Department of Mathematics
Final Exam

Math 39100 Spring 2006

Part I (71pts) Answer **all** problems.

1. (8pts) Find the general solution to
$$y' + 2ty = 2te^{t^2}$$
.

2. (9pts) Find the solution to
$$y' = (1 - 4x)y^2, \quad y(0) = -\frac{1}{2}.$$

3. (8pts) Find the general solution to
$$2y^2 + \sin x + (4xy + \cos y)\frac{dy}{dx} = 0.$$

4. (a) (5pts)Find the general solution to the differential equation:
$$y'' - 2y' - 3y = 0$$
(b) (7pts) Use part a) to help find the general solution to
$$y'' - 2y' - 3y = 3\sin t.$$

5. (8pts) Find all solutions to the differential equation:
$$2x^2y'' + 3xy' + 4y = 0, \ x > 0.$$

6. (9pts) Find an approximation of the general power series solution, around $x_0 = 0$, for the differential equation:
$$y'' + 2xy' + y = 0,$$
by giving the first five terms of the series solution (the series solution through the x^4 term).

7. (8pts) Use separation of variables to replace the partial differential equation:
$$tu_{xx} + xu_t = 0,$$
where u is a function of x and t with two ordinary differential equations. Do **NOT** solve the equations.

8. (9pts) Find the Fourier series of the function defined by
$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ -2x & -1 < x < 0 \end{cases}$$
and $f(x + 2) = f(x)$ for all x.

Exam continues on reverse side.

Part II Answer 3 out of 4 problems (21pts)

9. (7pts) At time t=0, a tank with capacity 10 gallons, contains 5 gallons of pure water (no salt in it). A solution containing 3 grams of salt per gallon enters the tank at a rate of 2 gallons per minute, and the well stirred mixture leaves the tank at a rate of 1 gallon per minute.

   a) Give the equation for the amount of salt in the tank at any time before the tank overflows.

   b) Find the amount of salt in the tank at the time it overflows.

10. (7pts) Given the differential equation

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0,$$

notice that one solution is $y = t$. Use this fact to help find ALL solutions to the above differential equation.

11. (7pts) Using the Laplace transform, solve the following differential equation:

$$y'' - 9y = -18 \qquad y(0) = -1, \quad y'(0) = 9.$$

12. (7pts) A mass with $m = 2\frac{\text{lb-sec}^2}{\text{ft}}$ is connected to a spring with spring constant $k = 4\frac{\text{lb}}{\text{ft}}$. The damping is such that when the mass is traveling at a velocity of $1\frac{\text{ft}}{\text{sec}}$ the resistance is 4lbs of force. Suppose the mass starts at its equilibrium position with a downward velocity of 3 $\frac{\text{ft}}{\text{sec}}$. Answer the following two questions:

   i) Find the general equation of motion for the system.

   ii) If, after 1 second, you held your hand 2 ft below the weight's starting position, would the weight eventually hit your hand? *To get credit, you will need to explain your answer.*

Part III (8pts) Answer **both** problems.

13. (5pts) Answer the following questions about the differential equation:

$$6x^2(x+1)^2y'' + \frac{1}{2}x(x+2)y' + y = 0.$$

   i) Does this equation have singular points? If so, where are they and are they regular or irregular?

   ii) Find all values of r for which there is a series solution of form $x^r \sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$. Find all values of r for which there is a series solution of form $x^r \sum_{n=0}^{\infty} a_n (x-2)^n$, with $a_0 \neq 0$. **DO NOT TRY TO SOLVE THE PROBLEM, JUST FIND THE VALUES OF r WITHOUT SOLVING FOR ANY CONSTANTS.**

14. (3pts) Show that if $y_p(t)$ is a solution of:

$$y'' + p(t)y' + q(t)y = g(t),$$

and $cy_1(t) + dy_2(t)$ (where c and d are arbitrary constants) represent all solutions of:

$$y'' + p(t)y' + q(t)y = 0,$$

then, $y_p(t) + cy_1(t) + dy_2(t)$ represent all solutions of

$$y'' + p(t)y' + q(t)y = g(t).$$