6.1 revised 11/29/2023

2) $y = 8\sqrt{x}$ $\qquad y = \frac{1}{4}x^3$ $\qquad$ intersects: $(0,0)$, $(4,16)$

$y\big|_{x=1} = 8\sqrt{(1)} = 8$ $\qquad y\big|_{x=1} = \frac{1}{4}(1)^3 = \frac{1}{4}$ $\qquad$ test at $x=1$

Upper Curve $\qquad$ lower Curve

$h(x) = \text{Upper} - \text{lower} = (8\sqrt{x}) - (\frac{1}{4}x^3) = (8\sqrt{x} - \frac{1}{4}x^3)$

$$A = \int_0^4 (8\sqrt{x} - \tfrac{1}{4}x^3)\,dx = \int_0^4 (8x^{\frac{1}{2}} - \tfrac{1}{4}x^3)\,dx$$

$$= \left[8\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] - \frac{1}{4}\left[\frac{x^4}{4}\right] + C\right]_0^4 = \left[\frac{16}{3}(\sqrt{x})^3 - \frac{1}{16}x^4 + C\right]_0^4$$

$$= \left[\frac{16}{3}(\sqrt{(4)})^3 - \frac{1}{16}(4)^4 + C\right] - \left[\frac{16}{3}(\sqrt{(0)})^3 - \frac{1}{16}(0)^4 + C\right]$$

$$= \left[\frac{16}{3}(2)^3 - \frac{1}{16}(4)^2(4)^2\right] - [0 - 0] = \frac{128}{3} - 16$$

$$= \frac{128}{3} - \frac{48}{3} = \underline{\underline{\frac{80}{3}\ \text{units}^2}}$$

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4) $y = \sqrt{x+2}$ $\qquad y = \frac{1}{x+1}$ $\qquad$ interval: $0 \le x \le 2$

$y\big|_{x=1} = \sqrt{(1)+2} = \sqrt{3}$ $\qquad y\big|_{x=1} = \frac{1}{(1)+1} = \frac{1}{2}$ $\qquad$ test at $x=1$

Upper Curve $\qquad$ lower Curve

$h(x) = \text{Upper Curve} - \text{lower Curve} = (\sqrt{x+2}) - (\frac{1}{x+1}) = (\sqrt{x+2} - \frac{1}{x+1})$

$$\int \sqrt{x+2}\,dx = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C_1 = \frac{2}{3}(\sqrt{u})^3 + C_1$$

$u = x+2$

$du = dx$

$$= \underbrace{\frac{2}{3}(\sqrt{x+2})^3}\, + C_1$$

4) continued...

$$\int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du = \int u^{-1}\,du = \ln|u| + C_2$$

$u = x+1$

$du = dx$

$$= \underbrace{\ln|x+1|}\, + C_2$$

$$A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx = \left[\frac{2}{3}\left(\sqrt{x+2}\right)^3 - \ln|x+1| + C\right]_0^2$$

$$= \left[\frac{2}{3}\left(\sqrt{(2)+2}\right)^3 - \ln|(2)+1| + C\right] - \left[\frac{2}{3}\left(\sqrt{(0)+2}\right)^3 - \ln|(0)+1| + C\right]$$

$$= \left[\frac{2}{3}(2)^3 - \ln|3|\right] - \left[\frac{2}{3}(\sqrt{2})^3 - \ln|1|\right]$$

$$= \left[\frac{16}{3} - \ln(3)\right] - \left[\frac{2}{3}(2\sqrt{2}) - 0\right] = \underline{\underline{\frac{16-4\sqrt{2}}{3} - \ln(3) \text{ units}^2}}$$

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6) $y = x^4$, $y = -x-1$ $\qquad x=0,\ x=1 \Rightarrow 0 \le x \le 1$

$y\big|_{x=\frac{1}{2}} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ $\qquad y\big|_{x=\frac{1}{2}} = -\left(\frac{1}{2}\right) - 1 = \frac{-3}{2}$ $\qquad$ test at $x = \frac{1}{2}$

Upper Curve $\qquad$ lower Curve

$$h(x) = \text{Upper} - \text{lower} = (x^4) - (-x-1) = (x^4 + x + 1)$$

$$A = \int_0^1 (x^4 + x + 1)\,dx = \left[\frac{x^5}{5} + \frac{x^2}{2} + x + C\right]_0^1$$

$$= \left[\frac{(1)^5}{5} + \frac{(1)^2}{2} + (1) + C\right] - \left[\frac{(0)^5}{5} + \frac{(0)^2}{2} + (0) + C\right]$$

$$= \left[\frac{1}{5} + \frac{1}{2} + 1\right] - [0] = \frac{2}{10} + \frac{5}{10} + \frac{10}{10} = \underline{\underline{\frac{17}{10} \text{ units}^2}}$$

8) $y = \sqrt{x}$, $y = 3x^2 + 1$ $\qquad x = 1, x = 4 \Rightarrow 1 \le x \le 4$

test at $x = 2$

$y\big|_{x=2} = \sqrt{(2)} = \sqrt{2}$ $\qquad y\big|_{x=2} = 3(2)^2 + 1 = 13$

Lower Curve $\qquad$ Upper Curve

$$h(x) = (3x^2+1) - (\sqrt{x}) = (3x^2 + 1 - \sqrt{x})$$

$$A = \int_1^4 (3x^2 + 1 - \sqrt{x})\,dx = \int_1^4 (3x^2 + 1 - x^{\frac{1}{2}})\,dx$$

$$= \left[3\left(\frac{x^3}{3}\right) + x - \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C\right]_1^4 = \left[x^3 + x - \frac{2}{3}(\sqrt{x})^3 + C\right]_1^4$$

$$= \left[(4)^3 + (4) - \frac{2}{3}(\sqrt{(4)})^3 + C\right] - \left[(1)^3 + (1) - \frac{2}{3}(\sqrt{(1)})^3 + C\right]$$

$$= \left[64 + 4 - \frac{16}{3}\right] - \left[1 + 1 - \frac{2}{3}\right] = \left[68 - \frac{16}{3}\right] - \left[2 - \frac{2}{3}\right]$$

$$= 68 - \frac{16}{3} - 2 + \frac{2}{3} = 66 - \frac{14}{3} = \frac{198}{3} - \frac{14}{3} = \underline{\underline{\frac{184}{3} \text{ units}^2}}$$

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10) $y = x^2 - 4x$, $y = 2x$

since the intervals are not given, we must find intersection points of these functions.

note: there may be more than 2 intersection points.

$x^2 - 4x = 2x$

$x^2 - 6x = 0$

$x(x-6) = 0$

$x = 0 \mid x - 6 = 0$

$\quad\;\; x = 6$

interval $\Rightarrow 0 \le x \le 6$

test at $x=1$:

$y\big|_{x=1} = (1)^2 - 4(1) = -3$ Lower Curve

$y\big|_{x=1} = 2(1) = 2$ Upper Curve

10) continued...

$h(x) = \text{Upper} - \text{lower} = (2x) - (x^2 - 4x) = (6x - x^2)$

$A = \int_0^6 (6x - x^2)\,dx = \left[6\left(\frac{x^2}{2}\right) - \frac{x^3}{3} + C\right]_0^6 = \left[3x^2 - \frac{1}{3}x^3 + C\right]_0^6$

$= \left[3(6)^2 - \frac{1}{3}(6)^3 + C\right] - \left[3(0)^2 - \frac{1}{3}(0)^3 + C\right]$

$= \left[3(36) - 2(36)\right] - \left[0 - 0\right] = 36\left[3 - 2\right] = 36\left[1\right] = \underline{\underline{36 \text{ units}^2}}$

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14) $y = x^2$, $y = 4x - x^2$

intersection:

$x^2 = 4x - x^2$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$2x = 0 \mid x - 2 = 0$

$x = 0 \mid x = 2$

interval

$\Rightarrow 0 \le x \le 2$

test at $x=1$:

$y|_{x=1} = (1)^2 = 1$ lower

$y|_{x=1} = 4(1) - (1)^2 = 3$ Upper

$h(x) = \text{Upper} - \text{lower} = (4x - x^2) - (x^2) = (4x - 2x^2)$

$A = \int_0^2 (4x - 2x^2)\,dx = \left[4\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right) + C\right]_0^2 = \left[2x^2 - \frac{2}{3}x^3 + C\right]_0^2$

$= \left[2(2)^2 - \frac{2}{3}(2)^3 + C\right] - \left[2(0)^2 - \frac{2}{3}(0)^3 + C\right]$

$= \left[8 - \frac{16}{3}\right] - \left[0 - 0\right] = \frac{24}{3} - \frac{16}{3} = \underline{\underline{\frac{8}{3} \text{ units}^2}}$

16) $y = x^3 - x, \ y = 3x$ $\quad 0 \leq x \leq 2$

since these are more complex functions, we intersection points and check if we get intersection between interval given above

$$x^3 - x = 3x$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$
$$(x+2)(x)(x-2) = 0$$

$x+2=0$, $x=-2$ | $x=0$ | $x-2=0$, $x=2$

test at $x=1$: $y\big|_{x=1} = (1)^3 - (1) = 0$ lower

$y\big|_{x=1} = 3(1) = 3$ Upper

$$h(x) = \text{Upper} - \text{lower} = (3x) - (x^3 - x) = (4x - x^3)$$

$$A = \int_0^2 (4x - x^3)\,dx = \left[4\left(\frac{x^2}{2}\right) - \left(\frac{x^4}{4}\right) + C\right]_0^2 = \left[2x^2 - \frac{1}{4}x^4 + C\right]_0^2$$

$$= \left[2(2)^2 - \frac{1}{4}(2)^4 + C\right] - \left[2(0)^2 - \frac{1}{4}(0)^4 + C\right]$$

$$= [8 - 4] - [0 - 0] = \underline{\underline{4 \text{ units}^2}}$$

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