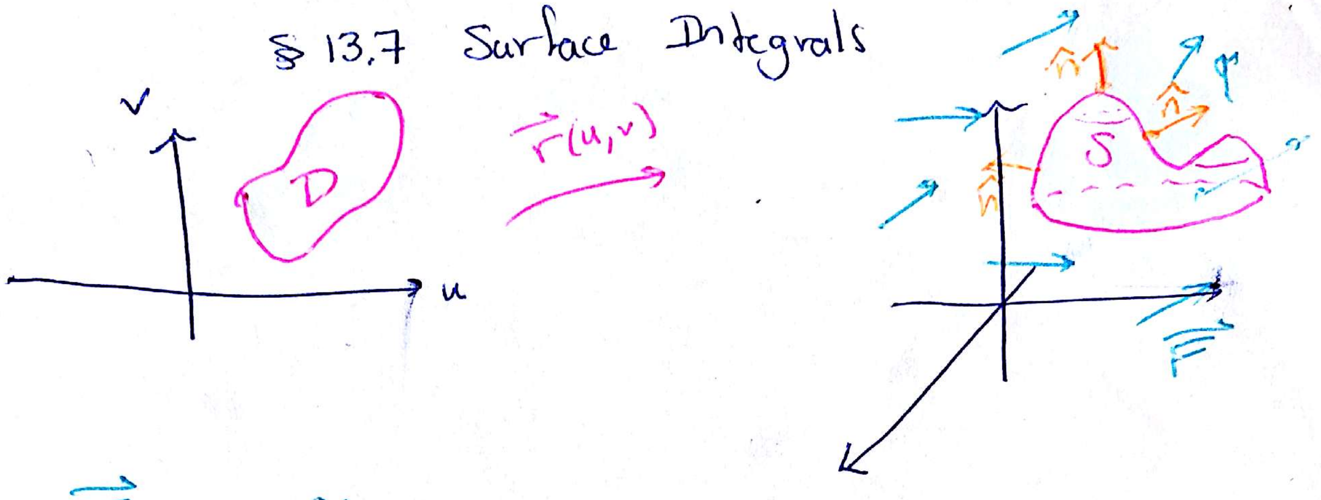
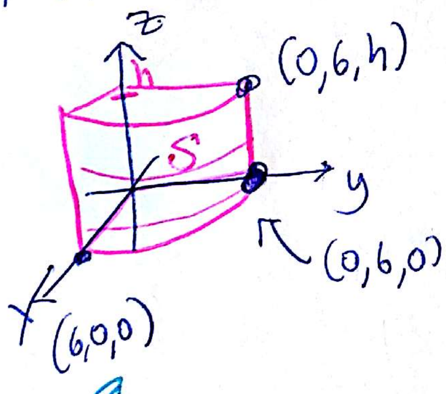
§ 13.7 Surface Integrals



$$\text{flux } \vec{F} \text{ through } S = \iint_S \vec{F} \cdot \underset{\hat{n}\,dS}{\overset{\shortparallel}{d\vec{S}}} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)\,du\,dv}_{\pm d\vec{S}}$$

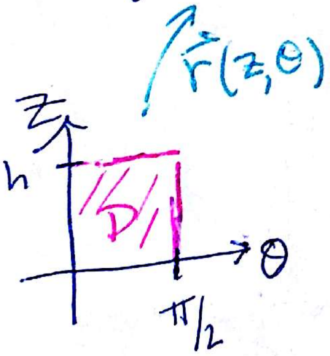
Ex 1) Find the flux of $\vec{F}(x,y,z) = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$ outward through the portion of the cylinder $x^2+y^2=36$ in the first octant below the plane $z=h$.



$$\vec{r}(z,\theta) = 6\cos\theta\,\hat{\imath} + 6\sin\theta\,\hat{\jmath} + z\hat{k}$$

$$\hat{n} = \frac{\langle x, y, 0\rangle}{6} \qquad dS = 6\,dz\,d\theta$$

$$\text{flux} = \iint_S \vec{F} \cdot \hat{n}\,dS = \iint_S \frac{xz+xy}{6}\,dS$$



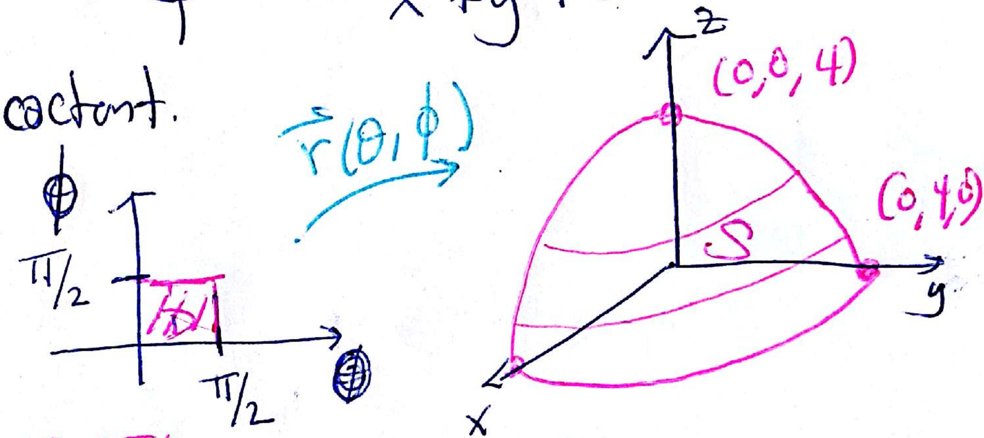
$$= \iint_D 6\cos\theta \cdot z + 6^2\cos\theta\sin\theta\,dz\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \frac{6\cos\theta\,z^2}{2} + 6^2\cos\theta\sin\theta\,z\,\Big|_{z=0}^{z=h}$$

$$= \int_{\theta=0}^{\theta=\pi/2} \left(3h^2 \cos\theta + 6^2 h \cos\theta \sin\theta \right) d\theta$$

$$= 3h^2 \sin\theta + 6^2 h \frac{\sin^2\theta}{2} \Bigg|_{\theta=0}^{\theta=\pi/2} = \boxed{3h^2 + 18h}$$

Ex2) Find the flux of $\vec{F} = \langle xz, yz, z^2 \rangle$ outward through the sphere $x^2+y^2+z^2=16$ lying in the first octant.



$$\hat{n} = \frac{\langle x, y, z \rangle}{4}$$

$$dS = 4^2 \sin\phi \, d\phi \, d\theta$$

$0 \le \theta \le \pi/2$

$0 \le \phi \le \pi/2$

$\vec{r}(\phi,\theta) = \langle 4\cos\theta \sin\phi, 4\sin\theta\sin\phi, \underbrace{4\cos\phi}_{= z} \rangle$

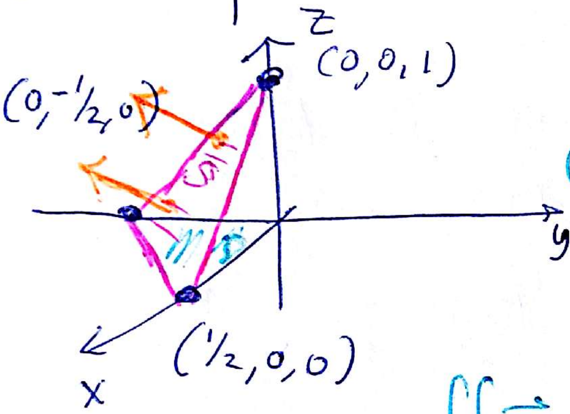
$$\text{flux} = \iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \frac{x^2 z + y^2 z + z^3}{4} \, dS$$

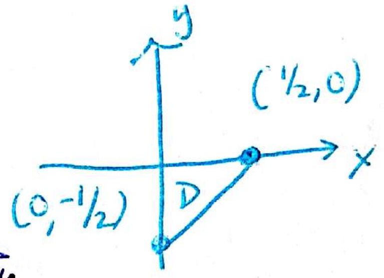
$$= \iint_D \frac{(x^2+y^2+z^2)z}{4} \, dS = \iint_D 4^3 z \sin\phi \, d\phi \, d\theta$$

$$= \iint_D 4^3 \cdot \underbrace{4\cos\phi}_{z} \sin\phi \, d\phi \, d\theta = \frac{\pi}{2} \cdot 4^4 \frac{\sin^2\phi}{2} \Bigg|_{\phi=0}^{\phi=\pi/2}$$

$$= \boxed{4^3 \pi}$$

Ex3) Find the flux of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ through the portion of the plane $2x - 2y + z = 1$ lying above the 4th quadrant. Let the normal to S point upward.





function

$$z = 1 - 2x + 2y$$

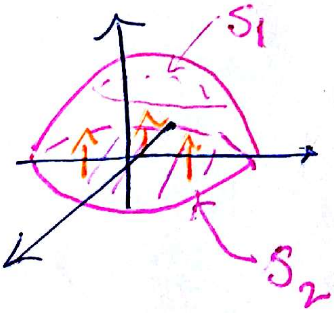
$$d\vec{S} = \pm\langle -z_x, -z_y, 1\rangle dA$$

$$d\vec{S} = \langle 2, -2, 1\rangle dx\, dy$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (2x - 2y + z)\, dx\, dy$$

$$= \iint_D 1\, dx\, dy = \text{Area}(D) = \boxed{\frac{1}{2^3}}$$

Ex 4) Compute the flux of $\vec{H}(x,y,z) = z\hat{k}$ inward. when S is the closed surface bounding the volume under $x^2+y^2+z^2 = a^2$ and above the plane $z = 0$.



$S = S_1 + S_2$

flux in S_1: $\hat{n} = -\dfrac{\langle x, y, z\rangle}{a}$

$$dS = a^2 \sin\phi\, d\phi\, d\theta$$

$0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi/2$

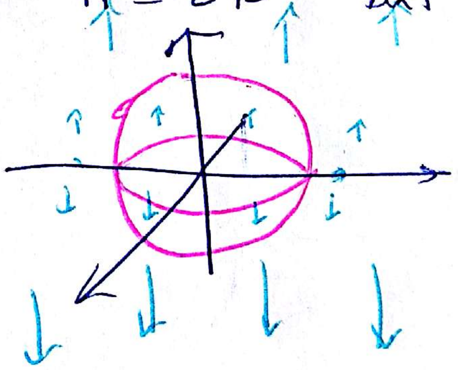
$$\text{flux} = \iint_{S_1} \vec{H}\cdot\hat{n}\, dS = \boxed{\iint_{S_1} -\frac{z^2}{a}\, dS}$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2} -\frac{a^2\cos^2\phi}{a}\, a^2\sin\phi\, d\phi\, d\theta = 2\pi\cdot\frac{\cos^3\phi}{3}\cdot a^3\Big|_{\phi=0}^{\phi=\pi/2}$$

$$= \boxed{-\frac{2\pi a^3}{3}}$$

$$\iint_{S_2} \vec{H}\cdot\hat{k}\, dS = \iint_{S_2} \overset{0}{\cancel{z}}\, dS = 0$$

Ex5) Same vector field $\vec{H} = z\hat{k}$ but S is sphere $x^2+y^2+z^2=a^2$



flux inward $= -\dfrac{4\pi a^3}{3}$

Surface Integral of scalar field $f(x,y,z)$

$$\iint_S f(x,y,z)\, dS$$