Intro. to Fourier Series (10.2) (F1)

Have already seen spaces of solutions (functs.) to certain ODE's form a finite dim'al vect. space, lin. indep means any funct. in space can be written as lin. comb., i.e. they form a "basis" for the space.

Build upon this similarly to $\mathbb{R}^n$:

in $\mathbb{R}^3$: $\vec{v} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$, $\{\hat{i}, \hat{j}, \hat{k}\}$ an orthonormal basis: $\hat{i}\cdot\hat{i} = 1$ ("normal"), $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$ ("orthog"), etc., and $\vec{v} = (\vec{v}\cdot\hat{i})\hat{i} + (\vec{v}\cdot\hat{j})\hat{j} + (\vec{v}\cdot\hat{k})\hat{k}$, $\forall \vec{v} \in \mathbb{R}^3$

in $\mathbb{R}^n$: $\hat{e}_i = (0, \ldots 0, \underset{i^{th}}{1}, 0, \ldots)$ similar, inner (or dot) prod. $\vec{v}\cdot\hat{e}_i \xrightarrow{i^{th}\text{ place}} \underset{\in\mathbb{R}}{i^{th}}$ component, i.e. $\vec{v} = \sum_i (\vec{v}\cdot\hat{e}_i)\hat{e}_i$

For funct. spaces, the basis vects. are functs. φ_i and the inner prod. of 2 fncs. f + g is given by $\int_\alpha^\beta f(x)g(x)\,dx$.
$= \langle f, g\rangle \in \mathbb{R}$ (or $\mathbb{C}$)

$\{\varphi_i\}_i$ as basis: orthog. cond. makes lin. indep.
want $\int_\alpha^\beta \varphi_i \varphi_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ (normal as well)

and any funct. f in the space can be written as
$$f = \sum_i \underbrace{\langle f, \varphi_i\rangle}_{\text{Fourier coeff}} \varphi_i,$$ the Fourier series of f

The classical one uses $\{e^{-inx}\}_{i=0}^{\infty} = \{\varphi_i\}$, equiv. to $\{\cos\frac{m\pi x}{L}\} \cup \{\sin\frac{m\pi x}{L}\}$. [Wavelets, etc.]

These are particularly useful when dealing w/ per. fncs. since they all are periodic

(F2)

Trig fnc. $\sin x$ + $\cos x$ have fund. per. 2π (see blc.), so $\frac{m\pi x}{L} = 2\pi \Rightarrow$

$T = \frac{2L}{m}$ is fund. per. of $\sin + \cos \frac{m\pi x}{L}$.

(any integral mult. of T is also a period)

Write series as $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \overbrace{\cos\frac{m\pi x}{L}}^{\varphi_m} + b_m \overbrace{\sin\frac{m\pi x}{L}}^{\psi_m}\right)$,
using inner prod. $\frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx = \langle f,g\rangle$, see

$\langle \varphi_m, \psi_n\rangle = \delta_{mn}$ by doing integrations (recalling trig. iden. + integ. by parts in blc)

Given a convergent Fourier series $f(x)$ above, the book derives (by integ. both sides of eq.) why the coeff's $a_m + b_m$ are what I claim:

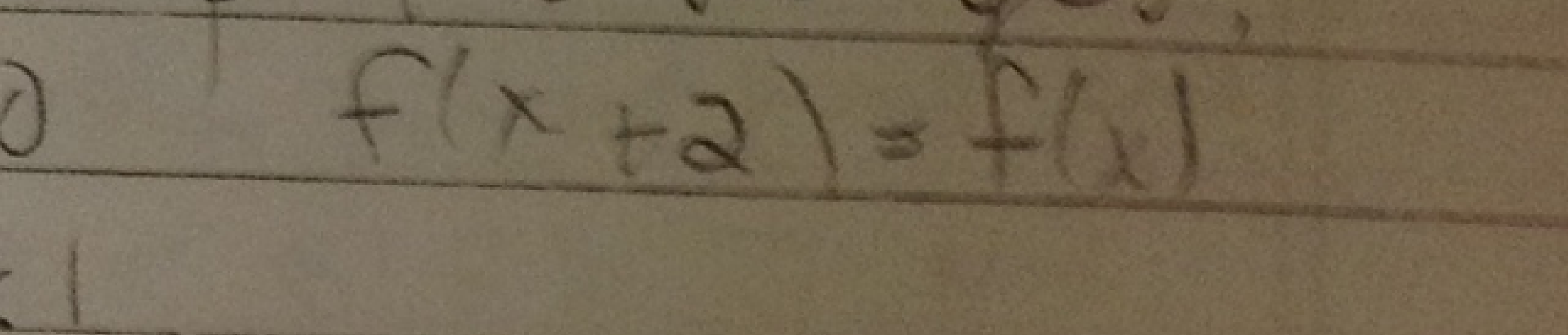
$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, $n \geq 1$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, $n \geq 1$

Ex: p. 593 #16 assuming it converges:

$f(x) = \begin{cases} x+1, & -1 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \end{cases}$ $\quad f(x+2) = f(x)$

So, fund. per. $T = 2L = 2$, $L = 1$



$a_0 = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 = 2 \times \text{area} = 2(1/2) = 1$

(F3)

$a_n \overset{n\geq 1}{=} \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}(1+x)\cos n\pi x\,dx + \int_{0}^{1}(1-x)\cos n\pi x\,dx$

$= \int_{-1}^{1}\cos n\pi x\,dx + 2\int_{0}^{1}(-x\cos n\pi x)\,dx$ (because of oddness)

by parts: $u = x,\ du = dx,\ dv = \cos n\pi x,\ v = \frac{1}{n\pi}\sin n\pi x$

$= \frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{1} - 2\left(\frac{1}{n\pi}x\sin n\pi x + \frac{1}{(n\pi)^2}\cos n\pi x\right)\Big|_{0}^{1}$

$= 0 + 2\left(0 - 0 - \frac{1}{(n\pi)^2}[\cos n\pi - 1]\right),\ n\geq 1$

$= \frac{2}{(n\pi)^2}\times\begin{cases}0 & n \text{ even}\\ 2 & n \text{ odd}\end{cases} = \begin{cases}0, & n \text{ even}\\ \left(\frac{2}{n\pi}\right)^2, & n \text{ odd}\end{cases}$

b_n's: calc. similarly $\rightarrow b_n = 0,\ \forall n\geq 1$, so

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\pi x + b_n\sin n\pi x)$

$= \frac{1}{2} + \sum_{\substack{n \text{ odd}\\ n\geq 1}}\left(\frac{2}{n\pi}\right)^2\cos n\pi x = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}$

Sec. 10.3: tells us when we get convergence:

Theor. (10.3.1) S'pose f & f' are piecewise cont. on $-L\leq x<L$ and f is periodic w/ per. $2L$ outside the int. $[-L, L)$, then the Fourier series for f converges to f at all pts. where f is cont., at the pts. of discty. $x=a$, it converges to

$\frac{1}{2}\left(\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x)\right)$ = avg. of left & rt. side limits at a

Recall: f is piecewise cts. on $[a,b]$ if $\exists$ part. of $[a,b)$ $a = x_0 < x_1 < x_2 < \dots < x_n = b$ s.t. f is cont. on ea. (x_{i-1}, x_i) and $\lim_{x\to x_i^+} f$ and $\lim_{x\to x_i^-} f$ exist & are finite

F4

Note: for a = pt. of cty., $\lim_{x\to a^+} f = \lim_{x\to a^-} f = \lim_{x\to a} f$

So, Fourier series converges everywhere to

$$\frac{1}{2}\left(\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x)\right) \quad [= f(a) \text{ if } f \text{ cts. at } a]$$

10.4 Recall: Even & Odd Functs.

odd $f(-x) = -f(x)$, **even** $f(-x) = f(x)$

$\int_{-L}^{L} \text{odd} = 0$ ($\sin x$) $\qquad \int_{-L}^{L} \text{even} = 2\int_{0}^{L}$ ($\cos x$)

prod of odd & even = odd $\qquad$ prod. of 2 odd = even

sum of odds = odd $\qquad$ prod. of 2 even = even

sum of evens = even

#13 p. 608: prove any funct f can be written as sum of an even fnc. g w/ and odd h:

Hint: what can you say about $f(x) + f(-x)$?

If $f(x) = g(x) + h(x)$, $f(-x) = g(x) - h(x)$

so $g(x) = \frac{1}{2}[f(x) + f(-x)] \Rightarrow h(x) = \frac{1}{2}[f(x) - f(-x)]$

Given a fnc. $f(x)$ on $[0, L]$ can extend it to an odd or even fnc. w/ per. $2L$ by letting

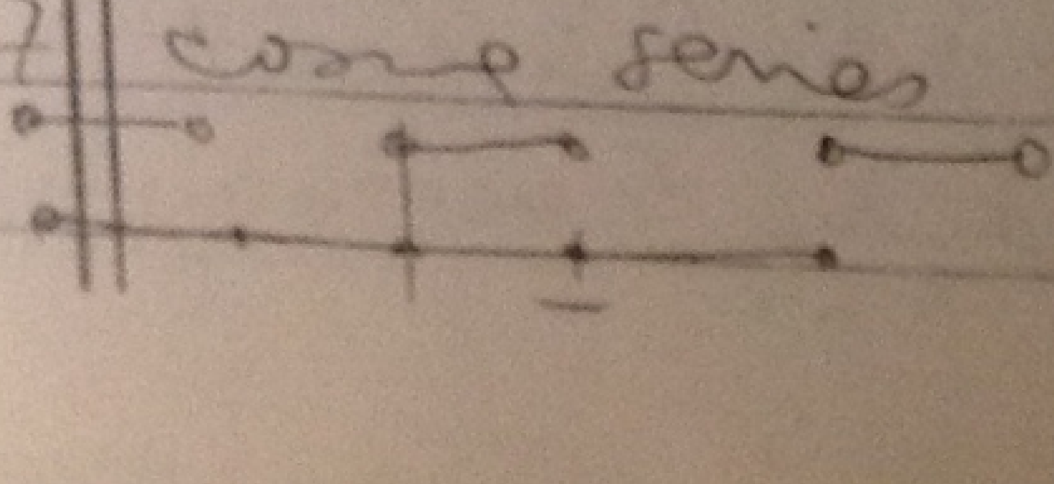
even: $g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases}$; **odd:** $h(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, L \\ -f(x), & -L < x < 0 \end{cases}$

→ rep. by cosine series $\qquad$ → rep. by sine series

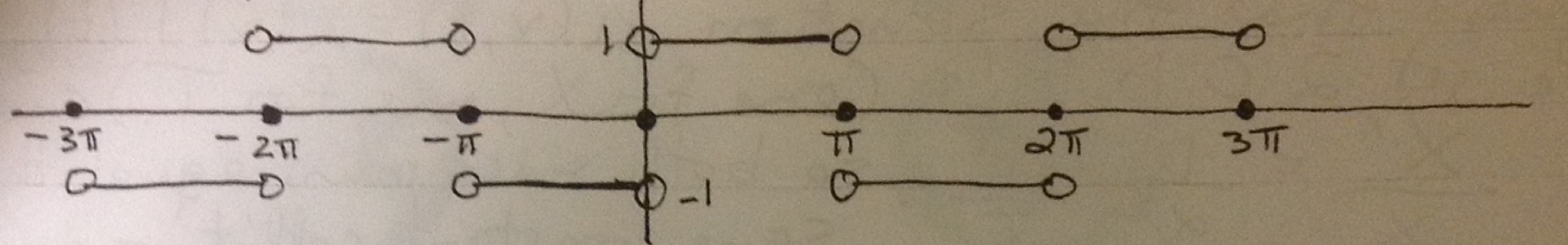
Ex. p. 608 #17 & 18 $f(x) = 1$ on $[0, \pi]$

#17 cosine series $f(x) = 1$

#18



#18 on bottom of previous page (F4) was (F5) cut-off upon photocopying. Here it is again:
The graph of the limiting function $F(x)$, which is the odd periodic extension of $f(x)=1$ on $(0,\pi)$ is



So the sine series for $f(x)=1$ on $(0,\pi)$ is given by $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin \pi x = \frac{2}{\pi}\int_0^{\pi} \sin \pi x \, dx$ (even integrand)

$$= \frac{-2}{n\pi}\cos nx \Big|_0^{\pi} = \frac{2}{n\pi}(1-\cos n\pi) = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}$$

$$\rightarrow \sum_{n=1}^{\infty} b_n \sin nx = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)}$$

Application to Simple P.D.E. (Heat Conduction 10.5)

Suppose you have a heat-conducting solid cylindrical rod whose length L is much greater than its diameter (eg. a wire), then the temperature in the rod is essentially a function of two independent variables: the distance along the rod x (ie. the spatial dimension along the axis of the cylinder) and the time t, call this function $u(x,t)$ and using the usual notation for partial derivatives, it turns out that

$$\alpha^2 u_{xx} = u_t \quad \text{for } 0 < x < L,\ t > 0$$

α^2 = constant (called thermal diffusivity)

Assuming the <u>initial condition</u> $u(x,0)=f(x)$ F6
on $0 \le x \le L$, and <u>boundary conditions</u>
$u(0,t)=u(L,t)=0$ for $t>0$, along with a guess that the solution $u(x,t)=X(x)T(t)$, we get 2 O.D.E.'s (one for X, one for T) and

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T},$$ so both sides must equal the same constant, call it $-\lambda$.

Solving these subject to the above boundary and initial conditions gives the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

where $$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx,$$

i.e. the c_n's are the Fourier coefficients of the sine series for $f(x)$!

Ex: Solve $\alpha^2 u_{xx} = u_t$ for $0 \le x \le \pi$, $t>0$
subject to $u(x,0)=1$ and $u(0,t)=u(\pi,t)=0$:
Using solution to #18 p. 608 (see page F4 + F5):

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\alpha^2 t} \sin(2n-1)x$$