Laplace Transforms (6.1+2) L1

In practical applications (such as circuit analysis), the nonhomogeneous function $g(t)$ is sometimes zero except over a small interval of (time) t when it is very large. In those, or other, discontinuous cases we apply a transform method to turn the function into something more tractible. (Sort of like using $\ln(a \cdot b) = \ln a + \ln b$ to get a handle on the product of two large numbers, and then exponentiating to reverse the process at the end.)

$$f(t) \longrightarrow \underset{\text{transform of } f}{F(s)} = \int_\alpha^\beta \underbrace{K(s,t)}_{\text{kernel}} f(t)\,dt$$ is called an integral transform

For $f(t)$ defined on $t \geq 0$, the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt, \text{ when it exists.}$$

So it is an integral transform with kernel e^{-st}.

Ex. $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} \lim_{b\to\infty} e^{-st}\Big|_0^b = \frac{1}{s} \quad (s>0)$

$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \quad (s>a)$

(see Table 6.2.1 p.317 for more)

Note: $\mathcal{L}$ is a linear operator.

(L2)

If $f(t)$ is piecewise continuous and $|f(t)| \le Ke^{at}$ for some $K, a \in \mathbb{R}$ and $0 \le t < \infty$, then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

By simply applying the integral definition, it is easy to see:

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$$

so $\mathcal{L}\{ay'' + by' + cy = f(t)\}$ gives, with $y(0) = y_0$, $y'(0) = y_0'$,

$$\mathcal{L}\{y\} = \frac{(as+b)y_0 + ay_0' + F(s)}{as^2 + bs + c} \equiv Y(s)$$

and the solution to the initial value problem is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, which you lookup on Table 6.2.1 (after doing some partial fraction decomposition, in all likelihood).

Ex. Solve $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$.

Letting $Y(s) = \mathcal{L}\{y(t)\}$ & taking Laplace transforms of both sides gives $s^2Y(s) - s - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s-3}$

so algebra yields

via partial fractions

$$Y(s) = \frac{1}{(s-3)(s^2-3s+2)} + \frac{s-3}{s^2-3s+2} = \frac{5/2}{s-1} - \frac{2}{s-2} + \frac{1/2}{s-3}$$
$$= \mathcal{L}\left\{\tfrac{5}{2}e^{t} - 2e^{2t} + \tfrac{1}{2}e^{3t}\right\}$$

so

$$y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t},$$ which you could already obtain via methods in ch. 3!