Modern Algebra Final

1. If x_1 and x_2 are solutions to $x^2 + bx + c = 0$, find formulas for:

a) $x_1^2 + x_2^2$ in terms of b and c.

b) $x_1^3 \cdot x_2^3$ in terms of b and c.

c) $x_1^3 + x_2^3$ in terms of b and c.

2. Identify the symmetry group of a baseball, taking the stitching into account.

3. Solve the cubic $x^3 + 6x - 4 = 0$. (Hint: you may want to make the substitution $x = z - \frac{p}{3z}$ that Dickson used to solve the cubic $x^3 + px + q = 0$.)

4. a) Find the subgroup H of $G_{4!}^{(4)}$ which leaves the polynomial $\Psi = x_1x_3 + x_2x_4$ unaltered.

b) Determine if the subgroup H from part a) is self-conjugate.

5. a) List all 3-cycles in the symmetric group of four letters $G_{4!}^{(4)}$. (Recall: 3-cycles are circular substitutions of three letters)

b) Do the three cycles form a subgroup of $G_{4!}^{4}$?

c) Are the three cycles even or odd substitutions?

d) How many three cycles are there in $G_{5!}^{5}$, the substitution group on five letters?

6. a) List the elements in the subgroup H of $G_{5!}^{(5)}$ which leave the polynomial $\Psi = (x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5)^5$ unaltered, when ω is a primitive fifth root of unity $\omega^5 = 1$.

b) Is the subgroup H from part a) self-conjugate?

7. a) Find, D_4, the symmetry group of the square as a subgroup of $G_{4!}^{4}$.

b) Find the subgroup H of D_4 that leaves one vertex of the square invariant (it does not matter which vertex you make invariant—or how you labeled the vertices).

c) Is H from part b) self-conjugate as a subgroup of $G_{4!}^{4}$?

d) Is H from part b) self-conjugate as a subgroup of D_4?

8. For each subgroup K of $G^3_{3!}$ find a polynomial on three letters $f(x_1, x_2, x_3)$ so that K is the subgroup belonging to K. (Hint: there are six subgroups of $G^3_{3!}$).

9. The order of a subgroup K of $G^n_{n!}$ is less than 300. K itself has subgroups L and M of order 45 and 75 respectively. What is the order of K?

10. Prove that (132)(45) is conjugate to (241)(35) by finding a substitution g on five letters so that $g^{-1}(123)(45)g = (241)(35)$.

11. a) Prove that if a self-conjugate subgroup H of $G^5_{5!}$ contains the circular substitutions (12) and (12345) then $H = G_{5!}$.

b) Prove that if a subgroup H (not necessarily self-conjugate) of $G^5_{5!}$ contains the circular substitutions (12) and (12345) then $H = G_{5!}$.

12. Find the subgroup of $G^4_{4!}$ corresponding to the symmetries of the cube that leave one of the four diagonals of the cube invariant (it does not matter which diagonal you choose to leave fixed).