Ch. 1 – 4 Review

(R1)

1st order
(gen. soln. has 1 arbit. const. c)

linear $y' + p(x)y = g(x)$ $\begin{cases} y(x_0) = y_0,\ x_0 \in (\alpha,\beta) + \\ p + g \text{ cont. on } (\alpha,\beta) \\ \exists!\ \text{soln. on } (\alpha,\beta) \end{cases}$

$$\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$$ integrating factor $\mu(x) = e^{\int p(x)dx}$

$$\int_{x_0}^{x} d(\mu(s)y) = \int_{x_0}^{x} \mu(s)g(s)ds$$

$$y(x) = \frac{1}{\mu(x)}\left[\mu(x_0)y_0 + \int_{x_0}^{x}\mu(s)g(s)ds\right]$$

gen. soln. $y(x) = \frac{1}{\mu(x)}[\ C\ +\ \text{anti-derivative}\ ]$

non-linear $y' = f(x,y)$ $\begin{cases} y(x_0) = y_0,\ x_0 \in (\alpha,\beta) \\ y_0 \in (\gamma,\delta) + f \text{ and } f_y \\ \text{cont. on } (\alpha,\beta) \times (\gamma,\delta) \\ \exists!\ \text{soln. on sub-interval} \end{cases}$

separable

$M(x)dx + N(y)dy = 0$

$\int N(y)dy = -\int M(x)dx$

soln. $F(y) = G(x) + C$

exact

$M(x,y)dx + N(x,y)dy = 0$

$M_y = N_x$ + M, N all cont. on $(\alpha,\beta) \times (\gamma,\delta)$

soln. $\psi(x,y) = C$ where $\psi_x = M$, $\psi_y = N$ so

$\int M dx = \psi + h(y)$
$\int N dy = \psi + k(x)$] compare

non-exact sometimes becomes separable with change of variable $y = vx$ ("homogeneous"), i.e. when $y' = F(y/x) = F(v) = x\frac{dv}{dx} + v$

$$\longrightarrow \ln|x| = \int \frac{dv}{F(v) - v} + C \quad \text{implicit soln.}$$

R2

2nd order linear $y'' + p(x)y' + q(x)y = g(x) = L[y]$ ← linear differential operator

(gen. soln. contains 2 arbit. consts. c_1 + c_2)

$y(x_0) = y_0$, $y'(x_0) = y_0'$, $x \in (\alpha, \beta)$ + p, q + g cont. on (α, β), $\exists!$ soln. on (α, β)

homogeneous $g(x) = 0$, $y = c_1 y_1 + c_2 y_2$ gen. soln

where $L[y_1] = L[y_2] = 0$ and $W(y_1, y_2) \neq 0$ on (α, β) ← linear independence

Wronskian $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

$= c \exp(-\int p(x)dx)$ **Abel's Theorem**

$c = 0$ when $c_1 y_1 + c_2 y_2 = 0$, ie. $y_2 = k y_1$ linearly dependent

reduction of order

given y_1, find y_2

try $y_2 = v y_1$

→ separable eq. for v'

$v' = \frac{c}{y_1^2} \exp(-\int p(x)dx)$

Const. coeff's. $ay'' + by' + cy = 0$

try $y = e^{rx}$ → **characteristic eq.**

$ar^2 + br + c = 0$

with roots r_1 + r_2:

1. $r_1 \neq r_2 \in \mathbb{R}$; $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$
2. $r_1 = r_2 = r \in \mathbb{R}$; $y_1 = e^{rx}$, $y_2 = xe^{rx}$ (by reduction of order)
3. $r_1 = \bar{r}_2 \in \mathbb{C}$, ie. $r_{1,2} = \alpha \pm i\beta$, $i^2 = -1$

$y_{1,2} = e^{r_{1,2}x} = e^{\alpha x}(\cos\beta x \pm i \sin\beta x)$

$y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$

2^{nd} order linear continued (R3)

<u>non-homogeneous</u> $g(x) \neq 0$, $y = c_1 y_1 + c_2 y_2 + y_p$

where $L[y_1] = L[y_2] = 0$, $W(y_1, y_2) \neq 0$, $L[y_p] = g(x)$

gen. $L[y]$ / \ const. coeff. $L[y]$

<u>variation of parameters</u>

try $y_p = u_1 y_1 + u_2 y_2$

with $u_1' y_1 + u_2' y_2 = 0$

↓

$$u_1' = -g y_2 / W(y_1, y_2)$$

$$u_2' = g y_1 / W(y_1, y_2)$$

$$y_p = -y_1 \int \frac{g y_2 \, dx}{W(y_1, y_2)} + y_2 \int \frac{g y_1 \, dx}{W(y_1, y_2)}$$

↑ anti-derivatives ↑

<u>undetermined coeff's.</u>

$$g(x) = e^{\alpha x} \times P_n(x) \times \begin{cases} \sin kx \\ \cos kx \end{cases}$$

↑ n^{th} order polynomial

try

$$y_p = x^s e^{\alpha x} [Q_n(x) \cos kx + R_n(x) \sin kx]$$

where $Q_n(x)$ & $R_n(x)$ are arbit. n^{th} order polynomials and s = multiplicity of $\alpha + ik$ as a root of char. eq.

<u>Higher Order Linear</u> $y^{(n)} + p_1(x) y^{(n-1)} + \dots + p_n(x) y = g(x)$

$y(x_0) = y_0$, $y'(x_0) = y_0'$, …, $y^{(n-1)}(x_0) = y_0^{(n-1)}$, $x_0 \in (\alpha, \beta)$ + $p_1(x), p_2(x), \dots, p_n(x)$ cont. on (α, β)

∴ ∃! soln. on (α, β)

(gen. soln. has n arbit. consts.)

<u>homogeneous</u> $g(x) = 0$, $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n = \sum_{i=1}^{n} c_i y_i$ gen. soln.

where $L[y_i] = 0$, $1 \leq i \leq n$, and $\{y_i\}$ linearly indep.

<u>non-homogeneous</u> $g(x) \neq 0$, $y = \sum_{i=1}^{n} c_i y_i + y_p$, $L[y_p] = g$

Higher Order Const. Coeff's. (R4)

$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$

try $y = e^{rx} \longrightarrow$ char. poly: $a_0 r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$

with roots $r_1, r_2, \dots, r_n$: 1. get $y_i = e^{r_i x}$ for unequal roots

2. for repeated roots r of multiplicity s, get $e^{rx}, xe^{rx}, x^2 e^{rx}, \dots, x^{(s-1)} e^{rx}$

when $r \in \mathbb{C}$, $\bar{r}$ also has multiplicity s, so get for $r = \alpha + i\beta$
$e^{\alpha x} \cos\beta x,\ e^{\alpha x} \sin\beta x,\ x e^{\alpha x} \cos\beta x,\ x e^{\alpha x} \sin\beta x, \dots, x^{(s-1)} e^{\alpha x} \cos\beta x,\ x^{(s-1)} e^{\alpha\beta} \sin\beta x$

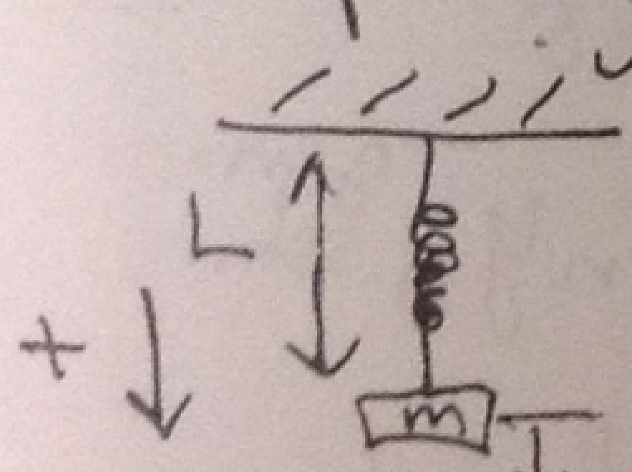
Computing roots: $(-1)^{1/n} = \left(e^{i(\pi + 2m\pi)}\right)^{1/n}$
$= \cos\pi\left(\frac{1+2m}{n}\right) + i \sin\pi\left(\frac{1+2m}{n}\right)$

$L[y] = g(x) = e^{\alpha x} \times P_n(x) \times \begin{cases} \sin kx \\ \cos kx \end{cases}$ try y_p using

undetermined coeff's as on p (R3) for 2nd order.

Modeling 1. Mixing $\frac{dQ}{dt}$ = rate in − rate out, $Q(0) = Q_0$

2. Springs $mu''(t) + \gamma u'(t) + ku(t) = F(t)$, $u(0) = u_0$



where $u(t)$ = displacement from equilibrium $u'(0) = u_0'$
weight = force of gravity = mg, m = mass
damping force = γv, v = velocity = $u'(t)$
Spring force = kx, x = displacement = $u(t)$
$F(t)$ = external force (this is an additional force applied separately from the previous ones)

L = natural length