Math 323 — **FINAL** — May 23, 2005

Please PRINT your name and ID# on the cover of each exam book you use, indicating if you are handing-in more than one. Write clearly and cross-out work not to be graded. Make your arguments rigorous and brief. In the cases where you have learned a definition that can be stated equivalently using sequences or using "$\epsilon - \delta$," you may give either. Partial credit will be given. Total: 100 pts.

1. Give **definitions** of ANY FOUR of the following five symbols or (boldface) words. (20 pts.)
   - (a) The sequence of reals (s_n) is **Cauchy**.
   - (b) $\liminf s_n$ of a sequence of reals (s_n).
   - (c) $\lim_{x\to a^+} f(x) = +\infty$, when $a \in R$ and $f(x)$ is a real-valued function.
   - (d) The series $\sum_{n=1}^{\infty} a_n$ of reals **converges**. (You may assume the definition for sequences, but define all terms peculiar to series.)
   - (e) The **lower Darboux sum** $L(f, P)$ of a bounded function f on the interval $[a, b]$ with respect to the partition $P = \{t_0 = a < t_1 < \ldots < t_n = b\}$.

2. (a) **State and use the definition** to prove that $f(x) = x^2$ is uniformly continuous on $[-1, 1]$. (5 pts.)
   - (b) **State and use the negation of the definition** to prove that $\lim s_n \neq 2/3$ when $s_n = 2/(3n)$. (10 pts.)

3. Let $f_n(x) = x^n/(n + x^n)$ for $x \in [0, \infty)$: (10 pts.)
   - (a) Find $f(x) = \lim f_n(x)$.
   - (b) Does $f_n \to f$ uniformly on $[0, 1]$? Justify your answer.
   - (c) Does $f_n \to f$ uniformly on $[1, \infty)$? Justify your answer.

4. **Give** a counterexample to each of the following false statements: (5 pts.)
   - (a) $\liminf s_n = \limsup s_n$ of any bounded sequence s_n.
   - (b) If $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ and $0 \in dom(f)$, then $f(x)$ is continuous at $x = 0$.
   - (c) If the sequence of functions $(f_n(x))$ converges to a discontinuous function $f(x)$ on a set S, then the sequence does not converge uniformly on S.
   - (d) If a function $f(x)$ is defined on $[-1, 1]$ and continuous at $x = 0$, then f is continuous on some open interval containing zero.
   - (e) Every sequence has a convergent subsequence.

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5. **Prove** ANY TWO OF the following THREE theorems: (20 pts.)

   Theorem 1 *If* (s_n) *is a sequence of reals with* $\limsup s_n = +\infty$ *and* $k > 0$ *is a real number, then* $\limsup(ks_n) = +\infty$.

   Theorem 2 *Every convergent sequence of reals is bounded.*

   Theorem 3 *If* f *is differentiable on* R *and* (s_n) *is a Cauchy sequence in a closed interval* $[a, b]$, *then the sequence* $(f(s_n))$ *converges.*

   Hint: this can be a two line proof.

6. **Prove** ANY TWO OF the following THREE theorems: (30 pts.)

   Theorem 4 *If* f *is a continuous function on a closed and bounded set* $S \subseteq R$, *then* f *is uniformly continuous on* S.

   Hint: assume not and use the Bolzano-Weierstrass Theorem.

   Theorem 5 *A bounded function* f *on* $[a, b]$ *is integrable if and only if for each* $\epsilon > 0$ *there exists a partition* P *of* $[a, b]$ *such that* $U(f, P) - L(f, P) < \epsilon$, *where* $U(f, P)$, *respectively* $L(f, P)$, *denotes the upper, respectively lower, Darboux sum of* f *with respect to* P.

   Theorem 6 *Suppose* f *is differentiable on* R, *select* $s_0 \in R$ *and define* $s_n = f(s_{n-1})$ *for* $n \geq 1$ *(thus* $s_1 = f(s_0)$, $s_2 = f(s_1)$, *etc.). Furthermore, suppose that* $a = \sup\{|f'(x)| : x \in R\} < 1$, *then* (s_n) *is a convergent sequence.*

   Hint: show that $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|$ for $n \geq 1$ and think Cauchy.