Math 324 — **Exam #2** — May 12, 2003

Reminder,the FINAL EXAM for this class is Monday, May 19, 6:15–8:30 pm, in NAC 6/113.

Please PRINT your name and ID# on the cover of each exam book you use. Write clearly and cross-out work not to be graded. Make your arguments rigorous and brief. Partial credit will be given. Total: 100 pts. plus extra credit.

1. Give **definitions** of EACH of the following (boldface) words. (20 pts.)
   (a) A subset K of a metric space (M, d) is **compact**.
   (b) A subset C of a metric space (M, d) is **path–connected**. (You must include the definition of a continuous path in M, though you may assume the definition of continuous functions on $\mathcal{R}$.)
   (c) A function $f : (M, d) \to (N, \rho)$ between metric spaces is **continuous at** the point x_o in its domain. (You may use $\epsilon - \delta$ or limits.)
   (d) A subset A of the metric space (M, d) is **disconnected**.

2. **True or false**? (For **extra credit**, give a counterexample for each false statement.) (30+ pts.)
   (a) Every compact set is closed.
   (b) Every connected set is compact.
   (c) The continuous image of every compact set is compact.
   (d) The continuous image of every compact set is closed.
   (e) The continuous image of every closed set is closed.
   (f) Every connected set is path connected.
   (g) A subset of $\mathcal{R}$ is path connected if and only if it is an interval.
   (h) If $f : (M, d) \to (N, \rho)$ is continuous on $A \subset M$, then for every convergent sequence $x_k \to x$ in A, $f(x_k) \to f(x)$.
   (i) If $f : (M, d) \to (N, \rho)$ is continuous and $S \subset M$ is connected, then the inverse image $f^{-1}(S)$ is connected.
   (j) If $C \subset N$ is compact, and $f : (M, d) \to (N, \rho)$ is continuous, then the inverse image $f^{-1}(C)$ is closed.

Exam continues. Please TURN OVER.

3. For each of the following sets, determine whether it is **compact**, and whether it is **connected**: (20 pts.)

   (a) $\{(x_1, x_2) \in \mathcal{R}^2 \mid |x_1| \leq 1\}$.

   (b) $\{x \in \mathcal{R}^n \mid \|x\| \leq 10\}$.

   (c) $\{x \in \mathcal{R}^n \mid 1 \leq \|x\| \leq 2\}$.

   (d) the integers: $\mathcal{Z} \subset \mathcal{R}$

   (e) a finite set in $\mathcal{R}$.

   (f) $\{x \in \mathcal{R} \mid \|x\| = 1\}$.

   (g) $\{x \in \mathcal{R}^n \mid n \geq 2 \text{ and } \|x\| = 1\}$.

   (h) the boundary of a bounded set in $\mathcal{R}$.

   (i) the rationals in $[0, 1]$.

   (j) a closed set in $[0, 1]$.

4. **Prove** ANY TWO OF the following THREE theorems: (30 pts.)

   Theorem 1 *A subset of $\mathcal{R}$ is connected if and only if it is an interval.*

   Theorem 2 *Let A be connected subset of the metric space (M, d) and $f : A \to \mathcal{R}$ be continuous with $f(x) \neq 0$ for all $x \in A$, then $f(x) > 0$ for all $x \in A$ or else $f(x) < 0$ for all $x \in A$.*

   Theorem 3 *If a subet A of a metric space (M, d) is connected and contains more than one point, then every point of A is an accumulation point of A.*