5/8 Review Eigenvectors Ch7
Systems of Differential Equations

Review Exam 3: Ch1, Ch2, 3, 4, 5, 7

Ch7: <u>Goal</u> Given a 2×2 matrix find its eigenvectors and eigenvalues.

Method: 1) Solve the characteristic polynomial to find eigenvalues
2) For each eigenvalue find an eigenvector.

Ex) Find eigenvectors and eigenvalues of

$$A=\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \rightarrow 1) \quad \boxed{\begin{vmatrix} (1-\lambda) & 5 \\ 2 & (4-\lambda) \end{vmatrix} = 0}$$

$$(1-\lambda)(4-\lambda)-10=0$$

$$\lambda^2-5\lambda-6=0$$

$$(\lambda-6)(\lambda+1)=0$$

There are two eigenvalues $\lambda_1=6$, $\lambda_2=-1$

2) $\lambda_1 = 6$ $\begin{bmatrix} -5 & 5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} -5x_1 + 5x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases}$ $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 6$

$\lambda_2 = -1$ $\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{cases} 2x_1 + 5x_2 = 0 \\ 2x_1 + 5x_2 = 0 \end{cases}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$.

What can go wrong? Answ: Characteristic equation may have complex roots or double roots.

Application: Differential Equations.

Ex 1) Solve $\frac{dx}{dt} = -3x$ so that $x(0) = 9$

$$\boxed{\frac{dx(t)}{dt} = -3x(t)}$$

Solution: $x(t) = Ce^{-3t}$ to find C

use $9 = x(0) = Ce^{-3(0)} = C$

Answer: $\boxed{x(t) = 9e^{-3t}}$

Check: $x' \stackrel{?}{=} -3x$

$$\underset{-27e^{-3t}}{\|} \qquad \underset{-27e^{-3t}}{\|}$$

Check: Initial Value $9 = x(0) = 9.1 = 9$

Formula: ALL SOLUTIONS

to $x' = ax$ are $x(t) = Ce^{at}$

$\frac{dx}{dt} = ax$

Ex2) Solve a linear 2x2 differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ if $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$

SAME

SAME

$$\begin{bmatrix} x_1'(t) = 2x_1(t) \\ x_2'(t) = 3x_1(t) - x_2(t) \end{bmatrix} \longleftrightarrow \begin{cases} x_1' = 2x_1 \\ x_2' = 3x_1 - x_2 \end{cases}$$

1) verify $\vec{x} = Ce^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution for any C = const.

$$\boxed{\begin{array}{ll} x_1 = Ce^{2t} & x_2 = Ce^{2t} \\ x_1' = 2Ce^{2t} & x_2' = 2Ce^{2t} \end{array}}$$

$\lambda = 2$ is an eigenvalue with corresp eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) verify $\vec{x} = De^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution for any D.

$$\boxed{\begin{array}{ll} x_1 = 0 & x_2 = De^{-t} \\ x_1' = 0 & x_2' = -De^{-t} \end{array}}$$

$\lambda = -1$ is an eigenvalue with eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ALL SOLUTIONS:

$$\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$$

☆ Verify that $\vec{x}$ solves the D.E $\vec{x}' = A\vec{x}$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$\frac{d\vec{x}}{dt} = 2c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} - c_2 e^{-t}\begin{bmatrix}0\\1\end{bmatrix}$$

$$A\vec{x} = A\left(c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= c_1 e^{2t}\underbrace{A\begin{bmatrix}1\\1\end{bmatrix}}_{=\begin{bmatrix}2\\2\end{bmatrix}} + c_2 e^{-t}\underbrace{A\begin{bmatrix}0\\1\end{bmatrix}}_{=\begin{bmatrix}0\\-1\end{bmatrix}}$$

$$A\vec{x} = c_1 e^{2t}\begin{bmatrix}2\\2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}0\\-1\end{bmatrix}$$

Ex) Solve IVP = D.E + Initial Condition

$$\begin{cases} x_1' = -3x_1 + 2x_2 \\ x_2' = 2x_1 - 3x_2 \end{cases}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

solution:

1. $$\begin{vmatrix} (-3-\lambda) & 2 \\ 2 & (-3-\lambda) \end{vmatrix} = 0 \longleftrightarrow (-3-\lambda)(-3-\lambda)-4=0$$

$$\lambda^2+6\lambda+5=0$$

$$(\lambda+5)(\lambda+1)=0$$

2. $$\boxed{\begin{matrix} \lambda_1 = -5 \\ \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{matrix}} \quad \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{cases} 2x_1 + 2x_2 = 0 \\ 2x_1 + 2x_2 = 0 \end{cases}$$

$$\boxed{\begin{matrix} \lambda_2 = -1 \\ \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}} \quad \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{cases} -2x_1 + 2x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases}$$

ALL SOLUTIONS TO $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$

are $\vec{x} = c_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix}4\\2\end{bmatrix} = \vec{x}(0) = c_1 e^0 \begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^0 \begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{cases} 4 = c_1 + c_2 \\ 2 = -c_1 + c_2 \end{cases}$$

$$6 = 2c_2 \rightarrow c_2 = 3, \; c_1 = 1$$

SOLUTION: $\vec{x} = e^{-5t} \begin{bmatrix}1\\-1\end{bmatrix} + 3e^{-t} \begin{bmatrix}1\\1\end{bmatrix}$

or
$$\boxed{\begin{aligned} x_1(t) &= e^{-5t} + 3e^{-t} \\ x_2(t) &= -e^{-5t} + 3e^{-t} \end{aligned}}$$

$x_1' =$

$x_2' =$

Exam Review: Dec 2006

#1) $A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$

a) find ~~the~~ A^{-1}

b) use answer (a) to solve $\begin{cases} 2x + 2z = 2 \\ 2x + y + z = 5 \\ 3x + 2y + 2z = 8 \end{cases}$

c) solve for x in (b) using Cramer's

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