3   Numerical Solution of Differential Equations

> *Principal Skinner:* Ah, there's nothing more exciting than science. You get all the fun of sitting still, being quiet, writing down numbers, paying attention...[chuckles] Science has it all.
>
> From: *The Simpsons*

3.1   *Introduction*

In Chapter 2 we discussed some techniques that can be used to find explicit solutions for some differential equations. With these and other techniques it is possible to solve many equations that arise in practice. However, a variety of equations, and in particular systems of differential equations, often cannot be solved in this way. In such situations we can still work with the solutions through the use of numerical approximations. In this chapter we describe how this is done. To understand these methods we need to recall how a function is approximated using its derivative.

Consider Figure 3.1 below showing the graph of a function and its tangent line at a particular point $(x, f(x))$. We wish to estimate the value of the function at the point $x + \Delta x$, where Δx is a small increment in x.
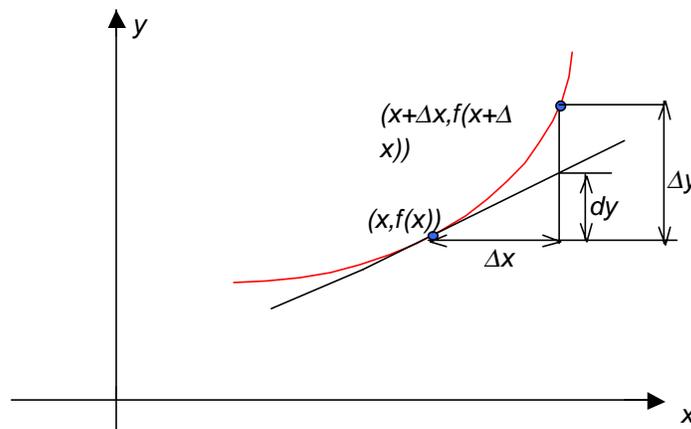

Figure 3.1

Suppose we know the value of $f'(x)$. Recall from elementary calculus that this number is the slope m of the tangent line and we know that for any line the slope is equal to the *rise/run*. Thus, in the picture above we have $f'(x) = m = dy / \Delta x$. Therefore, the rise dy in the tangent line is $dy = f'(x) \Delta x$. If Δx is small, (for the purposes of illustration this is obviously not the case in Figure 3.1, but use your imagination) the increment in the function Δy will be close to the rise dy in the tangent line. We thus obtain the fundamental approximation sometimes known as the differential approximation:

Theorem 3.1: If Δx is small and we know $f(x)$ and $f'(x)$ then we can estimate $f(x+\Delta x)$ as

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x \tag{3.1}$$

Proof:

In Figure 3.1 the quantity Δy is the difference $f(x+\Delta x) - f(x)$. Assuming Δx is small this difference is close to dy. Thus, $f(x+\Delta x) - f(x) \approx dy = f'(x)\Delta x$, from which we deduce the result stated in Theorem 3.1.∎

Although the approximation (3.1) is probably familiar to the reader, we consider a numerical illustration.

Example 3.1: Use (3.1) to estimate the value of $\sqrt{16.2}$ and compare the approximation to a more accurate estimate obtained using a scientific calculator.

Solution:

Consider the function $f(x) = \sqrt{x}$. We have $f'(x) = \dfrac{1}{2}x^{-1/2}$. In particular $f(16) = 4$ and $f'(16) = 1/8$ are both known. Applying Theorem 3.1 with $x = 16$ and $\Delta x = 0.2$ gives

$$f(16.2) = f(x+\Delta x) \approx f(x) + f'(x)\Delta x = 4 + \frac{0.2}{8} = 4.025.$$

From a calculator we find the estimate $\sqrt{16.2} \approx 4.02492$, so our result is correct to the third decimal place.∎

Figure 3.1 suggests, and this turns out to be quite correct, that as Δx gets smaller the accuracy of the approximation (3.1) improves. It is possible to strengthen Theorem 3.1 by giving an estimate for the error in the approximation, although we will not bother to do so. We turn rather to showing how Theorem 3.1 is used to construct approximate solutions to a differential equation, using a method first described by the 18th century Swiss mathematician Leonhard Euler, and usually referred to as Euler's method.

3.2   Euler's Method

In our work in Chapter 2 we solved an initial value problem by first finding the general solution of the ODE. The general solution involved an arbitrary constant, whose value was determined from the initial condition. Numerical procedures work differently, as they generate approximate solutions to the initial value problem directly, without finding general solutions of the equation. We illustrate the idea using an example for which the exact solution is known and can therefore be used as a comparison for accuracy.

Example 3.2: Estimate the value at $x=1$ of the solution to the initial value problem $\dfrac{dy}{dx} = y$, with $y(0) = 2$.

Solution:

From the work in Chapter 2 we know that the solution of the ODE is $y = Ce^x$. The constant C can be evaluated from the initial condition, yielding $y = 2e^x$. Thus when $x=1$ the value of the solution is $2e \approx 5.44$. To estimate this answer without solving the equation we must first decide on how much work we want to do, or more precisely how much computation we want to perform. The final estimate at $x=1$ is not obtained in one step, but follows from estimates of the solution over a sequence of small steps that terminate at $x=1$.

Suppose that we decide to use 10 steps. We divide the interval from 0 to 1 into 10 equal subintervals each of length 0.1. The endpoints of these intervals are 0, .1, .2, .3, … , .9, 1.0. We know something about the solution y at the beginning $x=0$. Namely, we know $y(0)=2$ from the initial condition. Moreover, the differential equation says that $y' = y$ so that at $x=0$ we also know that $y'(0) = 2$. We now have the information needed to apply Theorem 3.1, with $x=0$ and $\Delta x = 0.1$. According to that theorem the value of y (which we are using instead of $f(x)$) at $x + \Delta x = 0.1$ is approximately

$$y(.1) \approx y(0) + y'(0)\Delta x = 2 + 2(.1) = 2.2$$

We take this as the estimate for the solution of the ODE at $x = 0.1$. From the differential equation $y' = y$ we estimate that $y'(.1) = y \approx 2.2$. We can then use Theorem 3.1 again, but this time with $x = 0.1$ and $\Delta x = 0.1$. We obtain the estimate

$$y(.2) \approx y(.1) + y'(0.1)\Delta x = 2.2 + 2.2(0.1) = 2.42.$$

With the estimate $y(0.2) \approx 2.42$ we use the differential equation to obtain the value of $y'(0.2) \approx 2.42$. Now Theorem 3.1 is used again to get an estimate for $y(0.3)$. We find, with $x = 0.2$ and $\Delta x = 0.1$, that

$$y(0.3) \approx y(0.2) + y'(0.2)\Delta x = 2.42 + 2.42(0.1) = 2.662.$$

The table below completes the computation. All results have been rounded to three decimal places.

x	y	y'	x	y	y'
0	2.000	2.000	0.6	3.543	3.543
0.1	2.200	2.200	0.7	3.897	3.897
0.2	2.420	2.420	0.8	4.287	4.287
0.3	2.662	2.662	0.9	4.716	4.716
0.4	2.928	2.928	1	5.187	5.187
0.5	3.221	3.221			

The final estimate for $y(1) \approx 5.19$ is clearly a rather crude approximation to the true value 5.44 that we determined earlier. We will show below that the estimate can be improved by dividing the interval [0,1] into smaller pieces, in other words by doing more computation. This simply reflects the fact that the basic approximation (3.1), which we have repeatedly used, improves in accuracy as Δx approaches zero.∎

In order to clearly describe the construction used in Example 3.2 we need a little more notation. If we have a differential equation such as $y' = y - x$ then if we know that at $x = 2$ the value of y is 1, the differential equation enables us to find the value of y'. In this case $y' = 1 - 2 = -1$. This simple observation can be described more generally as follows. The right side of the differential equation is some expression involving x and y which we denote by $g(x, y)$. For the ODE $y' = y - x$, we have $g(x, y) = y - x$. **If we know that when $x = \tilde{x}$ the value of y is $\tilde{y}$ then the differential equation $y' = g(x, y)$ implies that the value of the derivative $y'(\tilde{x})$ is $g(\tilde{x}, \tilde{y})$.** With this notation we can describe the general procedure known as Euler's method.

Definition 3.1: (Euler's Method)

To construct an Euler approximation to the solution of an initial value problem $y' = g(x, y)$, $y(a) = y_0$ on an interval from a to b (where $b > a$) carry out the following steps:

a) Select a positive integer n.

b) Divide the interval $[a, b]$ into n equal pieces of length $\Delta x = (b - a)/n$.

c) Construct a table of values as shown below:

x	y	y'
$x_0 = a$ (given)	$y_0 = y(a)$ (given)	$y_0' = g(x_0, y_0)$
$x_1 = x_0 + \Delta x$	$y_1 = y_0 + y_0'\,\Delta x$	$y_1' = g(x_1, y_1)$
$x_2 = x_1 + \Delta x$	$y_2 = y_1 + y_1'\,\Delta x$	$y_2' = g(x_2, y_2)$
…	…	…
$x_n = x_{n-1} + \Delta x$	$y_n = y_{n-1} + y_{n-1}'\,\Delta x$	$y_n' = g(x_n, y_n)$

Table 3.1

The Euler approximation consists of the values $y_0, y_1, y_2, \ldots, y_n$ listed in the second column. The quantity Δx is called the *step size*.∎

The structure of the table in Definition 3.1 is very simple. The entries in the column marked y' are obtained by substituting the x and y values in the current row into the right side of the differential equation. To proceed to the next row we increment x by the amount Δx and then we

update y using the estimation process described by Theorem 3.1. This updating can be described succinctly by the equation

$$y_{new} = y_{old} + y'_{old}\, \Delta x \qquad\qquad (3.2)$$

which we call the *Euler update equation*. Here y_{new} refers to the y value in the current row and y_{old} and y'_{old} refer to the entries in the prior row in the columns labeled y and y' respectively. The reader should check that the entries listed in the table in Example 3.2 follow the pattern described in Definition 3.1, taking into account that all answers are rounded to three decimal places.

Example 3.3: Construct an Euler approximation on the interval [2,4] to the solution of the initial value problem $y' = y - x$, $y(2) = 1$. Use four subintervals.

Solution:

The value of n is the same as the number of subintervals, so $n = 4$. Then, from Definition 3.1b) we have $\Delta x = (4-2)/4 = 0.5$. We now construct the table containing the Euler approximation. The initial condition determines the first two entries in row one. The third column is obtained by substituting the current x and y values in the expression $y - x$ that gives the value of the derivative. Subsequent rows are updated according to (3.2).

x	y	y'
2	1	$y - x = 1 - 2 = -1$
$2 + \Delta x = 2.5$	$y_{old} + y'_{old}\,\Delta x = 1 - .5 = .5$	$y - x = .5 - 2.5 = -2$
$2.5 + \Delta x = 3$	$y_{old} + y'_{old}\,\Delta x = .5 - 1 = -.5$	$y - x = -.5 - 3 = -3.5$
$3 + \Delta x = 3.5$	$y_{old} + y'_{old}\,\Delta x = -.5 - 1.75 = -2.25$	$y - x = -2.25 - 3.5 = -5.75$
$3.5 + \Delta x = 4$	$y_{old} + y'_{old}\,\Delta x = -2.25 - 2.875 = -5.125$	$y - x = -5.125 - 4 = -9.125$

Of course with so few subintervals we would not expect that the result would be very close to the exact solution. This is example is considered further in the exercises.■

In both Example 3.2 and Example 3.3 we noted that the estimate obtained for the solution of the ODE using an Euler approximation with only a few subintervals was not particularly accurate. We consider what happens if we increase the number of subintervals used in the computation.

Example 3.4: Construct Euler approximations on the interval [0,1] to the solution of the initial value problem $y' = y$, $y(0) = 2$ using $n = 10$, 50, 100, 500, 1000, 10,000 subintervals. Compare the value $y(1)$ in each case to the exact value $2e$.

Solution:

Using the *Excel* file *ode.xls* (see section 3.4) with the given initial conditions and the listed values of n we obtain the following estimates for $y(1)$

| n	10	50	100	500	1000	10,000	∞ (exact)
$y(1)$	5.1874	5.3832	5.4096	5.4311	5.4338	5.4363	5.4366

The estimated value at $x=1$ slowly converges to the exact value, although it takes a very large number of subintervals to produce a result that is reasonably accurate. Unfortunately, having to carry out a large number of computations, besides taking a longer time, introduces another possible error source from accumulated roundoff in the numerical calculations. Because of the slow rate of convergence of Euler's method, more complicated but efficient computational schemes are used in practice. We will describe one such method in the next section.■

According to our definition, an Euler approximation to an initial value problem is a sequence of y values that estimate the true solution at the x values listed in Table 3.1. We can use these y values to produce a graphical representation of the approximation. Namely, for each row in Table 3.1 plot the point (x_k, y_k) and then join these points with line segments in order of increasing x. As n increases the resulting graph will get closer and closer to the graph of the exact solution. This is illustrated in

Example 3.5: Plot the Euler solution on $[0,1]$ of the initial value problem $y' = y$, $y(0) = 2$ using $n = 5, 10, 25, 50$ subintervals. Compare with the graph of the exact solution $y = 2e^x$.

Solution:

We use *ode.xls* to generate the graphs shown below.

36
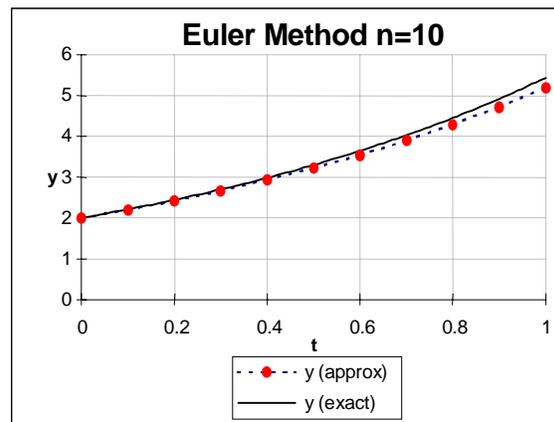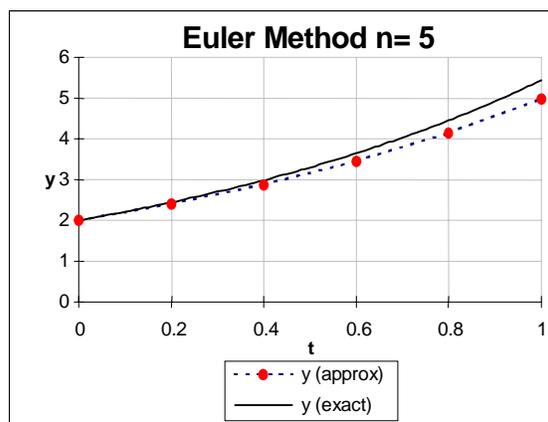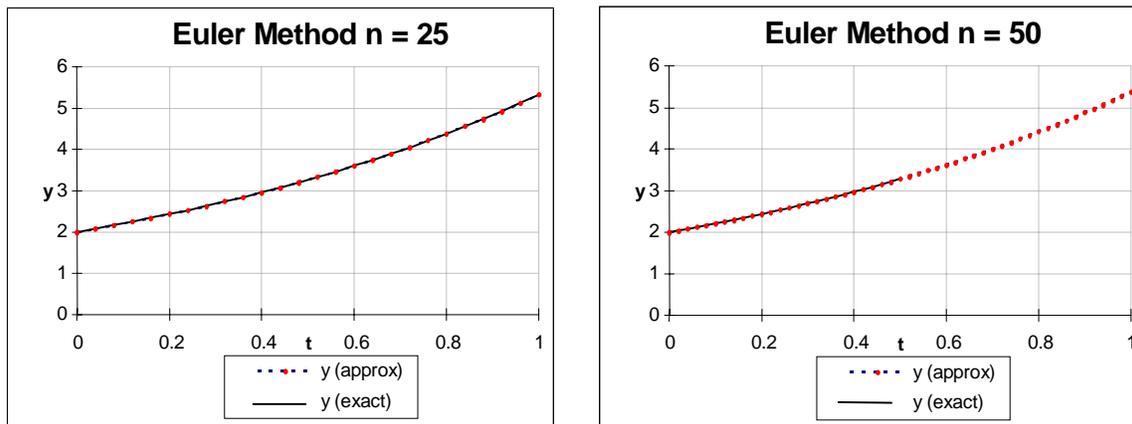
When $n = 25$ and $n = 50$ the approximation graph is barely distinguishable from the exact solution, at the resolution provided by the picture. As noted though in Example 3.4, when $n = 50$ the Euler estimate for $y(1)$ only agrees with exact answer to one decimal place. ∎

3.3   The Modified Euler Method

The Euler method is the conceptually simplest approach to computing a numerical approximation to the solution of an initial value problem. We have seen though that as a practical procedure it has definite drawbacks. To overcome these deficiencies a number of more accurate methods have been devised. We will discuss a procedure known as the modified Euler method. For a given number of subintervals this method requires approximately twice as much computation as the Euler method. However, its accuracy more than pays for the additional work.

The key to the modified Euler method is a more accurate method of approximating the change in a function from knowledge of its derivative. The result is embodied in Theorem 3.2.

Theorem 3.2: If Δx is small and we know $f(x)$ and $f'(x + \frac{\Delta x}{2})$, then we can estimate $f(x + \Delta x)$ using

$$f(x + \Delta x) \approx f(x) + f'(x + \tfrac{\Delta x}{2})\Delta x \tag{3.3}$$

Proof:

The estimate (3.3) is sometimes called the midpoint estimation for $f(x + \Delta x)$, since it uses the value of the derivative at the point $x + \Delta x / 2$ mid-way between x and $x + \Delta x$. In general (3.3) provides a much more accurate estimate for $f(x + \Delta x)$ than is provided by the differential estimate (3.1). Although we will not prove this claim, it is certainly suggested by Figure 3.2 below in which the quantity labeled dy is much closer to the increment of the function Δy than was the case in Figure 3.1.
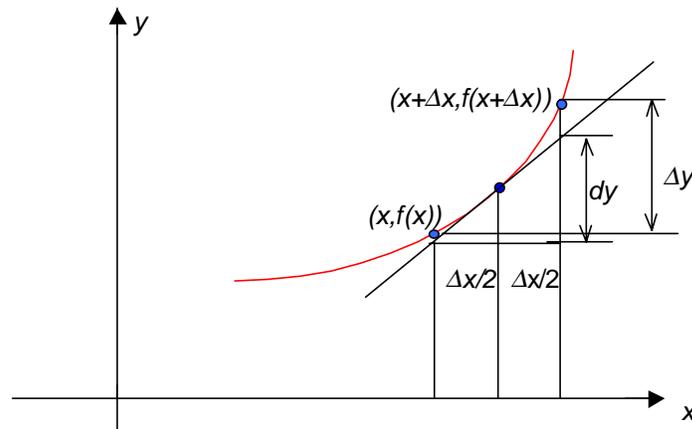
Figure 3.2

In Figure 3.2 we have drawn the tangent line to the graph at the point $(x+\Delta x/2, f(x+\Delta x/2))$. The quantity dy is the rise in that tangent line over the interval from x to $x+\Delta x$. This rise is the slope $f'(x+\Delta x/2)$ multiplied by the run Δx. Using dy as an estimate for $\Delta y = f(x+\Delta x) - f(x)$ we obtain (3.3).■

The chief difficulty in using Theorem 3.2 is the need for information regarding not only the function value at x but also the derivative value at $x+\Delta x/2$. Useful information regarding the latter may often be obtained using Theorem 3.1. We illustrate this idea, which is the key to the modified Euler method, by re-examining Example 3.1.

Example 3.6: Use (3.3) to estimate the value of $\sqrt{16.2}$ and compare the approximation to a more accurate estimate obtained using a scientific calculator.

Solution:

We let $f(x) = \sqrt{x}$ so $f'(x) = \dfrac{1}{2\sqrt{x}}$. We take $x=16$ so that $f(x)=4$, and $\Delta x = 0.2$. In order to apply (3.3) we need an estimate for $f'(16.1)$. Since $f'(x) = \dfrac{1}{2\sqrt{x}}$, to estimate $f'(16.1)$ we need to estimate $\sqrt{16.1}$ and the latter estimate may be found using Theorem 3.1 with $x=16$ and $\Delta x = .1$. We obtain from (3.1) that $\sqrt{16.1} \approx \sqrt{16} + \dfrac{.1}{2\sqrt{16}} = 4.0125$. Hence for $f'(16.1)$ we get the approximation $f'(16.1) = \dfrac{1}{2\sqrt{16.1}} \approx \dfrac{1}{8.025} \approx .1246106$. We now use this in (3.3). Our final estimate is $\sqrt{16.2} \approx \sqrt{16} + .2(.1246106) = 4.02492212$, compared to the value from a scientific calculator of 4.02492235. Theorem 3.2 gives an answer that is accurate to the sixth decimal place as compared to the three decimal place accuracy we found in Example 3.1.■

The key point in Example 3.6 was to use the simpler estimate provided by Theorem 3.1 to approximate the derivative at the halfway point $x + \Delta x / 2$. We then combined this with our knowledge of $f(x)$ to obtain a quite accurate estimate for $f(x + \Delta x)$. The modified Euler method uses a similar strategy. The interval from a to b is divided into n equal subintervals and, as in Euler's method, our goal is to obtain approximations $y_0, y_1, \ldots y_n$ to the value of the solution at the end points $x_0, x_1, \ldots, x_n$ of the n subintervals.

The value y_0 is given as part of the data of the initial value problem. To approximate y_1 we first estimate the derivative at $x_0 + \Delta x / 2$. This is done by using Euler's method, but with half the step size. This simple Euler method gives us an estimate for the solution at $x + \Delta x / 2$ and we can substitute this estimate, and the value $x = x_0 + \Delta x / 2$ into the right side of the differential equation to approximate the derivative at $x + \Delta x / 2$. Once we have the latter estimate, we can use (3.3) with the original step size Δx to obtain an estimate for y_1. The two-stage process is then repeated to obtain an estimate for y_2 from y_1, etc. We illustrate the details in

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Example 3.7: Use the modified Euler method to approximate the solution of the initial value problem $y' = y - x$, $y(2) = 1$ on the interval $[2, 4]$ using 4 subintervals. (See Example 3.3 for the corresponding calculation using Euler's method.)

Solution:

We have $\Delta x = (4 - 2) / 4 = .5$. We then repeat the following steps:

a) Set $x = x_0 = 2$, $y = y_0 = y(2) = 1$, $y' = y - x = 1 - 2 = -1$.

b) Now use Euler's method with step size $\Delta x / 2 = .25$ to estimate the solution and its derivative at $x + \Delta x / 2 = 2.25$. (We use $x_{1/2}$ and $y_{1/2}$ to stand for the x and y values computed at the mid points of the approximation intervals.)

i) y value using Euler method with half step size: $y_{1/2} = y + y' \Delta x / 2 = 1 - .25 = .75$

ii) y' from the differential equation: $y'_{1/2} = y_{1/2} - x_{1/2} = .75 - 2.25 = -1.5$

c) Use the value $y'_{1/2}$ and (3.3) to estimate y_1. Here we use the full step size.

i) $y_1 = y + y'_{1/2} \Delta x = 1 - 1.5(.5) = .25$

ii) Increment x: $x_1 = x + \Delta x = 2.5$

Now return to step a) with x_1 and y_1 replacing x_0 and y_0 respectively. The process ends when we reach y_4 and x_4 in step c). The five values y_0, y_1, y_2, y_3, y_4 of principal interest are shown below

with the auxiliary mid-point values on the right. The reader is urged to verify the results, which have been rounded to the third decimal place.

x	y	y'	$x_{1/2}$	$y_{1/2}$	$y'_{1/2}$
2.000	1.000	-1.000	2.250	0.750	-1.500
2.500	0.250	-2.250	2.750	-0.313	-3.063
3.000	-1.281	-4.281	3.250	-2.352	-5.602
3.500	-4.082	-7.582	3.750	-5.978	-9.728
4.000	-8.946	-12.946	4.250	-12.182	-16.432

Table 3.2

How accurate is the computed value $y(4) \approx -8.946$? The exact value is -9.778 so the answer is still quite inaccurate in the absolute sense, although the relative error is only about 10%. Note, however, that the result is much closer to the correct value than the estimate obtained in Example 3.3.∎

We will not formalize the description of the modified Euler method. However, to get a sense of its accuracy we examine Example 3.4 using this method.

Example 3.8: Using the modified Euler method on the interval [0,1] construct approximate solutions to the initial value problem $y' = y$, $y(0) = 2$ using $n = 10$, 20, 50, 100, 200, 400 subintervals. In each case, compare the value $y(1)$ to the exact value $2e$.

Solution:

We use the file *ode.xls* to perform the calculations. The results are shown below.

n	10	20	50	100	200	400	∞ (exact)
$y(1)$	5.42816	5.43438	5.43621	5.43647	5.43654	5.43656	5.43656

If we are interested in 4-digit accuracy then the results obtained with 50 subintervals and 100 subintervals both round to 5.436. The true answer rounded to the same number of places is 5.437, so we would not be far off had we taken the result obtained with 50 subintervals as a 4-digit estimate for the answer. By comparison, using 50 subintervals with the straightforward Euler method in Example 3.4 produces an answer in which the only reliable digits are the first two. Clearly, the modified Euler method is computationally superior to the plain vanilla Euler method. The latter should not be used when efficiency and numerical accuracy are important. ∎

3.4   Tech Notes

All the computations referred to in sections 3.2 and 3.3 were carried out using the spreadsheet file *ode.xls*. To use this file to solve a differential equation, note first that the independent variable

must be entered as t and not x. Thus, to solve an equation such as $\frac{dy}{dx} = yx$ we have to rewrite this as $\frac{dy}{dt} = yt$. You interact with the program through a so-called user form box, also known as a dialog box. To activate the dialog box click on the button labeled "Click to Enter Data".

Example 3.9: Fill in the entries in the *ode.xls* dialog box to numerically solve the initial value problem $\frac{dy}{dt} = yt$, $y(0) = 4$ on the interval $[0,3]$. Use the modified Euler method.

Solution:

The filled in dialog box is shown below.

Figure 3.3

The formula for the right side of the differential equation is entered on the first line. ***Be careful to insert explicit multiplication signs and appropriate parentheses***. The solution interval $[0,3]$ determines the entries for the initial time and final time. The initial value $y(0) = 4$ is entered for the initial y value. The Modified Euler method is checked.

The number of subintervals is set at 50, but this may not be an adequate number for the accuracy desired. Without delving into sophisticated estimations, there is a simple rule-of-thumb for determining whether you have selected an adequate number of subintervals:– ***try doubling the number of subintervals. If the computed value of y at the interval endpoint does not change in the decimal places of interest to you then the original solution was likely to be accurate to that***

number of places throughout the interval in question. In this problem we find that starting with $n = 50$ and doubling the number of subintervals produces the following estimates for $y(3)$.
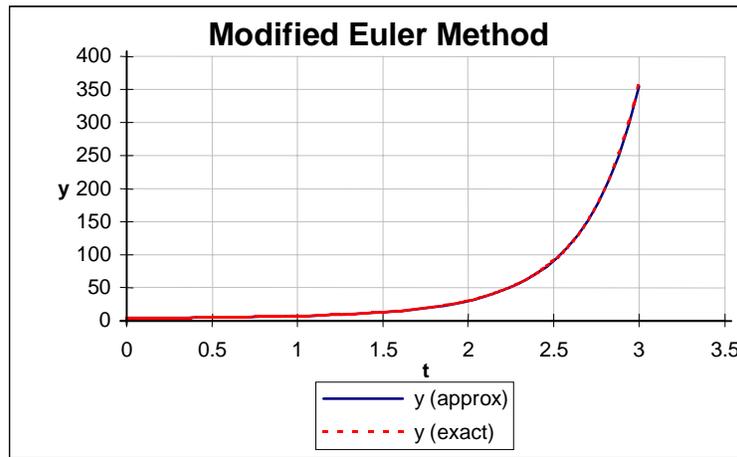
n	50	100	200	400
$y(3)$	354.881	358.691	359.714	359.979

From this table it would appear that using $n = 200$ subintervals produces a solution that has three digits (not three decimal places) of accuracy, so $y(3) \approx 360$. Note that it would make no sense to report that $y(3) \approx 359.714$, because you have no evidence that the last three decimal places are correct. ***The computer faithfully reports its work, but you must analyze what is valid in relation to the problem you are trying to solve***. If we want an estimate of the solution over a larger interval we must repeat the same sort of experiment to determine an appropriate number of subintervals to use.

The dialog box in Figure 3.3 contains a check box that may be used to enter an exact solution, if one is known. In this case since the equation is separable we can find an explicit solution to the initial value problem, namely $y = 4e^{t^2/2}$. This solution would be entered in the dialog box as shown below.

Although we noted above that 50 subintervals does not provide a particulary accurate estimation of the exact solution (in fact $y(3) = 4e^{9/2} = 360.07$ compared to the estimate of 354.88) the error is only about 1%. This is small enough so that the graphs of the approximation and exact solution are indistinguishable.

Modified Euler Method

[Figure: plot with y on vertical axis (0 to 400) and t on horizontal axis (0 to 3.5), showing two curves labeled "y (approx)" (solid line) and "y (exact)" (dashed line)]

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The analysis in Example 3.9 might lead one to use a large number of subintervals to start with, rather than experiment in selecting an appropriate number. This is a bad practice. A massive number of computations is no guarantee against inaccuracy and in fact may introduce subtle errors because of accumulated roundoff in the finite precision arithmetic used by the computer. The best advice we can give is to:

i)   Decide on the number of digits of accuracy you would like in the solution.

ii)   With the modified Euler method (or a variety of more accurate techniques), use the interval doubling procedure discussed in Example 3.9 until the approximate solution stabilizes for the decimal places you are interested in.

The top sheet of the *ode.xls* workbook gives the summary information for plotting the approximation. The calculation details can be viewed on the Calculation sheet. Just click the tab marked "Calculation" at the bottom of the main sheet to bring this sheet into view.

3.5 Summary

When an initial value problem for a differential equation cannot be solved by exact methods, we can approximate the solution using numerical methods. The simplest of these methods is the *Euler approximation*. This uses the observation that for an equation $\frac{dy}{dx} = g(x, y)$, if a current value is known for both x and y then the value $g(x, y)$ gives the slope of the unknown function $f(x)$. With the slope known, we can follow along the tangent direction to the curve $y = f(x)$ for a short increment Δx and we shall land at an approximate value of $y(x + \Delta x)$. The new y value is given by the Euler update equation, $y_{new} = y_{old} + y'_{old}\, \Delta x$. The process is then repeated starting at the point $(x + \Delta x, y_{new})$. Smaller steps Δx will yield more accurate approximations to the solution

of the differential equation. Alternatively, a more accurate solution may be obtained using a modification of the basic Euler method.

3.6 Exercises

1. Use the method of Theorem 3.1 to approximate $\sqrt{26}$ using the known value of $\sqrt{25}$. What is the error compared with the true value as found from a calculator?

2. Use the method of Theorem 3.1 to estimate $\ln(1.5)$ using the known value of $\ln(1)$. What is the error compared with the true value as found from a calculator?

3. Given the differential equation $\dfrac{dy}{dt} = y^3 - .75(1+t)y$ with initial condition $y(0) = 1$,

    a) What is the exact value of $y'(0)$?

    b) Using only $y(0)$ and $y'(0)$ find an estimate for $y(0.5)$.

    c) Using the answer to b) and the method of Theorem 3.2, find an estimate for $y(1.0)$.

4. The function $y = f(x)$ is a solution of the initial value problem $y' = x + 2 + e^y$, with $y(0) = 0$.

    a) What is the value of $f'(0)$?

    b) Using only $f(0)$ and $f'(0)$ find an estimate for $f(0.1)$.

    c) Using the answer to b) and the method of Theorem 3.2, find an estimate for $f(0.2)$.

5. Consider the initial value problem $\dfrac{dy}{dx} = y - x^2$, $y(0) = 1$. Use Euler's method on the interval $[0,1]$ with 4 subintervals to estimate the value of $y(1)$. Make a table similar to Table 3.1 showing your calculations.

6. a) The value of e can be obtained from the solution of the initial value problem $y' = y$, $y(0) = 1$. How?

    b) Based on your answer to a) use Euler's method with 4 subintervals to estimate the value of e. Make a table showing your calculations. What is the error compared with the true value of e as found from a calculator?

7. a) The value of $\ln(2)$ can be obtained from the solution of the initial value problem $y' = 1/x$, $y(1) = 0$. How?

    b) Based on your answer to a) use Euler's method with 4 subintervals to estimate the value of $\ln(2)$. Make a table showing your calculations. What is the error compared with the true value of $\ln(2)$ as found from a calculator?

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8. a) Use Euler's method and 5 subintervals to estimate the solution of the initial value problem $y' = \dfrac{x}{y}$, $y(0) = 2$ at $x = 1$. Make a table showing your calculations.

   b) Using separation of variables, solve the initial value problem in 8a)and compare the exact value of the solution at $x = 1$ with the approximation computed in a).

9. Consider the differential equation given by $y' = \dfrac{1}{1+y} - x$, with initial value $y(0) = 1$.

   Applying Euler's method on the interval $[0, 2.5]$ using 5 subintervals gives the following table:

x	y	y'
0	1.000	0.500
	1.250	-0.056
	1.222	-0.550
	0.947	-0.986
	0.454	-1.312

   a) Fill in the missing entries in the first column and the last row, indicating clearly how the numbers are obtained.

   b) Based on the data in the completed table sketch the graph of the solution y on the interval $[0, 2.5]$.

10. Consider the initial value problem $\dfrac{dy}{dt} = \dfrac{y^2}{2} - 20t^2$, $y(0) = 2$. Applying Euler's method on the interval $[0, 0.5]$ with 5 subintervals produces the following table.

t	y	y'
0	2.000	2.000
0.1	2.200	2.220
0.2	2.422	2.133
0.3	2.635	1.672
0.4	2.803	0.727
0.5		

   a) Fill in the missing entries in the last row showing clearly how they are obtained.

   b) Based on the estimates from the table, is the solution y increasing or decreasing at $t = 0.5$. Explain.

11. Use the method of Theorem 3.2 to approximate $\sqrt{26}$ using the known value of $\sqrt{25}$ and an estimate for $\sqrt{25.5}$. What is the error compared with the true value as found from a calculator?

12. Use the method of Theorem 3.2 to estimate $\ln(1.5)$ using the known value of $\ln(1)$. What is the error compared with the true value as found from a calculator?

13. a) Referring to Exercise 6, use the modified Euler method on the interval $[0,1]$ with four subintervals to obtain an estimate for e. Prepare a table similar to Table 3.2 showing your calculations.

    b) What is the error compared to the true value of e as found from a calculator?

14. Repeat the calculations asked for in Exercise 8, but using the modified Euler method. Display the calculations in a table similar to Table 3.2 and compare the value of $y(1)$ to the exact value found using 8b).

15. a) Verify that for any C the function $y = Ce^t + 1 + t$ is a solution of the differential equation $y' = y - t$. Determine the value of C so that the solution satisfies the initial condition $y(2) = 1$.

    b) Using the file *ode.xls* determine approximately how many subintervals are needed using the Euler method and the modified Euler method in order to estimate to three decimal places the solution at $t = 4$ of the initial value problem in a). Show appropriate data (using the technique of Example 3.9) to support your answer.

16. a) The differential equation $\dfrac{dy}{dt} = (1+t)y + 1 - 3t + t^2$ was studied by Newton. With an initial value $y(0) = .0656$ use the modified Euler method to estimate the value of $y(2)$ to two decimal places. Show appropriate data as in Example 3.9 to support the accuracy of your answer.

    b) Consider the same differential equation as in a) but with the initial condition $y(0) = .0661$. Use the modified Euler method to estimate the value of $y(2)$ to two decimal places. Show appropriate data as in Example 3.9 to support the accuracy of your answer.

    c) Generate (and print) graphs of the two solutions in a) and b) that show the very different behavior of these two solutions as t increases, even though they start out with almost identical initial values. (You will not want to make t too large in these experiments.)

17. Consider the initial value problem $y' = y^2 - t$, $y(0) = 0.5$.

    a) Print out the solution graph obtained using Euler's method on the interval $[0, 30]$ using 100 subintervals.

b) Print out the solution graph to the initial value problem on the interval [0, 30] using the modified Euler method with 100 subintervals.

c) Determine which if any of the graphs found in a) and b) shows an approximation that agrees with the true solution of the initial value problem to two decimal places. Provide a graph of such a correct approximation and explain how you are sure it has the desired accuracy. (See Example 3.9.)

18. Consider the initial value problem $\dfrac{dy}{dt} = y - t^2$, $y(0) = 1$ (See also exercise 5).

a) Using the modified Euler method fill in the estimate obtained for $y(3)$ using the number of subintervals specified in the table below. The y value should be written to the fourth decimal place.

# of subintervals	100	200	400	800
$y(3)$				

b) Based on the table obtained in a), what is the smallest number of subintervals you should use to produce an approximation that has three decimal place accuracy throughout the interval $[0,3]$?

c) Using the number of subintervals determined in b) (and the modified Euler method) use the graph of the solution $y(t)$ to estimate a value t_0 in $[0,3]$ such that $y(t_0) = 0$. (Such a value is called a root of the solution.) Type the estimate in the comments box provided in *ode.xls*.

d) Use the output table to find as small as possible an interval that contains the root t_0. If you were asked to estimate the root, what is the most accurate estimate you could make with this data? What would be the largest possible discrepancy between your estimate and the exact root? State your answer, together with an explanation, in the comments box.

e) The solution $y(t)$ also has a maximum in the interval $[0,3]$. Using the graph determine an estimate for this maximum and type your estimate in the comments box.

f) Use the output table to locate the maximum as accurately as you can. State a single estimate for this maximum. Explain how you obtained this estimate.

g) Print the first sheet of the output file containing the solution graph and your answers to parts b) - f).

19. a) Use separation of variables to solve the initial value problem $\dfrac{dy}{dt} = y \sin(t)$, $y(0) = 1$. (Note: you will need the formula $\int \sin(t)dt = -\cos(t) + C$.) The remainder of this exercise illustrates the computational difficulties in using Euler's method to approximate oscillating solutions of differential equations.

b) Open the file *ode.xls*. Check the box indicating you know an exact solution and then enter the formula for the exact solution that you found in a). We want to approximate this exact solution on the interval $[0,20]$ using Euler's method. Experiment with the number of subintervals until you are satisfied with the fit of the <u>graphs</u> of the approximate solution and the exact one. Print a single page of output showing this final graph.

c) Repeat the experiment in b) but using the modified Euler method. You should find a satisfactory fit of the two graphs using considerably fewer subintervals. Again, print a single page of output showing your final graph.