Math 325 — **Final** — May 24, 1999

Please PRINT your name on the cover of the exam booklet. Write clearly and cross-out work not to be graded. Use the standard basis for all vector spaces.

1. **Define** ANY FOUR OF the following FIVE: (20 pts.)

   (a) the **derivative**, $Df(x_0)$, of a differentiable function $f : R^n \to R^m$ at $x_0 \in R^n$.

   (b) the **gradient** of a function $f : R^n \to R$ at $x_0 \in R^n$

   (c) the bounded function $f : R^n \to R$ is **integrable** on the bounded rectangle $R \subset R^n$ (include the definitions of upper and lower sums, etc.).

   (d) the unbounded, nonnegative function $f : R^n \to R$ is **integrable** on the bounded set $A \subset R^n$.

   (e) the **volume** of a bounded set in R^n.

2. Given $x_0 = (1, 3, 2) \in R^3$, and $u = (1, -1, 1) \in R^3$, let $f : R^3 \to R^2$ be such that $f(x_1, x_2, x_3) = (x_1^2 x_2, 1 + 2x_3)$, let $g : R^2 \to R^3$ be given by $g(x_1, x_2) = (x_1^2 + x_2^2, x_2, x_1)$, let $h : R^3 \to R$ such that $h(x_1, x_2, x_3) = x_1^3 - 3x_1^2 + x_2^2 + x_3^2$, and let $k : R^2 \to R$ be given by $k(x, y) = x^2 - y$. (30 pts.)

   (a) Find the directional derivative of h at x_0 in the direction u.

   (b) Compute the Hessian of h.

   (c) Find the critical points of h.

   (d) Identify these critical points as local maxima, minima, or saddle points.

   (e) Compute the matrix representing $D(g \circ f)(x_0)$.

   (f) Does the Change of Variables Theorem apply to $g \circ f : R^3 \to R^3$, taking (x_1, x_2, x_3) to (y_1, y_2, y_3), in some neighborhood of x_0? If so, express $dx_1 dx_2 dx_3$ in terms of $dy_1 dy_2 dy_3$.

   (g) Find $k^+(x, y)$, the positive part of k, as a function of $(x, y) \in R^2$.

3. **Prove** ANY ONE OF the following TWO: (10 pts.)

   Theorem 1 *If $f : R^n \to R^m$ is differentiable on the open set $A \subset R^n$, then $Df(x_0)$ is uniquely determined by f at $x_0 \in A$.*

   Theorem 2 *Let $f : R^n \to R$ be differentiable with A convex, and suppose $||gradf|| \leq M$ for all $x \in A$. Then $|f(x) - f(y)| \leq M||x - y||$ for $x, y \in A$.*

4. Let $f : [0, 2] \to R$ be defined by $f(x) = 0$ for $0 \leq x \leq 1$, and by $f(x) = 1$ for $1 < x \leq 2$. Compute, *using the definition or Riemann's condition*, $\int_0^2 f(x)\, dx$. (10 pts.)

5. Determine which of the following sets has **measure zero**. (5 pts.)

   (a) $\{(x, y) \in R^2 : x^2 + y^2 \leq 1; x, y \in Q\}$
   (b) xy-plane in R^3
   (c) the irrationals in $[0, 1] \subset R$
   (d) the irrationals in $[0, 1] \subset R^2$
   (e) the integers $Z \subset R$

6. Does the set $\{(x, y) \in R^2 : x^2 + y^2 \leq 1; x, y \in Q\}$ from part (a) have volume? Explain. (5 pts.)

Prove ANY ONE OF the following TWO: (10 pts.)

Theorem 3 *If $f : R^n \to R$ is continuous at $x_0 \in R^n$, then the oscillation of f at x_0, $osc(f, x_0) = 0$.*

Prove ANY ONE OF the following TWO: (10 pts.)

Theorem 4 *Suppose $A \subset R^n$ is convex, closed and bounded, and such that $\int_A \|_A$ exists and $0 < \int_A \|_A < \infty$. Suppose $f : R^n \to R$ is continuous and $\int_A f = 0$. Then there exists $x_0 \in A$ such that $f(x_0) = 0$. (Hint: this can be a two line proof.)*

Theorem 5 *Let $f : [a, b] \to R$ be continuous, then f is integrable on $[a, b]$. (Hint: use Riemann's condition.)*