Math 325 — **Midterm** — April 24, 2002

Please PRINT your name on the cover of the exam booklet. Write clearly and cross-out work not to be graded. Use the standard basis e_i such that $(e_i)_j = 0$ if $i \neq j$ and 1 if $i = j$, for all vector spaces.

1. **Define** ANY **3 out of 4** of the following: (15 pts.)
   (a) the **partial derivative** of the differentiable function $f : R^n \to R$ **with respect to** the kth component, for $1 \leq k \leq n$
   (b) an **affine** function $f : R^n \to R$
   (c) the **gradient, or derivative vector,** of a differentiable function $f : R^n \to R$ at $x_0 \in R^n$.
   (d) the **norm** $||A||$ of the $n \times m$ matrix $A = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$

2. Given the vectors $x_0 = (1,1)$ and $v = (2,0)$ in $\mathbf{R}^2$, let $f : \mathbf{R}^2 \to \mathbf{R}^3$ be such that $f(x_1, x_2) = (x_1^2 x_2^2, e^{x_1}, x_1 + x_2)$, and let $g : \mathbf{R}^3 \to \mathbf{R}^2$ be given by $g(x_1, x_2, x_3) = (x_1, 2x_2 + x_3^2)$.
   (a) Compute the matrix representing $Df(x_0)$. (5 pts.)
   (b) Find $Df(x_0)(v)$. (5 pts.)
   (c) Compute the matrix representing $D(f \circ g)(x_0)$. (10 pts.)
   (d) Is g invertible in some neighborhood of x_0? Why or why not? (10 pts.)

3. Use the method of Lagrange Multipliers to find the extrema of $f(x, y) = 3x + 2y$ subject to the constraint $2x^2 + 3y^2 = 3$. (10 pts.)

4. Given the bilinear form $H : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ whose matrix is (15 pts.)

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   compute $H(x, x)$ where $x = (1, -1, 1) \in \mathbf{R}^3$. Is H positive definite? Why or why not?

5. **Prove** ANY TWO OF the following THREE: (20 pts.)

   Theorem 1 *Let $f : B \subset \mathbf{R}^n \to \mathbf{R}$ be continuous, and suppose it is differentiable on the interior, int(B), of $B = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$. Further, suppose $f(x) = 0$ on the boundary of B, bd(B). Then there is a point $x_0 \in$ int(B) for which $Df(x_0) = 0$. (Hint: this can be a two line proof.)*

   Theorem 2 *If $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable on the open set $A \subset \mathbf{R}^n$, then f is continuous at $x_0 \in A$.*

Theorem 3 *Let $f : R^n \to R^m$ and suppose there is a constant M such that for $x \in R^n$, $\|f(x)\| \le M\|x\|^2$, then f is differentiable at $x_0 = 0$ and $Df(x_0) = 0$.*

6. **True or false** : (10 pts.)

   (a) The Implicit Function Theorem applies at points where the function is non-zero.

   (b) Every continuous function is differentiable.

   (c) The derivative of an everywhere differentiable function is zero at a point where a local maximum occurs.

   (d) The Inverse Function Theorem applies at points where the function is non-zero.

   (e) If the index of a critical point x_0, to which the Morse Lemma applies, is equal to zero, then x_0 is the location of a local minimum.

7. **Extra credit:** Is the parabola $y = x^2$ in the xy-plane is a one-manifold? If so, what mapping on R^2 could you use to prove your assertion? (+5 pts.)