CONIC SECTIONS AND ROTATION OF AXES

CONICS

Concerning parabolas, ellipses, hyperbolas, and circles, and their (as may apply) vertices, asymptotes, foci, and all that good stuff, here's what you need to know.

Parabola

The equation of a parabola with a vertical axis (upward or downward opening) can be expressed as:

$$\boxed{y = a(x-h)^2 + k}$$

This is called the "Standard form (of a parabola)". When in this form, the following hold:

Vertex: (h, k) Focus: $\left(h, k + \frac{1}{4a}\right)$ Directrix: $y = k - \frac{1}{4a}$

If $a > 0$, the parabola opens upward.

$a > 0$

If $a < 0$, the parabola opens downward.

$a < 0$

The equation of a parabola with a horizontal axis (leftward or rightward opening) can be expressed as:

$$\boxed{x = a(y-h)^2 + k}$$

This is called the "Standard form (of a parabola)". When in this form, the following hold:

Vertex: (k, h) Focus: $\left(k + \frac{1}{4a}, h\right)$ Directrix: $x = k - \frac{1}{4a}$

If $a > 0$, the parabola opens rightward.

$a > 0$

If $a < 0$, the parabola opens leftward.

$a < 0$

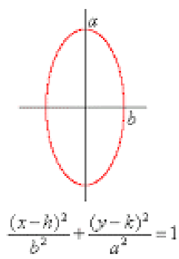
Ellipse

The equation of an ellipse with a vertical major axis can be expressed as:

$$\boxed{\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1} \text{ for } a \geq b > 0$$

When in this form, the following hold:

Center: (h, k) (Major) Vertices: $(h, k \pm a)$ Foci: $(h, k \pm c)$ .....where $c^2 = a^2 - b^2$

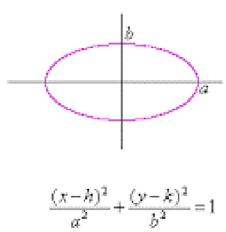


The equation of an ellipse with a horizontal major axis can be expressed as:

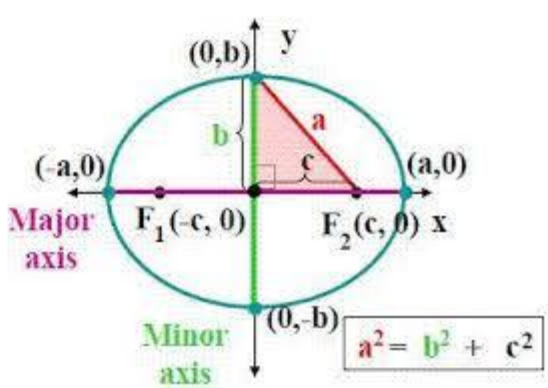
$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1} \text{ for } a \geq b > 0$$

When in this form, the following hold:

Center: (h, k) (Major) Vertices: $(h \pm a, k)$ Foci: $(h \pm c, k)$ .....where $c^2 = a^2 - b^2$



To illustrate where the equation $c^2 = a^2 - b^2$ comes from, the image below may be helpful:



Hyperbola

The equation of an East-West opening hyperbola can be expressed as:

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$

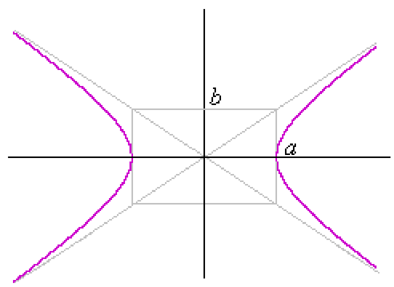
When in this form, the following hold:

Center: (h, k)

Vertices: $(h \pm a, k)$

Asymptotes: $y = \pm \frac{b}{a}(x-h) + k$

Foci: $(h \pm c, k)$ ......where $c^2 = a^2 + b^2$



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The equation of a North-South opening hyperbola can be expressed as:

$$\boxed{\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1}$$

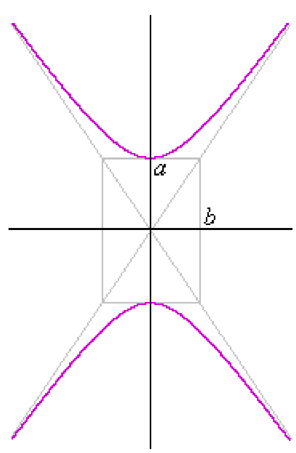
When in this form, the following hold:

Center: (h, k)

Vertices: $(h, k \pm a)$

Asymptotes: $y = \pm \frac{a}{b}(x-h) + k$

Foci: $(h, k \pm c)$ ......where $c^2 = a^2 + b^2$



$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

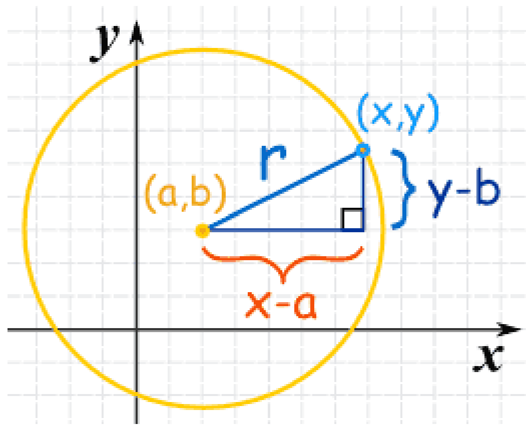
Circle

The most famous kind of conic section; you're probably familiar with it, but for completeness:

The equation of a circle can be expressed in the form:

$$(x-h)^2+(y-k)^2=r^2$$

where the center is (h,k) and the radius is r.



Now on to some examples!

Problem 1: What is the directrix of the parabola whose equation is $y+3=\frac{1}{10}(x+2)^2$?
Ans: $y=-\frac{11}{2}$

Problem 2: Identify the conic and state its center: $9x^2+16y^2-18x+64y=71$.
Ans: It is an ellipse with center (1,-2). Try sketching this. For fun.

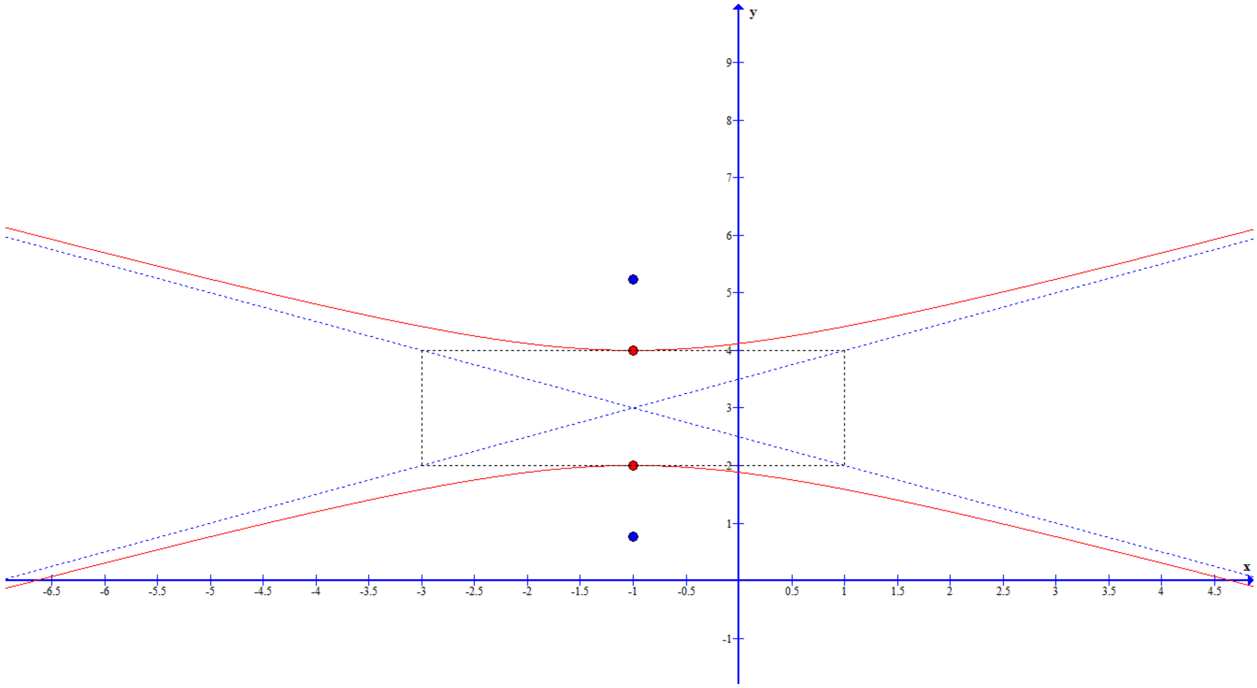
Problem 3: What are the foci of the given equation: $7(x-2)^2+3(y-2)^2=21$?
Ans: This is an ellipse, its foci are (2,4) and (2,0). Again, sketch.

Problem 4: (From Fall 2014 final, problem 10(a)) Draw a sketch of the conic whose equation is

$$4y^2-x^2-2x-24y+31=0$$

Identify which sort of conic it is. On your sketch, show and label whichever of the following are present: vertices, asymptotes, and foci.

Ans: Note that this equation can be simplified to $(y-3)^2-\frac{(x+1)^2}{4}=1$, hence it is a hyperbola with center $(-1,3)$, vertices $(-1,4)$ and $(-1,2)$, foci $\left(-1,3\pm\sqrt{5}\right)$ and asymptotes $y=\pm\frac{1}{2}(x+1)+3$. The sketch is below. Note that the graph is the red line, the foci are the blue dots and the vertices are the red dots. The rectangle is to aid in graphing, and the extended diagonals of the rectangle are the asymptotes.



ROTATION OF AXES

There is more to this topic (as you can see from the handout on the website), but something that would be nice to know is how to rotate a general conic so that it looks like (after a change of variable) one of the conics we've dealt with in this lecture.

The general equation of a conic section is:

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

(provided it is not degenerate) if the xy-term has a non-zero coefficient, your conic will be rotated. To figure out what angle you must rotate it by to eliminate the xy-term, do the following:

> To eliminate the xy-term in the general quadratic (conic) equation
>
> $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
>
> Rotate the coordinate axes through an angle of θ satisfying:
>
> $$\cot 2\theta = \frac{A - C}{B}$$

Let's do an example.

Problem 5: Given the equation $x^2 + 8\sqrt{3}x + 2\sqrt{3}xy + 3y^2 - 8y = 0$, find the angle of rotation needed to eliminate the xy-term in the equation above.
Ans: $\theta = -\frac{\pi}{6}$

(If you're interested in seeing more problems for Rotation of Axes, let me know, I can oblige ☺)