391 Ch. 5 Summary (5-1)

Power Series (5.1) $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

1. Always converges for $x = x_0$
2. Ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= L|x-x_0|$$

converges if $|x-x_0|L < 1$, diverges if $|x-x_0|L > 1$ (if $=1$ cannot determine via this method)

3. $\rho = \frac{1}{L}$ (defined to be ∞ if $L = 0$ and 0 if $L = \infty$)

is called the radius of convergence so the series converges for $|x-x_0| < \rho$. One must test $x = x_0 \pm \rho$ by hand to get the full interval of convergence.

Series Solutions at an Ordinary Point (5.2)

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \text{where}$$

P, Q & R are polynomials in x with <u>no</u> mutually common factors

$P(x_0) \neq 0 \iff x_0$ an ordinary point, then try solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, so

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$$y' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

& substitute these back into diff. eq. (being careful to get powers of x correctly), adjust summations/indices so that the same power of $(x-x_0)$ appears in all (eg. $(x-x_0)^n$) and combine into one summation, setting the general coefficient of the n^{th} power of $(x-x_0)$ equal to zero gives the recursion relation. Determining which coefficients (usually a_0, a_1) are arbitrary allows you to pick-out 2 linearly indep. solutions. Applying initial conditions determines a_0 & a_1: $y(0) = a_0$ & $y'(0) = a_1$.

<u>Euler's Eq. (5.4)</u> $P(x_0) = 0 \Leftrightarrow x_0 =$ <u>singular point</u>

If $\lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} < \infty$ and

$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} < \infty$, x_0 is <u>regular</u>.

Ex. Euler's Eq. $x^2 y'' + \alpha x y' + \beta y = 0 = L[y]$

with $\alpha, \beta \in \mathbb{R}$ has a regular singular pt. $x_0 = 0$.

$L[x^r] = (r^2 + (\alpha-1)r + \beta) x^r = 0$ requires

$F(r) = r^2 + (\alpha-1)r + \beta = 0$, with roots r_1 & r_2)

giving solutions of the form $x^{r_1} + x^{r_2}$: so for any interval NOT containing zero, we get

1. $r_1 \neq r_2 \in \mathbb{R}$ then solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

2. $r_1 = r_2 = r \in \mathbb{R}$, $y = (c_1 + c_2 \ln|x|)|x|^r$

3. $r_1 = \bar{r}_2 \in \mathbb{C}$ with $r_1, r_2 = \lambda \pm i\mu$, $x^r = e^{r \ln x} = e^{(\lambda \pm i\mu)\ln x}$

and $y = |x|^\lambda [c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)]$

General Series Solutions Near a Regular Singular Pt.

(5.5) $P(x)y'' + Q(x)y' + R(x)y = 0$ with $x = 0$ a regular singular point $(P(0) = 0)$
[P, Q & R polynomials with no mutually common factors]

then $\lim_{x \to 0} x\frac{Q(x)}{P(x)} = p_0 < \infty$ and $\lim_{x \to 0} x^2 \frac{R(x)}{P(x)} = q_0 < \infty$

so $x\frac{Q}{P} \equiv xp(x) = \sum_{n=0}^{\infty} p_n x^n$ and $x^2 \frac{R}{Q} \equiv x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$,

both with non-zero radius of convergence. The d.e. becomes

$$y'' + p(x)y' + q(x)y = 0$$
$$x^2 y'' + x(xp(x))y' + x^2 q(x)y = 0$$
$$x^2 y'' + x(p_0 + p_1 x + \ldots)y' + (q_0 + q_1 x + \ldots)y = 0$$

The Associated Euler Eq. is:

$$x^2 y'' + xp_0 y' + q_0 y = 0$$

whose

$F(r) = r(r-1) + p_0 r + q_0 = 0$ is called the indicial equation, with roots r_1 and r_2.

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Let r be a root of the indicial equation, seek solutions of the form $x^r \sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$

so try

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} \text{ and}$$

$$y'' = \sum_{n=0}^{\infty} (r+n-1)(r+n) a_n x^{r+n-2},$$ substituted

back into the d.e. as in 5.2, gives a recursion relation for the coefficients a_n.

If ρ = minimum of the radii of convergence of $xp(x)$ and $x^2 q(x)$

then

for the larger root $r_1 \geq r_2$ and $0 < x < \rho$, there is a solution

$$y_1 = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n\right].$$