_2]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2]{} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+R_2]{} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = r$ $\quad x_2 = 0$ $\quad x_1 = r$ $\quad e_r = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P^{-1}AP = D$

3.5.1a $\quad f_1' = 2f_1 + 4f_2 \qquad f_1(0) = 0$

$\qquad\quad f_2' = 3f_1 + 3f_2 \qquad f_2(0) = 1$

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \qquad xI - A = \begin{pmatrix} x-2 & -4 \\ -3 & x-3 \end{pmatrix}$$

det $= x^2 - 5x + 6 - 12 = x^2 - 5x - 6 = (x-6)(x+1)$ roots $6, -1$

$x = 6$
$$\begin{pmatrix} 4 & -4 \\ -3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \qquad e_\lambda = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x_2 = r \quad x_1 = r$

$\lambda = -1$
$$\begin{pmatrix} -3 & -4 \\ -3 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4/3 \\ 0 & 0 \end{pmatrix} \qquad e_{\lambda=-1} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$x_2 = r \quad x_1 = -\frac{4}{3} r$

$$P = \begin{pmatrix} 1 & -4 \\ 1 & 3 \end{pmatrix} \qquad P^{-1}AP = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(t) = c_1 e^{+6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$t = 0$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 \\ 1 & 3 \end{pmatrix}^{-1} = \frac{1}{7} \begin{pmatrix} 3 & 4 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \\ \frac{1}{7} \end{pmatrix}$$

Diagonalize $A = \begin{pmatrix} 2 & 4 & 8 & 0 \\ 3 & 3 & -6 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$

$$xI - A = \begin{pmatrix} x-2 & -4 & -8 & 0 \\ -3 & x-3 & 6 & 0 \\ 0 & 0 & x-1 & -1 \\ 0 & 0 & 0 & x-6 \end{pmatrix}$$

$$\det(xI - A) = \det\begin{pmatrix} x-2 & -4 \\ -3 & x-3 \end{pmatrix} \det\begin{pmatrix} x-1 & -1 \\ 0 & x-6 \end{pmatrix}$$

$= (x-6)(x+1)(x-1)(x-6)$ roots $6, -1, 1$

$x = 6$ $\begin{pmatrix} 4 & -4 & -8 & 0 \\ -3 & 3 & 6 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\to \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & -2/5 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$x_4 = r$ $x_3 = \frac{r}{5}$ $x_2 = s$ $x_1 = s + \frac{2r}{5}$

$e_{\lambda=5} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 5 \end{pmatrix}$ $e_s = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x = -1$ $\begin{pmatrix} -3 & -4 & -8 & 0 \\ -3 & -4 & 6 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 4/3 & 8/3 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to$

$\begin{pmatrix} 1 & 4/3 & 8/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$x_4 = 0$ $x_3 = 0$ $x_2 = r$ $x_1 = -4/3\, r$

$e_{\lambda=-1} \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$x = 1$

$$\begin{pmatrix} -1 & -4 & -8 & 0 \\ -3 & -2 & 6 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 8 & 0 \\ 0 & 10 & 30 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 8 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_4 = 0 \quad x_3 = r \quad x_2 = -3r \quad x_1 = 4r$

$$e_4 = \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & 1 & -4 & 4 \\ 0 & 1 & 3 & -3 \\ 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} A P = D$$

8.1.1c

$B = \{(1,-1,1), (1,0,1), (1,1,2)\}$
$\qquad\quad v_1 \qquad\quad v_2 \qquad\quad v_3$

$w_1 = v_1 = (1 \ -1 \ 1)$

$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1 \ 0 \ 1) - \frac{2}{3}(1 \ -1 \ 1)$

$= (\frac{1}{3} \ \frac{2}{3} \ \frac{1}{3})$ mult by 3

$w_2 = (1 \ 2 \ 1)$

$w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1$

$= (1 \ 1 \ 2) - \frac{5}{6}(1 \ 2 \ 1) - \frac{4}{6}(1 \ -1 \ 1)$

$(-\frac{3}{6} \ 0 \ \frac{3}{6})$ mult by 2

$w_3 = (-1 \ 0 \ 1)$

orthogal basis w_1, w_2, w_3. For orthonormal divide by length

$u_1 = \frac{1}{\sqrt{3}}(1 \ -1 \ 1)$, $u_2 = \frac{1}{\sqrt{6}}(1 \ 2 \ 1)$, $u_3 = \frac{1}{\sqrt{2}}(-1, 0, 1)$

8.1.4 e

$\{v_1, v_2, v_3\} = (1\,0\,1\,0)\ (1\,1\,1\,0)\ (1\,1\,0\,0)$

$w_1 = v_1 = (1\,0\,1\,0)$

$w_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = (1\,1\,1\,0) - \frac{2}{2}(1\,0\,1\,0) =$

$w_2 = (0\,1\,0\,0)$

$w_3 = v_3 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 = (1\,1\,0\,0) - \frac{1}{1}(0\,1\,0\,0) - \frac{1}{2}(1\,0\,1$

$(\frac{1}{2}\ 0\ -\frac{1}{2}\ 0)$ mult by 2

$w_3 = (1\ 0\ -1\ 0)$

orthogon w_1, w_2, w_3

$P_U(x) = \frac{x \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{x \cdot w_2}{w_2 \cdot w_2} w_2 + \frac{x \cdot w_3}{w_3 \cdot w_3} w_3$

$\frac{1}{2}(1\,0\,1\,0) + 0(0\,1\,0\,0) + \frac{3}{2}(1\ 0\ -1\ 0)$

$= (2, 0, -1, 0)$

Notice: $(1\,0\,1)\quad(0\,1\,0)\quad(1\ 0\ -1)\qquad w_1'\quad w_2'\quad w_3'$

orthogonal basis for $\mathbb{R}^3$

$x' = (2, 0, -1)$ in $\mathbb{R}^3$

So projection of $x' = x'$

8.2.5d $\quad A = \begin{bmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 7 & 0 & 3 \end{bmatrix}$

$xI - A = \begin{bmatrix} x-3 & 0 & -7 \\ 0 & x-5 & 0 \\ -7 & 0 & x-3 \end{bmatrix}$ for deter expand along row 2

$\det = (x-5)\det\begin{pmatrix} x-3 & -7 \\ -7 & x-3 \end{pmatrix} = (x-5)(x^2-6x+9-49)$

$= (x-5)(x^2-6x-40) = (x-5)(x-10)(x+4)$

roots $= 5, 10, -4$

$x = 5 \quad \begin{pmatrix} 2 & 0 & -7 \\ 0 & 0 & 0 \\ -7 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -7/2 \\ 0 & 0 & -45/2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = 0 \quad x_2 = s \quad x_1 = 0 \qquad e_u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$x = 10 \quad \begin{pmatrix} 7 & 0 & -7 \\ 0 & 5 & 0 \\ -7 & 0 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = s \quad x_2 = 0 \quad x_1 = s \qquad e_u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$x = -4 \quad \begin{pmatrix} -7 & 0 & -7 \\ 0 & -9 & 0 \\ -7 & 0 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x_3 = s \quad x_2 = 0 \quad x_1 = -s \qquad e_u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For orthonormal basis divide by lengths

$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -4 \end{pmatrix}$

$P^{-1} = P^T$

8.2.5e

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad xI - A = \begin{pmatrix} x-1 & -1 & 0 \\ -1 & x-1 & 0 \\ 0 & 0 & x-2 \end{pmatrix}$$

$$\det(xI-A) = (x-2)\det\begin{pmatrix} x-1 & -1 \\ -1 & x-1 \end{pmatrix} =$$

$$(x-2)(x^2-2x+1-1) = (x-2)x(x-2) \qquad \text{root } 2, 0$$

$$x=0 \qquad \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = 0 \qquad x_2 = r \qquad x_1 = -r \qquad e_r = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$x=2 \qquad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = r \qquad x_2 = s \qquad x_1 = s \qquad e_r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad e_s = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P^{-1} = P^T$$