Math 323 — **Midterm** — April 13, 2005

Please PRINT your name and ID# on the cover of your exam book. Write clearly and cross-out work not to be graded. Make your arguments rigorous and brief. Partial credit will be given.

1. Give **definitions** of ANY FOUR of the following five symbols or **(boldface)** words. (20 pts.)

   (a) The sequence (s_n) of reals is **nonincreasing**.

   (b) The real-valued function f is **continuous at** $x_o \in dom(f)$. (You may use either sequences or $\epsilon - \delta$.)

   (c) The sequence (s_n) of reals **converges to** $s \in R$.

   (d) The sequence (s_n) of reals has **limit (i.e. diverges to)** $-\infty$.

   (e) The **series** $\sum a_n$ **satisfies the Cauchy criterion**. (Do not presume any definitions associated with sequences.)

2. For each of the following statements, decide whether is true or false. If it is true, **prove it**; if false, **give a counterexample**. (30 pts.)

   (a) If (s_n) is sequence of reals and $\lim(1/s_n) = 0$, then $\lim s_n = +\infty$.

   (b) All bounded nondecreasing sequences of reals converge.

   (c) All sequences of reals have a convergent subsequence.

   (d) If (s_n) is a convergent sequence, then it is Cauchy.

3. Give an **example** of each of the following: (5 pts.)

   (a) A single sequence (s_n) of reals such that $\liminf s_n = -\infty$, $\limsup s_n = +\infty$, and (s_n) has a convergent subsequence.

   (b) A function that is continuous, but not uniformly continuous, on $(0, 1)$.

   (c) A bounded set of reals with a minimum, but no maximum.

   (d) A convergent series Σa_n such that $\Sigma|a_n|$ diverges, but Σa_n^2 converges.

   (e) An unbounded sequence of reals that has no limit.

4. **State**, but do not prove, the **Ratio Test** for convergence of series (Theorem 14.8). (5 pts.)

5. Use the **negation of the definition** to prove that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$. (10 pts.)

EXAM CONTINUES, please TURN OVER.

6. **Prove** EITHER Theorem 1, OR Theorems 2 and 3: (30 pts.)

Theorem 1 (Generalized Extreme Value Theorem) *If f is continuous on a* **closed and bounded set** *$S \subset R$, then f is bounded on S. Moreover, f attains its bounds, i.e. there exist $x_0, y_0 \in S$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in S$.*

(Hint: the proof proceeds in the same way as that for the theorem on a closed interval. By the way, you are actually proving that in the metric space R, a continuous function on a compact set is bounded and attains its bounds. Isn't that exciting?)

Theorem 2 *If f is uniformly continuous on a set $S \subset R$ and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.*

Theorem 3 *Let (a_n) and (b_n) be two sequences of reals such that (b_n) converges to $b \in R$, and there exist real numbers a and M, and a natural number N, such that*

$$|a_n - a| \le M|b_n - b| \quad \forall n > N,$$

then the sequence (a_n) converges to a.

(**Note:** you MAY NOT presume the "Sandwich Theorem." To get credit, your proof must use only basic principles and definitions.)